Commun. Korean Math. Soc. **31** (2016), No. 3, pp. 507–518
http://dx.doi.org/10.4134/CKMS.c150173
pISSN: 1225-1763 / eISSN: 2234-3024

COMMON FIXED POINTS FOR SINGLE-VALUED AND MULTI-VALUED MAPPINGS IN COMPLETE ℝ-TREES

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Abstract. The aim of this paper is to prove some strong convergence theorems for the modified Ishikawa iteration process involving a pair of a generalized asymptotically nonexpansive single-valued mapping and a quasi-nonexpansive multi-valued mapping in the framework of ℝ-trees under the gate condition.

1. Introduction

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map ϕ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\phi(0) = x$, $\phi(l) = y$, and $d(\phi(t_1), \phi(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, ϕ is an isometry and $d(x, y) = l$. The image of ϕ is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. For each $x, y \in X$ and $\alpha \in (0, 1)$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$. The space (X, d) is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A nonempty subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points. A nonempty subset D of X is said to be *gated* if for any point $x \notin D$ there is a unique point y_x such that for any $z \in D$,

$$d(x, z) = d(x, y_x) + d(y_x, z).$$

Clearly, gate sets in a complete geodesic space are always closed and convex. The point y_x is called the *gate* of x in D. It is easy to see that y_x is also the unique nearest point of x in D.

Definition 1.1. An ℝ-tree is a geodesic metric space X such that:

Received September 16, 2015; Revised November 16, 2015.
2010 *Mathematics Subject Classification.* Primary 47H09, 47H10.
Key words and phrases. generalized asymptotically nonexpansive mapping, quasi-nonexpansive mapping, ℝ-trees, gate condition.
This work was financially supported by Research and Development Institute, Nakhon Pathom Rajabhat University, Thailand.

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in X$;

(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

It follows by (i) and (ii) that

(iii) if $u, v, w \in X$, then $[u, v] \cap [u, w] = [u, z]$ for some $z \in X$.

An $\mathbb{R}$-tree is a special case of a CAT(0) space. For a thorough discussion of CAT(0) spaces and their applications, see [5]. Note that a metric space X is a complete $\mathbb{R}$-tree if and only if X is hyperconvex with unique geodesic segments, see [8].

$\mathbb{R}$-trees were introduced by Tits [18] in 1977. Fixed point theory for single-valued mappings in $\mathbb{R}$-trees was first studied by Kirk [9]. He proved that every continuous single-valued mappings defined on a geodesically bounded complete $\mathbb{R}$-tree always has a fixed point. Since then fixed point theorems for various types of single-valued and multi-valued mappings in $\mathbb{R}$-trees has been rapidly developed and many of papers have appeared (*e.g.*, see [2, 3, 4, 7, 9, 11]).

In 2009, Shahzad and Zegeye [16] proved strong convergence theorems of the Ishikawa iteration for a quasi-nonexpansive multi-valued mapping satisfying the endpoint condition in Banach spaces. They also constructed a modified Ishikawa iteration and proved strong convergence theorems of the proposed iteration without the endpoint condition. Later in 2010, Puttasontiphot [13] obtained similar results in complete CAT(0) spaces. In 2012, Samanmit and Panyanak [15] introduced a new condition on mappings in $\mathbb{R}$-trees which is more general than the endpoint condition, call it the *gate condition*, and proved strong convergence theorems of a modified Ishikawa iteration for a quasi-nonexpansive multi-valued mapping satisfying such condition in $\mathbb{R}$-trees.

In 2011, Sokhuma and Kaewkhao [17] introduced the following modified Ishikawa iterative process for finding a common fixed point of a pair of a non-expansive single-valued mapping T and a nonexpansive multi-valued mapping S in uniformly convex Banach spaces. For an initial point $x_1 \in D$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$(1) \qquad \begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n z_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \ n \in \mathbb{N}, \end{cases}$$

where $z_n \in Sx_n$, $0 \leq \alpha_n, \beta_n \leq 1$, and S satisfies the endpoint condition. They proved that the sequence $\{x_n\}$ generated by (1) converges strongly to a common fixed point of T and S under some suitable conditions.

Recently, Akkasriworn and Sokhuma [1] extended the results of Sokhuma and Kaewkhao [17] to a pair of an asymptotically nonexpansive single-valued mapping and a nonexpansive multi-valued mapping in CAT(0) spaces. They also proposed the following iterative process for finding a common fixed point of a pair of an asymptotically nonexpansive single-valued mapping T and a nonexpansive multi-valued mapping S in CAT(0) spaces. For an initial point

$x_1 \in D$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n z_n, \\ x_{n+1} = (1-\beta_n)x_n \oplus \beta_n T^n y_n, \ n \in \mathbb{N}, \end{cases} \tag{2}$$

where $z_n \in ST^n x_n$, $0 \leq \alpha_n, \beta_n \leq 1$, S satisfies the endpoint condition, and T, S are commuting.

Remark 1.2. We note that the iterative process (2) is very complicated. The condition that $z_n \in ST^n x_n$ and T, S are commuting may not be necessary.

In this paper, motivated by the above results and Remark 1.2, we introduce a new iterative process which is a modification of (2) and obtain the strong convergence theorems for finding a common fixed point of a pair of a generalized asymptotically nonexpansive single-valued mapping and a quasi-nonexpansive multi-valued mapping in the framework of $\mathbb{R}$-trees under the gate condition. Our results extend and improve the results of Sokhuma and Kaewkhao [17], Akkasriworn and Sokhuma [1], Samanmit and Panyanak [15], and the corresponding results given by many authors.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. Let D be a nonempty subset of a metric space X. Let $T : D \to D$ be a single-valued mapping. The set of all fixed points of T will be denoted by $F(T) = \{x \in D : x = Tx\}$.

Definition 2.1. A single-valued mapping $T : D \to D$ is said to be

(i) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$;

(ii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $d(T^n x, T^n y) \leq k_n d(x, y)$ for all $x, y \in D$ and $n \in \mathbb{N}$;

(iii) *generalized asymptotically nonexpansive* if there exist two sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ and $d(T^n x, T^n y) \leq k_n d(x, y) + s_n$ for all $x, y \in D$ and $n \in \mathbb{N}$;

(iv) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L d(x, y)$ for all $x, y \in D$ and $n \in \mathbb{N}$.

In the case of $s_n = 0$ for all $n \in \mathbb{N}$, the mapping T will be called *an asymptotically nonexpansive mapping*. In particular, if $k_n = 1$ and $s_n = 0$ for all $n \in \mathbb{N}$, a single-valued mapping T reduce to a *nonexpansive mapping*. The fixed point property for generalized asymptotically nonexpansive single-valued mappings can be found in [12]. The next example shows that there is a generalized asymptotically nonexpansive mapping which is not asymptotically nonexpansive and its fixed point set is not necessarily closed.

Example 2.2 ([12])**.** Define a single-valued mapping $T : \left[-\frac{2}{3}, \frac{2}{3}\right] \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ by

$$Tx = \begin{cases} x, & \text{if } x \in \left[-\dfrac{2}{3}, 0\right), \\ \dfrac{16}{81}, & \text{if } x = 0, \\ x^4, & \text{if } x \in \left(0, \dfrac{2}{3}\right]. \end{cases}$$

Then T is generalized asymptotically nonexpansive. It is clear that T is not asymptotically nonexpansive and $F(T) = \left[-\frac{2}{3}, 0\right)$ which is not closed.

Remark 2.3. It is worth mentioning that if T is uniformly L-Lipschitzian and generalized asymptotically nonexpansive, then $F(T)$ is always closed.

We shall denote the family of nonempty closed bounded subsets of D by $CB(D)$, the family of nonempty closed convex subsets of D by $CC(D)$, and the family of nonempty compact convex subsets of D by $KC(D)$. The *Pompeiu-Hausdorff distance* [19] on $CB(D)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\right\} \text{ for } A, B \in CB(D).$$

where $\operatorname{dist}(x, D) = \inf\{d(x, y) : y \in D\}$ is the distance from a point x to a subset D. Let S be a multi-valued mapping of D into $CB(D)$. The set of all fixed points of S will be denoted by $F(S) = \{x \in D : x \in Sx\}$. A point $x \in D$ is called an *endpoint* of S if x is a fixed point of S and $Sx = \{x\}$. The set of all endpoints of S will be denoted by $End(S)$. We see that for each mapping S, $End(S) \subseteq F(S)$ and the converse is not true in general. A multi-valued mapping S is said to satisfy the *endpoint condition* if $End(S) = F(S)$. A point x is called a *common fixed point* of T and S if $x = Tx \in Sx$.

Definition 2.4. A multi-valued mapping $S : D \to CB(D)$ is said to

(i) be *nonexpansive* if $H(Sx, Sy) \le d(x, y)$ for all $x, y \in D$;
(ii) be *quasi-nonexpansive* if $F(S) \ne \emptyset$ and $H(Sx, Sz) \le d(x, z)$ for all $x \in D$ and $z \in F(S)$;
(iii) be *hemicompact* if for any sequence $\{x_n\}$ in D such that
$$\lim_{n\to\infty} \operatorname{dist}(x_n, Sx_n) = 0,$$
there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in D$. We note that if D is compact, then every multi-valued mapping S is hemicompact.
(iv) satisfy *condition* (E_μ) where $\mu \ge 0$ if for each $x, y \in D$,
$$\operatorname{dist}(x, Sy) \le \mu \operatorname{dist}(x, Sx) + d(x, y).$$
We say that S satisfies *condition* (E) whenever S satisfies (E_μ) for some $\mu \ge 1$.

Remark 2.5. From Definition 2.4, it is clear that

(i) every nonexpansive multi-valued mapping S with $F(S) \neq \emptyset$ is quasi-nonexpansive but there exist quasi-nonexpansive mappings that are not nonexpansive;
(ii) if S is nonexpansive, then S satisfies the condition (E_1);
(iii) if S is quasi-nonexpansive, then $F(S)$ is closed.

The next example shows that there is a quasi-nonexpansive mapping which is not nonexpansive.

Example 2.6. Let $D = [0, \infty)$ with the usual metric and $S : D \to CB(D)$ be a multi-valued mapping defined by

$$Sx = \begin{cases} \{0\}, & \text{if } x \in [0,2], \\ \left[x - \dfrac{7}{4}, x - \dfrac{4}{3}\right], & \text{if } x \in (2, \infty). \end{cases}$$

Then S is quasi-nonexpansive and $F(S) = \{0\}$. It is easy to see that S is not nonexpansive since $H(S(4), S(2)) = H([\frac{9}{4}, \frac{8}{3}], \{0\}) = \frac{8}{3} > 2 = |4 - 2|$.

Let $S : D \to CC(D)$ be a multi-valued mapping with $F(S) \neq \emptyset$. We say that a point $u \in D$ is a *key* of S if, for each $x \in F(S)$, x is the gate of u in Sx. We say that S satisfies the *gate condition* if S has a key in D. It is clear that the endpoint condition implies the gate condition but the converse is not true. The following example shows that there is a mapping satisfying the gate condition but does not satisfy the endpoint condition.

Example 2.7 ([15])**.** Let $D = [0, 1]$ and $S : D \to CC(D)$ be defined by

$$Sx = [0, x] \text{ for all } x \in D.$$

We see that $F(S) = [0, 1]$ and $u = 1$ is a key of S. It is obvious that $End(S) = \{0\}$. Then S does not satisfy the endpoint condition.

We now collect some basic properties of $\mathbb{R}$-trees.

Lemma 2.8. *Let X be a complete $\mathbb{R}$-tree. Then the following statements hold*:

(i) [6] *If $x, y, z \in X$ and $\alpha \in [0, 1]$, then*
$$d(z, \alpha x \oplus (1-\alpha)y)^2 \leq \alpha d(z, x)^2 + (1-\alpha)d(z, y)^2 - \alpha(1-\alpha)d(x, y)^2.$$
(ii) [6] *If $x, y, z \in X$, then $d(x, z) + d(z, y) = d(x, y)$ if and only if $z \in [x, y]$.*
(iii) [7] *The gate subsets of X are precisely its closed and convex subsets.*
(iv) [11] *If A and B are bounded closed convex subsets of X, then*
$$d(P_A(u), P_B(u)) \leq H(A, B)$$
for any $u \in X$, where $P_A(u), P_B(u)$ are respectively the unique nearest points of u in A and B.

The following result is a characterization of CAT(0) spaces. It can be applied to an $\mathbb{R}$-tree as well.

Lemma 2.9 ([10]). *Let X be a CAT(0) space, and let $x \in X$. Suppose that $\{t_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and that $\{x_n\}, \{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(x_n, x) \le R$, $\limsup_{n\to\infty} d(y_n, x) \le R$ *and*

$$\lim_{n\to\infty} d((1 - t_n)x_n \oplus t_n y_n, x) = R \text{ for some } R \ge 0.$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

The following facts are needed for proving our main results.

Definition 2.10 ([14]). Let F be a nonempty subset of a complete metric space X and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is of *monotone type* (I) *with respect to F* if there exist sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $d(x_{n+1}, p) \le (1 + \delta_n)d(x_n, p) + \varepsilon_n$ for all $n \in \mathbb{N}$ and $p \in F$.

Proposition 2.11 ([14]). *Let F be a nonempty closed subset of a complete metric space X and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is of monotone type (I) with respect to F and* $\liminf_{n\to\infty} dist(x_n, F) = 0$, *then* $\lim_{n\to\infty} x_n = p$ *for some $p \in F$.*

Lemma 2.12 ([20]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfying*:

$$a_{n+1} \le (1 + c_n)a_n + b_n \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$. *Then*

(i) $\lim_{n\to\infty} a_n$ *exists.*
(ii) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T : D \to D$ be a generalized asymptotically nonexpansive single-valued mapping with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $S : D \to KC(D)$ be a quasi-nonexpansive multi-valued mapping satisfying the gate condition. Assume that $F(T) \cap F(S)$ is nonempty and closed. Let u be a key of S. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1 - \alpha_n)x_n \oplus \alpha_n z_n \text{ for all } n \in \mathbb{N},$$

where z_n is the gate of u in Sx_n, and

$$x_{n+1} = (1 - \beta_n)x_n \oplus \beta_n T^n y_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Then $\lim_{n\to\infty} d(x_n, p)$ *exists for all $p \in F(T) \cap F(S)$.*

Proof. Let $p \in F(T) \cap F(S)$, we have

$$\begin{aligned}
d(x_{n+1}, p) &\le (1-\beta_n)d(x_n, p) + \beta_n d(T^n y_n, p) \\
&\le (1-\beta_n)d(x_n, p) + \beta_n (k_n d(y_n, p) + s_n) \\
&= (1-\beta_n)d(x_n, p) + \beta_n k_n d(y_n, p) + \beta_n s_n \\
&\le (1-\beta_n)d(x_n, p) + \beta_n k_n((1-\alpha_n)d(x_n, p) + \alpha_n d(z_n, p)) + \beta_n s_n \\
&= (1-\beta_n + \beta_n k_n(1-\alpha_n))d(x_n, p) + \beta_n \alpha_n k_n d(z_n, p) + \beta_n s_n \\
&\le (1-\beta_n + \beta_n k_n(1-\alpha_n))d(x_n, p) + \beta_n \alpha_n k_n d(P_{Sx_n}(u), P_{Sp}(u)) \\
&\quad + \beta_n s_n \\
&\le (1-\beta_n + \beta_n k_n(1-\alpha_n))d(x_n, p) + \beta_n \alpha_n k_n H(Sx_n, Sp) + \beta_n s_n \\
&\le (1-\beta_n + \beta_n k_n(1-\alpha_n))d(x_n, p) + \beta_n \alpha_n k_n d(x_n, p) + \beta_n s_n \\
&= (1-\beta_n + \beta_n k_n(1-\alpha_n) + \beta_n \alpha_n k_n)d(x_n, p) + \beta_n s_n \\
&= (1-\beta_n + \beta_n k_n)d(x_n, p) + \beta_n s_n \\
&= (1+\beta_n(k_n - 1))d(x_n, p) + \beta_n s_n \\
&\le (1+(k_n - 1))d(x_n, p) + s_n.
\end{aligned}$$

By Lemma 2.12, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, we conclude that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T) \cap F(S)$. □

Lemma 3.2. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T : D \to D$ be a generalized asymptotically nonexpansive single-valued mapping with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $S : D \to KC(D)$ be a quasi-nonexpansive multi-valued mapping satisfying the gate condition. Assume that $F(T) \cap F(S)$ is nonempty and closed. Let u be a key of S. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1-\alpha_n)x_n \oplus \alpha_n z_n \text{ for all } n \in \mathbb{N},$$

where z_n is the gate of u in Sx_n, and

$$x_{n+1} = (1-\beta_n)x_n \oplus \beta_n T^n y_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$. Then, we have $\lim_{n\to\infty} d(x_n, z_n) = 0$ and $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. Moreover, if a single-valued mapping T is also uniformly L-Lipschitzian, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Let $p \in F(T) \cap F(S)$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists. Set

$$\lim_{n\to\infty} d(x_n, p) = c.$$

If $c = 0$, then all the conclusions are trivial. Therefore we will assume that $c > 0$. By the definition of the sequence $\{x_n\}$, we have

$$d(T^n y_n, p) \le k_n d(y_n, p) + s_n$$

$$
\begin{aligned}
&\le k_n(1-\alpha_n)d(x_n,p) + k_n\alpha_n d(z_n,p) + s_n \\
&\le k_n(1-\alpha_n)d(x_n,p) + k_n\alpha_n d(P_{Sx_n}(u), P_{Sp}(u)) + s_n \\
&\le k_n(1-\alpha_n)d(x_n,p) + k_n\alpha_n H(Sx_n, Sp) + s_n \\
&\le k_n(1-\alpha_n)d(x_n,p) + k_n\alpha_n d(x_n,p) + s_n \\
&= k_n d(x_n,p) + s_n.
\end{aligned}
$$

It follows from $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$ that

$$\limsup_{n\to\infty} d(T^n y_n, p) \le \limsup_{n\to\infty} d(y_n,p) \le \limsup_{n\to\infty} d(x_n,p). \tag{3}$$

Since $c = \limsup_{n\to\infty} d(x_{n+1},p) = \limsup_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n T^n y_n, p)$, it follows by Lemma 2.9 that

$$\lim_{n\to\infty} d(x_n, T^n y_n) = 0. \tag{4}$$

Consider

$$
\begin{aligned}
d(x_{n+1},p) &\le (1-\beta_n)d(x_n,p) + \beta_n d(T^n y_n, p) \\
&\le (1-\beta_n)d(x_n,p) + \beta_n(k_n d(y_n,p) + s_n).
\end{aligned}
$$

This implies that

$$d(x_{n+1},p) - d(x_n,p) \le \beta_n(k_n d(y_n,p) - d(x_n,p) + s_n).$$

Therefore,

$$
\begin{aligned}
\frac{d(x_{n+1},p) - d(x_n,p)}{b} + d(x_n,p) &\le \frac{d(x_{n+1},p) - d(x_n,p)}{\beta_n} + d(x_n,p) \\
&\le k_n d(y_n,p) + s_n.
\end{aligned}
$$

It implies by (3) that

$$
\begin{aligned}
c &= \liminf_{n\to\infty} \left(\frac{d(x_{n+1},p) - d(x_n,p)}{b} + d(x_n,p) \right) \\
&\le \liminf_{n\to\infty} (k_n d(y_n,p) + s_n) \\
&= \liminf_{n\to\infty} d(y_n,p) \\
&\le \limsup_{n\to\infty} d(y_n,p) \le c.
\end{aligned}
$$

Thus,

$$c = \lim_{n\to\infty} d(y_n,p) = \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n z_n, p).$$

Since

$$d(z_n,p) = d(P_{Sx_n}(u), P_{Sp}(u)) \le H(Sx_n, Sp) \le d(x_n,p),$$

it implies that

$$\limsup_{n\to\infty} d(z_n,p) \le \limsup_{n\to\infty} d(x_n,p) = c.$$

Using Lemma 2.9, we get

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{5}$$

Next, we show that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. Since T is generalized asymptotically nonexpansive, we have

$$\begin{aligned} d(T^n x_n, x_n) &\le d(T^n x_n, T^n y_n) + d(T^n y_n, x_n) \\ &\le k_n d(x_n, y_n) + s_n + d(T^n y_n, x_n) \\ &= k_n \alpha_n d(x_n, z_n) + d(T^n y_n, x_n) + s_n \\ &\le k_n d(x_n, z_n) + d(T^n y_n, x_n) + s_n. \end{aligned}$$

Then, by (4) and (5), we get

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0. \tag{6}$$

Finally, if T is uniformly L-Lipschitzian, then we have

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\le (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(T^n x_n, x_n) \\ &\le (1+L)\beta_n d(x_n, T^n y_n) + d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(T^n x_n, x_n) \\ &\le (1+L)b d(x_n, T^n y_n) + d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(T^n x_n, x_n). \end{aligned}$$

By (4) and (6), we conclude that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

By Remarks 2.3 and 2.5(iii), $F(T) \cap F(S)$ is always closed. Then we have the following strong convergence theorem in complete $\mathbb{R}$-trees.

Theorem 3.3. *Let D be a nonempty compact convex subset of a complete $\mathbb{R}$-tree X. Let $T : D \to D$ be a uniformly L-Lipschitzian and generalized asymptotically nonexpansive single-valued mapping with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $S : D \to KC(D)$ be a quasi-nonexpansive multi-valued mapping satisfying the gate condition and the condition (E). Assume that $F(T) \cap F(S)$ is nonempty. Let u be a key of S. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1-\alpha_n)x_n \oplus \alpha_n z_n \ \textit{for all } n \in \mathbb{N},$$

where z_n is the gate of u in Sx_n, and

$$x_{n+1} = (1-\beta_n)x_n \oplus \beta_n T^n y_n \ \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and S.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded. Since D is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges strongly to p in D. By condition (E), there exists $\mu \geq 1$ such that

$$\begin{aligned} \operatorname{dist}(p, Sp) &\leq d(p, x_{n_i}) + \operatorname{dist}(x_{n_i}, Sp) \\ &\leq d(x_{n_i}, p) + \mu \operatorname{dist}(x_{n_i}, Sx_{n_i}) + d(x_{n_i}, p) \\ &= 2d(x_{n_i}, p) + \mu \operatorname{dist}(x_{n_i}, Sx_{n_i}) \\ &\leq 2d(x_{n_i}, p) + \mu d(x_{n_i}, z_n). \end{aligned}$$

Then, by Lemma 3.2, we have $p \in F(S)$. Since T is uniformly L-Lipschitzian, we have

$$\begin{aligned} d(Tp, p) &\leq d(Tp, Tx_{n_i}) + d(Tx_{n_i}, x_{n_i}) + d(x_{n_i}, p) \\ &\leq (L+1)d(x_{n_i}, p) + d(Tx_{n_i}, x_{n_i}). \end{aligned}$$

By Lemma 3.2, it implies that $p \in F(T)$.

Therefore, $p \in F(T) \cap F(S)$.

Since $\lim_{n\to\infty} d(x_n, p)$ exists, we get $\lim_{n\to\infty} d(x_n, p) = \lim_{i\to\infty} d(x_{n_i}, p) = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of T and S. □

The compactness of D can be dropped if a multi-valued mapping S is hemicompact. Then the following theorem is obtained immediately from Theorem 3.3.

Theorem 3.4. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T : D \to D$ be a uniformly L-Lipschitzian and generalized asymptotically nonexpansive single-valued mapping with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $S : D \to KC(D)$ be a quasi-nonexpansive multi-valued mapping satisfying the gate condition and the condition (E). Assume that $F(T) \cap F(S)$ is nonempty. Let u be a key of S. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1 - \alpha_n)x_n \oplus \alpha_n z_n \ \text{ for all } n \in \mathbb{N},$$

where z_n is the gate of u in Sx_n, and

$$x_{n+1} = (1 - \beta_n)x_n \oplus \beta_n T^n y_n \ \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. If S is hemicompact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and S.

Proof. Since S is hemicompact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges strongly to p in D. As in the proof of Theorem 3.3, we obtain that the sequence $\{x_n\}$ converges strongly to a common fixed point of T and S. □

In our next theorem, we show that the condition (E) and hemicompactness of S in Theorem 3.4 can be omitted if $\liminf_{n\to\infty} \operatorname{dist}(x_n, F(T) \cap F(S)) = 0$.

Theorem 3.5. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T : D \to D$ be a uniformly L-Lipschitzian and generalized asymptotically nonexpansive single-valued mapping with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $S : D \to KC(D)$ be a quasi-nonexpansive multi-valued mapping satisfying the gate condition. Assume that $F(T) \cap F(S)$ is nonempty. Let u be a key of S. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1 - \alpha_n)x_n \oplus \alpha_n z_n \text{ for all } n \in \mathbb{N},$$

where z_n is the gate of u in Sx_n, and

$$x_{n+1} = (1 - \beta_n)x_n \oplus \beta_n T^n y_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of T and S if and only if $\liminf_{n\to\infty} dist(x_n, F(T) \cap F(S)) = 0$.

Proof. The necessity is obvious and then we prove only the sufficiency. Suppose that $\liminf_{n\to\infty} \operatorname{dist}(x_n, F(T) \cap F(S)) = 0$. In the proof of Lemma 3.1, we obtain that the sequence $\{x_n\}$ is of monotone type (I) with respect to $F(T) \cap F(S)$. By the closedness of $F(T) \cap F(S)$ and Proposition 2.11, we have $\{x_n\}$ converges strongly to a point in $F(T) \cap F(S)$. □

Remark 3.6. Theorems 3.3-3.5 extend and improve the results of Sokhuma and Kaewkhao [17] to a pair of a generalized asymptotically nonexpansive single-valued mapping and a quasi-nonexpansive multi-valued mapping in $\mathbb{R}$-tree without assuming the endpoint condition. Theorems 3.3-3.5 improve the results of Samanmit and Panyanak [15] to a pair of a generalized asymptotically nonexpansive single-valued mapping and a quasi-nonexpansive multi-valued mapping.

Acknowledgements. The authors would like to thank the referees for valuable suggestions on the research paper.

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