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STABILITY OF TWO FUNCTIONAL EQUATIONS ARISING FROM DETERMINANT OF MATRICES

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Abstract. Let $f:\mathbb{R}^3\to\mathbb{R}$. In this paper we prove the stability of functional inequalities

$$|f(ux+vy,uy-vx,zw)-f(x,y,z)f(u,v,w)|\le\phi(u,v,w) \text{ or } \phi(x,y,z),$$

$$|f(ux-vy,uy-vx,zw)-f(x,y,z)f(u,v,w)|\le\phi(u,v,w) \text{ or } \phi(x,y,z)$$

for all $x,y,z,u,v,w\in\mathbb{R}$. Furthermore, we give refined descriptions of bounded functions satisfying the inequalities as in Albert and Baker [1].

1. Introduction

Throughout this paper we denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^n$ the set of real numbers, nonnegative real numbers and the n-dimensional Euclidean space, respectively, and consider the functions $f,g:\mathbb{R}^2\longrightarrow\mathbb{R}$ and $\phi:\mathbb{R}^2\longrightarrow\mathbb{R}^+$. A function $M:\mathbb{R}\longrightarrow\mathbb{R}$ is called *a multiplicative function* provided that $M(xy)=M(x)M(y)$ for all $x,y\in\mathbb{R}$ and $E:\mathbb{R}\longrightarrow\mathbb{R}$ is called *an exponential function* provided that $E(x+y)=E(x)E(y)$ for all $x,y\in\mathbb{R}$.

In the problems column of the *News Letter* of the European Mathematical Society, Sahoo posed the problem to determine the general solutions $g:\mathbb{R}^2\longrightarrow\mathbb{R}$ of functional equations

$$g(ux+vy,uy-vx)=g(x,y)+g(u,v)+g(x,y)g(u,v), \tag{1.1}$$

$$g(ux-vy,uy-vx)=g(x,y)+g(u,v)+g(x,y)g(u,v) \tag{1.2}$$

for all $x,y,u,v\in\mathbb{R}$. Houston and Sahoo in [8] showed that the general solutions of functional equations (1.1) and (1.2) are given by

$$g(x,y)=M\left(x^2+y^2\right)-1,$$

and

$$g(x,y)=M\left(x^2-y^2\right)-1$$

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for all $x, y \in \mathbb{R}$, respectively, where $M : \mathbb{R} \longrightarrow \mathbb{R}$ is a multiplicative function. Replacing $g(x, y)$ by $f(x, y) - 1$ in (1.1) and (1.2), functional equations (1.1) and (1.2) are reduced to

$$f(ux + vy, uy - vx) = f(x, y)f(u, v), \tag{1.3}$$

$$f(ux - vy, uy - vx) = f(x, y)f(u, v) \tag{1.4}$$

for all $x, y, u, v \in \mathbb{R}$. The functional equations (1.3) and (1.4) are connected with the characterizations of determinant and permanent of two-by-two matrices. Furthermore, functional equation (1.3) arises from number theory (see Jung-Bae [9] or Chung-Chang [4]). Let us define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$f(x, y, z) = \det \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ -y & 0 & x \end{pmatrix}$$

for all $x, y, z \in \mathbb{R}$. Then, since

$$\det \begin{pmatrix} ux + vy & 0 & uy - vx \\ 0 & zw & 0 \\ -(uy - vx) & 0 & ux + vy \end{pmatrix} = \det \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ -y & 0 & x \end{pmatrix} \cdot \det \begin{pmatrix} u & 0 & v \\ 0 & w & 0 \\ -v & 0 & u \end{pmatrix}$$

for all $x, y, z, u, v, w \in \mathbb{R}$, we obtain the functional equation

$$f(ux + vy, uy - vx, zw) = f(x, y, z)f(u, v, w) \tag{1.5}$$

for all $x, y, z, u, v, w \in \mathbb{R}$, where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Similarly, defining $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$f(x, y, z) = \det \begin{pmatrix} x & 0 & -y \\ 0 & z & 0 \\ -y & 0 & x \end{pmatrix}$$

for all $x, y, z \in \mathbb{R}$, we obtain the functional equation

$$f(ux - vy, uy - vx, zw) = f(x, y, z)f(u, v, w) \tag{1.6}$$

for all $x, y, z, u, v, w \in \mathbb{R}$, where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$. In other word, $f(x, y, z) = \left(x^2 + y^2\right) z$ is a solution of functional equation (1.5) and $f(x, y, z) = \left(x^2 - y^2\right) z$ is a solution of functional equation (1.6). In this paper, we first discuss the general solutions of functional equations (1.5) and (1.6) and then as main results of the paper we consider the Ulam-Hyers stability (see [7]) of functional equations (1.5) and (1.6), i.e., we investigate both unbounded functions and bounded functions f satisfying the functional inequalities

$$|f(ux + vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \phi(u, v, w), \tag{1.7}$$

$$|f(ux + vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \phi(x, y, z), \tag{1.8}$$

$$|f(ux - vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \phi(u, v, w), \tag{1.9}$$

$$|f(ux - vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \phi(x, y, z) \tag{1.10}$$

for all $x, y, z, u, v, w \in \mathbb{R}$, where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $\phi : \mathbb{R}^3 \longrightarrow [0, \infty)$ is a fixed function. In particular, we give refined descriptions of bounded functions satisfying the functional inequalities (1.7)∼ (1.10) as in Albert and Baker [1] and Chung [3] in which refined descriptions of bounded functions satisfying exponential functional inequalities are given.

2. General solutions of (1.5) and (1.6)

In this section, we find general solutions of functional equations (1.5) and (1.6). The following result can be found in [4, 8, 9].

Lemma 2.1. *Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy the functional equation*

$$g(ux + vy, uy - vx) = g(x, y)g(u, v) \tag{2.1}$$

for all $x, y, u, v \in \mathbb{R}$. Then $g = 1$ or has the form

$$g(x, y) = M\left(\sqrt{x^2 + y^2}\right) \tag{2.2}$$

for all $x, y \in \mathbb{R}$, where $M : \mathbb{R} \longrightarrow \mathbb{R}$ is a multiplicative function.

The following result can be found in [5].

Lemma 2.2. *Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy the functional equation*

$$g(ux - vy, uy - vx) = g(x, y)g(u, v) \tag{2.3}$$

for all $x, y, u, v \in \mathbb{R}$. Then $g = 1$ or has the form

$$g(x, y) = M\left(x^2 - y^2\right) \tag{2.4}$$

for all $x, y \in \mathbb{R}$, where $M : \mathbb{R} \longrightarrow \mathbb{R}$ is a multiplicative function.

Theorem 2.3. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfy* (1.5)*. Then f has the form*

$$f(x, y, z) = M_1\left(\sqrt{x^2 + y^2}\right) M_2(z) \tag{2.5}$$

for all $x, y, z \in \mathbb{R}$, where $M_1, M_2 : \mathbb{R} \longrightarrow \mathbb{R}$ are multiplicative functions.

Proof. Define $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $M_2 : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$g(x, y) = f(x, y, 1),\ M_2(z) = f(1, 0, z) \tag{2.6}$$

for all $x, y, z \in \mathbb{R}$. Then from (1.5) and (2.1) we have

$$g(ux + vy, uy - vx) = g(x, y)g(u, v),\ M_2(zw) = M_2(z)M_2(w) \tag{2.7}$$

for all $x, y, u, v, z, w \in \mathbb{R}$. Thus, from (1.5) we have

$$f(x, y, z) = f(x, y, 1)f(1, 0, z) = g(x, y)M_2(z) \tag{2.8}$$

for all $x, y, z \in \mathbb{R}$. Using Lemma 2.1 we get (2.5). □

Using Lemma 2.2 we obtain the following.

Theorem 2.4. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfy* (1.6)*. Then f has the form*

$$f(x, y, z) = M_1\left(x^2 - y^2\right) M_2(z) \tag{2.9}$$

for all $x, y, z \in \mathbb{R}$, where $M_1, M_2 : \mathbb{R} \longrightarrow \mathbb{R}$ are multiplicative functions.

3. Unbounded functions satisfying the functional inequalities (1.7) ∼ (1.10)

In this section, we investigate unbounded functions satisfying each of functional inequalities (1.7) ∼ (1.10).

Theorem 3.1. *Let* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *be an unbounded function satisfying* (1.7) *or* (1.8)*. Then there exist multiplicative functions* $M_1, M_2 : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$f(x, y, z) = M_1\left(\sqrt{x^2 + y^2}\right) M_2(z) \tag{3.1}$$

for all $x, y, z \in \mathbb{R}$.

Proof. We first consider the case when f satisfies

$$|f(ux + vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \phi(u, v, w) \tag{3.2}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Choose a sequence $(p_n, q_n, r_n) \in \mathbb{R}^3$ $(n = 1, 2, 3, \ldots)$ such that $|f(p_n, q_n, r_n)| \to \infty$ as $n \to \infty$. Replacing (x, y, z) by (p_n, q_n, r_n) in (3.2) and dividing the result by $|f(p_n, q_n, r_n)|$ we have

$$\left| f(u, v, w) - \frac{f(up_n + vq_n,\, uq_n - vp_n, wr_n)}{f(p_n, q_n, r_n)} \right| \le \frac{\phi(u, v, w)}{|f(p_n, q_n, r_n)|} \tag{3.3}$$

for all $u, v, w \in \mathbb{R}$. Letting $n \to \infty$ in (3.3) we have

$$f(u, v, w) = \lim_{n\to\infty} \frac{f(up_n + vq_n,\, uq_n - vp_n, wr_n)}{f(p_n, q_n, r_n)} \tag{3.4}$$

for all $u, v, w \in \mathbb{R}$. Multiplying both sides of (3.4) by $f(x, y, z)$ and using (1.7) and (3.4) we have

(3.5)

$$\begin{aligned} &f(u, v, w)f(x, y, z) \\ &= \lim_{n\to\infty} \frac{f(up_n + vq_n,\, uq_n - vp_n, wr_n)f(x, y, z)}{f(p_n, q_n, r_n)} \\ &= \lim_{n\to\infty} \frac{f\big((ux - vy)p_n + (uy + vx)q_n,\, (ux - vy)q_n - (uy + vx)p_n, zwr_n\big)}{f(p_n, q_n, r_n)} \\ &= f(ux - vy, uy + vx, zw) \end{aligned}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Replacing v by $-v$ in (3.5) we have

$$f(u, -v, w)f(x, y, z) = f(ux + vy, uy - vx, zw) \tag{3.6}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. From (3.2) and (3.6) we have

$$|f(x, y, z)||f(u, v, w) - f(u, -v, w)| \le \phi(u, v, w) \tag{3.7}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Since f is unbounded, we have

$$f(u, v, w) = f(u, -v, w) \tag{3.8}$$

for all $u, v, w \in \mathbb{R}$. Thus, from (3.6) and (3.8) we get the functional equation

$$f(ux + vy, uy - vx, zw) = f(x, y, z)f(u, v, w) \tag{3.9}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Secondly, we consider the case when f satisfies

$$|f(ux + vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \leq \phi(x, y, z) \tag{3.10}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Choose a sequence $(p_n, q_n, r_n) \in \mathbb{R}^3$ $(n = 1, 2, 3, \ldots)$ such that $|f(p_n, q_n, r_n)| \to \infty$ as $n \to \infty$. Replacing (u, v, w) by (p_n, q_n, r_n) in (3.10), dividing the result by $|f(p_n, q_n, r_n)|$ and letting $n \to \infty$ we have

$$f(x, y, z) = \lim_{n\to\infty} \frac{f(p_n x + q_n y,\, p_n y - q_n x, r_n z)}{f(p_n, q_n, r_n)} \tag{3.11}$$

for all $x, y, z \in \mathbb{R}$. Multiplying both sides of (3.11) by $f(u, v, w)$ and using (1.8) and (3.11) we have

(3.12)

$$\begin{aligned}
&f(u, v, w)f(x, y, z) \\
&= \lim_{n\to\infty} \frac{f(u, v, w)f(p_n x + q_n y,\, p_n y - q_n x, r_n z)}{f(p_n, q_n, r_n)} \\
&= \lim_{n\to\infty} \frac{f\big(p_n(ux + vy) + q_n(uy - vx),\, q_n(uy + vx) - p_n(uy - vx), r_n wz\big)}{f(p_n, q_n, r_n)} \\
&= f(ux + vy, uy - vx, zw)
\end{aligned}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. By Theorem 2.3, we get (3.1). This completes the proof. □

Theorem 3.2. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be an unbounded function satisfying* (1.9) *or* (1.10). *Then there exist multiplicative functions $M_1, M_2 : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$f(x, y, z) = M_1\left(x^2 - y^2\right) M_2(z) \tag{3.13}$$

for all $x, y, z \in \mathbb{R}$.

Proof. We first consider the case when f satisfies

$$|f(ux - vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \leq \phi(u, v, w) \tag{3.14}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Choose a sequence $(p_n, q_n, r_n) \in \mathbb{R}^3$ $(n = 1, 2, 3, \ldots)$ such that $|f(p_n, q_n, r_n)| \to \infty$ as $n \to \infty$. Replacing (x, y, z) by (p_n, q_n, r_n) in (3.14) and dividing the result by $|f(p_n, q_n, r_n)|$ we have

$$\left| f(u, v, w) - \frac{f(up_n - vq_n,\, uq_n - vp_n, wr_n)}{f(p_n, q_n, r_n)} \right| \leq \frac{\phi(u, v, w)}{|f(p_n, q_n, r_n)|} \tag{3.15}$$

for all $u, v, w \in \mathbb{R}$. Letting $n \to \infty$ in (3.15) we have

$$f(u, v, w) = \lim_{n\to\infty} \frac{f(up_n - vq_n,\, uq_n - vp_n, wr_n)}{f(p_n, q_n, r_n)} \tag{3.16}$$

for all $u, v, w \in \mathbb{R}$. Multiplying both sides of (3.16) by $f(x, y, z)$ and using (1.9) and (3.16) we have

(3.17)

$$\begin{aligned} & f(u,v,w)f(x,y,z) \\ = & \lim_{n\to\infty} \frac{f(up_n - vq_n,\, uq_n - vp_n, wr_n)f(x,y,z)}{f(p_n,q_n,r_n)} \\ = & \lim_{n\to\infty} \frac{f\big((ux+vy)p_n - (uy+vx)q_n,\, (ux+vy)q_n - (uy+vx)p_n, zwr_n\big)}{f(p_n,q_n,r_n)} \\ = & f(ux+vy, uy+vx, zw) \end{aligned}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Replacing v by $-v$ in (3.17) we have

$$f(u,-v,w)f(x,y,z) = f(ux-vy, uy-vx, zw) \tag{3.18}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. From (3.14) and (3.18) we have

$$|f(x,y,z)||f(u,-v,w) - f(u,v,w)| \le \phi(u,v,w) \tag{3.19}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Since f is unbounded, we have

$$f(u,-v,w) = f(u,v,w) \tag{3.20}$$

for all $u, v, w \in \mathbb{R}$. Thus, from (3.18) and (3.20) we get

$$f(ux-vy, uy-vx, zw) = f(x,y,z)f(u,v,w) \tag{3.21}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Secondly, we consider the case when f satisfies

$$|f(ux-vy, uy-vx, zw) - f(x,y,z)f(u,v,w)| \le \phi(x,y,z) \tag{3.22}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Putting $(x, y, z) = (1, 0, 1)$ in (3.22) we have

$$|f(u,-v,w) - f(1,0,1)f(u,v,w)| \le \phi(1,0,1) \tag{3.23}$$

for all $u, v, w \in \mathbb{R}$. Choose a sequence $(p_n, q_n, r_n) \in \mathbb{R}^3$ $(n = 1, 2, 3, \ldots)$ such that $|f(p_n, q_n, r_n)| \to \infty$ as $n \to \infty$. Replacing (u, v, w) by (p_n, q_n, r_n) in (3.22), dividing the result by $|f(p_n, q_n, r_n)|$ and letting $n \to \infty$ we have

$$f(x,y,z) = \lim_{n\to\infty} \frac{f(p_n x - q_n y,\, p_n y - q_n x, r_n z)}{f(p_n,q_n,r_n)} \tag{3.24}$$

for all $x, y, z \in \mathbb{R}$. Multiplying both sides of (3.24) by $f(u, v, w)$ and using (1.10) and (3.24) we have

(3.25)

$$\begin{aligned} & f(u,v,w)f(x,y,z) \\ = & \lim_{n\to\infty} \frac{f(u,v,w)f(p_n x - q_n y,\, p_n y - q_n x, r_n z)}{f(p_n,q_n,r_n)} \\ = & \lim_{n\to\infty} \frac{f\big(p_n(ux-vy) - q_n(uy-vx),\, p_n(vx-uy) - q_n(vy-ux), r_n zw\big)}{f(p_n,q_n,r_n)} \end{aligned}$$

$$= \lim_{n\to\infty} \frac{f(1,0,1)f\big(p_n(ux-vy)-q_n(uy-vx),\, p_n(uy-vx)-q_n(ux-vy), r_nzw\big)}{f(p_n,q_n,r_n)}$$
$$= f(1,0,1)f(ux-vy, uy-vx, zw)$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Putting $(x, y, z) = (1, 0, 1)$ in (3.25) we have

$$f(u, v, w) = f(u, -v, w) \tag{3.26}$$

for all $u, v, w \in \mathbb{R}$. From (3.23) and (3.26) we have

$$|f(u, v, w)||1 - f(1, 0, 1)| \le \phi(1, 0, 1) \tag{3.27}$$

for all $u, v, w \in \mathbb{R}$. Since f is unbounded, from (3.27) we have $f(1, 0, 1) = 1$. Thus, from (3.25) we get

$$f(ux - vy, uy - vx, zw) = f(x, y, z)f(u, v, w) \tag{3.28}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. By Theorem 2.4, we get (3.13). This completes the proof. □

4. Bounded functions satisfying the inequalities (1.7) ∼ (1.10)

In this section, we investigate bounded functions satisfying each of functional inequalities (1.7) ∼ (1.10).

Theorem 4.1. *Let* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *be a bounded function satisfying* (1.7) *or* (1.8)*. Then* f *satisfies*

$$|f(x, y, z)| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\phi(x, y, z)}\right) \tag{4.1}$$

for all $(x, y, z) \in \mathbb{R}^3$*. Furthermore, let*

$$K = \left\{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2)z \ne 0, \phi(x, y, z) < \frac{1}{4}\right\}.$$

Then f *satisfies*

$$\frac{1}{2}\left(1 + \sqrt{1 - 4\phi(x, y, z)}\right) \le |f(x, y, z)| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\phi(x, y, z)}\right) \tag{4.2}$$

for all $(x, y, z) \in K$*, or else*

$$|f(x, y, z)| \le \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(x, y, z)}\right) \tag{4.3}$$

for all $(x, y, z) \in K$*.*

Proof. We first consider the case when f satisfies

$$|f(ux + vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \phi(u, v, w) \tag{4.4}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Let

$$M_f = \sup_{(x,y,z)\in\mathbb{R}^3} |f(x, y, z)|.$$

Using the triangle inequality with (4.1) we have

$$\begin{aligned} |f(x,y,z)||f(u,v,w)| &\le |f(ux+vy, uy-vx, zw)| + \phi(u,v,w) \\ &\le M_f + \phi(u,v,w) \end{aligned} \tag{4.5}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. From (4.5) we have

$$M_f|f(u,v,w)| \le M_f + \phi(u,v,w) \tag{4.6}$$

for all $u, v, w \in \mathbb{R}$. Thus, we have

$$M_f(|f(u,v,w)| - 1) \le \phi(u,v,w) \tag{4.7}$$

for all $u, v, w \in \mathbb{R}$, which implies

$$|f(u,v,w)|(|f(u,v,w)| - 1) \le \phi(u,v,w) \tag{4.8}$$

for all $u, v, w \in \mathbb{R}$ and (4.1) follows. Replacing (x, y, z) by $\left(\frac{ux-vy}{u^2+v^2}, \frac{uy+vx}{u^2+v^2}, \frac{z}{w}\right)$ in (4.4) for $\left(u^2+v^2\right)w \neq 0$ and using the triangle inequality with the result we have

$$\begin{aligned} |f(x,y,z)| &\le \left| f\left(\frac{ux-vy}{u^2+v^2}, \frac{uy+vx}{u^2+v^2}, \frac{z}{w}\right)\right| |f(u,v,w)| + \phi(u,v,w) \\ &\le M_f|f(u,v,w)| + \phi(u,v,w) \end{aligned} \tag{4.9}$$

for all $x, y, z, u, v, w \in \mathbb{R}$ with $\left(u^2+v^2\right)w \neq 0$. From (4.9) we have

$$M_f \le M_f|f(u,v,w)| + \phi(u,v,w) \tag{4.10}$$

for all $u, v, w \in \mathbb{R}$ with $(u^2+v^2)w \neq 0$. Thus, we have

$$M_f\big(1 - |f(u,v,w)|\big) \le \phi(u,v,w) \tag{4.11}$$

for all $u, v, w \in \mathbb{R}$ with $(u^2+v^2)w \neq 0$. From (4.7) and (4.11) we have

$$M_f\big||f(u,v,w)| - 1\big| \le \phi(u,v,w) \tag{4.12}$$

for all $u, v, w \in \mathbb{R}$ with $(u^2+v^2)w \neq 0$. Thus, we get

$$|f(x,y,z)|\big||f(x,y,z)| - 1\big| \le \phi(x,y,z) \tag{4.13}$$

for all $x, y, z \in \mathbb{R}$ with $(x^2+y^2)z \neq 0$. For each fixed $(x, y, z) \in K$, solving (4.13) we have

$$\frac{1}{2}\left(1 + \sqrt{1 - 4\phi(x,y,z)}\right) \le |f(x,y,z)| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\phi(x,y,z)}\right) \tag{4.14}$$

or

$$|f(x,y,z)| \le \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(x,y,z)}\right). \tag{4.15}$$

Now, assume that there exist $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}_0^3$ such that

$$\begin{aligned} |f(x_1,y_1,z_1)| &\le \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(x_1,y_1,z_1)}\right), \\ |f(x_2,y_2,z_2)| &\ge \frac{1}{2}\left(1 + \sqrt{1 - 4\phi(x_2,y_2,z_2)}\right). \end{aligned} \tag{4.16}$$

Then, from (4.13) we have

$$|f(x_2,y_2,z_2)|\big(1-|f(x_1,y_1,z_1)|\big) \le M_f\big(1-|f(x_1,y_1,z_1)|\big) \le \phi(x_1,y_1,z_1). \tag{4.17}$$

On the other hand, from (4.16) we have

$$\begin{aligned} &|f(x_2,y_2,z_2)|\big(1-|f(x_1,y_1,z_1)|\big) \\ &\ge \frac{1}{2}\left(1+\sqrt{1-4\phi(x_2,y_2,z_2)}\right)\left(1-\frac{1}{2}\left(1-\sqrt{1-4\phi(x_1,y_1,z_1)}\right)\right) \\ &> \frac{1}{2}\left(1-\sqrt{1-4\phi(x_1,y_1,z_1)}\right)\left(1-\frac{1}{2}\left(1-\sqrt{1-4\phi(x_1,y_1,z_1)}\right)\right) \\ &= \phi(x_1,y_1,z_1), \end{aligned} \tag{4.18}$$

which contradicts (4.17). Thus, $f(x,y,z)$ satisfies (4.2) for all $(x,y,z)\in K$ or satisfies (4.3) for all $(x,y,z)\in K$. Similarly, we can prove the case when f satisfies (1.8). This completes the proof. □

Corollary 4.2. *Let $0\le \epsilon < \frac{1}{4}$. Assume that $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ be a bounded function satisfying*

$$|f(ux+vy,uy-vx,zw)-f(x,y,z)f(u,v,w)| \le \epsilon \tag{4.19}$$

for all $x,y,z,u,v,w\in\mathbb{R}$. Then f satisfies

$$|f(x,y,z)| \le \frac{1}{2}\left(1+\sqrt{1+4\epsilon}\right) \tag{4.20}$$

for all $(x,y,z)\in\mathbb{R}^3$. Furthermore, let $K=\{(x,y,z)\in\mathbb{R}^3 : (x^2+y^2)z\ne 0\}$. Then f satisfies

$$\frac{1}{2}\left(1+\sqrt{1-4\epsilon}\right) \le |f(x,y,z)| \le \frac{1}{2}\left(1+\sqrt{1+4\epsilon}\right) \tag{4.21}$$

for all $(x,y,z)\in K$, or else

$$|f(x,y,z)| \le \frac{1}{2}\left(1-\sqrt{1-4\epsilon}\right) \tag{4.22}$$

for all $(x,y,z)\in K$.

Theorem 4.3. *Let $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ be a bounded function satisfying* (1.9) *or* (1.10)*. Then f satisfies*

$$|f(x,y,z)| \le \frac{1}{2}\left(1+\sqrt{1+4\phi(x,y,z)}\right) \tag{4.23}$$

for all $(x,y,z)\in\mathbb{R}^3$. Furthermore, let

$$K=\left\{(x,y,z)\in\mathbb{R}^3 : (x^2-y^2)z\ne 0, \phi(x,y,z)<\frac{1}{4}\right\}.$$

Then f satisfies

$$\frac{1}{2}\left(1+\sqrt{1-4\phi(x,y,z)}\right) \le |f(x,y,z)| \le \frac{1}{2}\left(1+\sqrt{1+4\phi(x,y,z)}\right) \tag{4.24}$$

for all $(x, y, z) \in K$*, or else*

$$|f(x, y, z)| \le \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(x, y, z)}\right) \tag{4.25}$$

for all $(x, y, z) \in K$*.*

Proof. We first consider the case when f satisfies

$$|f(ux - vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \phi(u, v, w) \tag{4.26}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. Let

$$M_f = \sup_{(x,y,z)\in\mathbb{R}^3} |f(x, y, z)|.$$

Using the triangle inequality with (4.26) we have

$$\begin{aligned} |f(x, y, z)||f(u, v, w)| &\le |f(ux - vy, uy - vx, zw)| + \phi(u, v, w) \\ &\le M_f + \phi(u, v, w) \end{aligned} \tag{4.27}$$

for all $x, y, z, u, v, w \in \mathbb{R}$. From (4.27) we have

$$M_f|f(u, v, w)| \le M_f + \phi(u, v, w) \tag{4.28}$$

for all $u, v, w \in \mathbb{R}$. Thus, we have

$$M_f\big(|f(u, v, w)| - 1\big) \le \phi(u, v, w) \tag{4.29}$$

for all $u, v, w \in \mathbb{R}$, which implies

$$|f(u, v, w)|\big(|f(u, v, w)| - 1\big) \le \phi(u, v, w) \tag{4.30}$$

for all $u, v, w \in \mathbb{R}$ and (4.23) follows. Replacing (x, y, z) by $\left(\frac{ux+vy}{u^2-v^2}, \frac{uy+vx}{u^2-v^2}, \frac{z}{w}\right)$ in (4.26) for $\big(u^2 - v^2\big)w \ne 0$ and using the triangle inequality with the result we have

$$\begin{aligned} |f(x, y, z)| &\le \left|f\left(\frac{ux + vy}{u^2 - v^2}, \frac{uy + vx}{u^2 - v^2}, \frac{z}{w}\right)\right| |f(u, v, w)| + \phi(u, v, w) \\ &\le M_f|f(u, v, w)| + \phi(u, v, w) \end{aligned} \tag{4.31}$$

for all $x, y, z, u, v, w \in \mathbb{R}$ with $\big(u^2 - v^2\big)\, w \ne 0$. Now, the remaining part of the proof is the same as that of Theorem 4.1 (see (4.10) $\sim$ (4.18)). Similarly, we can prove the case when f satisfies (1.10). This completes the proof. □

Corollary 4.4. *Let* $0 \le \epsilon < \frac{1}{4}$*. Assume that* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *be a bounded function satisfying*

$$|f(ux - vy, uy - vx, zw) - f(x, y, z)f(u, v, w)| \le \epsilon \tag{4.32}$$

for all $x, y, z, u, v, w \in \mathbb{R}$*. Then* f *satisfies*

$$|f(x, y, z)| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon}\right) \tag{4.33}$$

for all $(x,y,z) \in \mathbb{R}^3$. *Furthermore, let* $K = \big\{(x,y,z) \in \mathbb{R}^3 : (x^2-y^2)z \neq 0\big\}$. *Then* f *satisfies*

$$\frac{1}{2}\left(1+\sqrt{1-4\epsilon}\right) \le |f(x,y,z)| \le \frac{1}{2}\left(1+\sqrt{1+4\epsilon}\right) \tag{4.34}$$

for all $(x,y,z) \in K$, *or else*

$$|f(x,y,z)| \le \frac{1}{2}\left(1-\sqrt{1-4\epsilon}\right) \tag{4.35}$$

for all $(x,y,z) \in K$.

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