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RESULTS ON MEROMORPHIC FUNCTIONS SHARING THREE VALUES CM IN SOME ANGULAR DOMAINS

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Abstract. We study the uniqueness question of transcendental meromorphic functions that share three values CM in some angular domains instead of the whole complex plane. The results in this paper extend the corresponding results in Zheng [13, 14] and Yi [12]. Some examples are given to show that the results in this paper, in a sense, are the best possible.

1. Introduction and main results

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a transcendental meromorphic function, where $\mathbb{C}$ is the complex plane. We assume that the readers are familiar with the Nevanlinna theory of meromorphic functions and the standard notations such as Nevanlinna deficiency $\delta(a, f)$ of f with respect to $a \in \mathbb{C}$ and Nevanlinna characteristic $T(r, f)$ of f. Moreover, the lower order $\mu(f)$, the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined as

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}$$

respectively. For the references, see, for example, Hayman [7]. Let f and g be two meromorphic functions in the complex plane, and let $a \in \mathbb{C} \cup \{\infty\}$ be a value. We say that f and g share a IM (ignoring multiplicities) in a domain $X \subseteq \mathbb{C}$ if in X, $f(z) = a$ if and only if $g(z) = a$. We say that f and g share a CM (counting multiplicities), if f and g share a IM (ignoring multiplicities) in a domain $X \subseteq \mathbb{C}$. In 1929, Nevanlinna [9] proved that if

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two meromorphic functions f and g have five distinct IM shared values in $X = \mathbb{C}$, then $f = g$. Later on, many mathematicians in the world treated some uniqueness questions of meromorphic functions with shared values in the whole complex plane, see, for example, Yang-Yi [10]. In this paper, we will consider the uniqueness question of meromorphic functions sharing three values in some angular $X \subset \mathbb{C}$. Next we consider q pair of real numbers $\{\alpha_j, \beta_j\}$ such that

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_q < \beta_q \leq \pi, \tag{1.1}$$

and define

$$\omega = \max\left\{\frac{\pi}{\beta_1 - \alpha_1}, \frac{\pi}{\beta_2 - \alpha_2}, \ldots, \frac{\pi}{\beta_q - \alpha_q}\right\}. \tag{1.2}$$

We recall the following result from Nevanlinna [9]:

Theorem A ([9])**.** *Let f and g be two nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct values in the extended complex plane. If f and g share a_1, a_2, a_3, a_4, a_5 IM, then $f = g$.*

Yi [12] proved the following result to consider the uniqueness question of meromorphic functions sharing three values in the complex plane:

Theorem B ([12, Theorem 1])**.** *Let f and g be two distinct nonconstant meromorphic functions such that f and g share* 0, 1, ∞ *CM, and let a be a finite complex number such that $a \neq 0, 1, \infty$. If*

$$T(r, f) \neq N\left(r, \frac{1}{f - a}\right) + S(r, f),$$

then a is a Picard exceptional value of f, and that one of the following three equations holds:

(i) $(f - a)(g + a - 1) = a(1 - a)$;
(ii) $f = (1 - a)g + a$;
(iii) $f = ag$.

Zheng [13, 14] first took into the uniqueness question of meromorphic functions with shared values in an angular domain or some angular domains instead of the complex plane. We recall the following result from Zheng [14]:

Theorem C ([14, Theorem 1])**.** *Let f and g be transcendental meromorphic functions. Suppose that f is of finite lower order μ and that for some $a \in \mathbb{C} \cup \{\infty\}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. Assume that q pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfies* (1.1) *and*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}, \tag{1.3}$$

where $\sigma = \max\{\omega, \mu\}$, ω is defined as in (1.2)*, and assume that f and g share a_1, a_2, a_3, a_4, a_5 IM in $\overline{X} = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$, where a_1, a_2, a_3,*

a_4, a_5 are five distinct values in the extended complex plane. If $\rho(f) > \omega$, then $f = g$.

Theorem D ([14, Theorem 2]). *Let f and g be transcendental meromorphic functions such that for some $a \in \mathbb{C}\cup\{\infty\}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. Assume that q radii $\arg z = \alpha_j$ $(1 \leq j \leq q)$ satisfies*

$$-\pi \leq \alpha_1 < \alpha_2 < \cdots < \alpha_q < \pi, \quad \alpha_{q+1} = \alpha_1 + 2\pi, \tag{1.4}$$

and assume that f and g share a_1, a_2, a_3, a_4, a_5 IM in $X = \mathbb{C} \setminus \bigcup_{j=1}^{q}\{z : \arg z = \alpha_j\}$, where a_1, a_2, a_3, a_4, a_5 are five distinct values in the extended complex plane. If $\rho(f) > \frac{\pi}{\min\limits_{1\leq j\leq q}\{\alpha_{j+1}-\alpha_j\}}$, then $f = g$.

Regarding Theorems B-D, one may ask the following question:

Question 1.1. What can be said about the relationship between f and g, if two distinct transcendental meromorphic functions f and g share 0, 1, ∞ CM in an angular domain or some angular domains instead of the complex plane ?

We will prove the following result which deals with Question 1.1, improves Theorem B and extends Theorems C and D:

Theorem 1.1. *Let f and g be two distinct transcendental meromorphic functions. Suppose that f is of finite lower order μ and that there exists some $a \in \mathbb{C} \cup \setminus\{0, 1\}$ such that $\delta = \delta(a, f) > 0$. Assume that q pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfies* (1.1) *and* (1.3)*, where $\sigma = \max\{\omega, \mu\}$, ω is defined as in* (1.2)*, and assume that f and g share* 0, 1, ∞ *CM in $\overline{X} = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\rho(f) > \omega$, then one of the three equations* (i)-(iii) *of Theorem B holds.*

We recall the following three examples:

Example 1.1. Let $f(z) = e^{2z} + e^{z} + 1$, $g(z) = e^{-2z} + e^{-z} + 1$, let ε be any positive number and let

$$\alpha_1 = -\frac{\pi}{2} - \varepsilon, \quad \beta_1 = -\frac{\pi}{2} + \varepsilon; \quad \alpha_2 = -\frac{\pi}{3} - \varepsilon, \quad \beta_2 = -\frac{\pi}{3} + \varepsilon;$$
$$\alpha_3 = \frac{\pi}{3} - \varepsilon, \quad \beta_3 = \frac{\pi}{3} + \varepsilon; \quad \alpha_4 = \frac{\pi}{2} - \varepsilon, \quad \beta_4 = \frac{\pi}{2} + \varepsilon.$$

Then, f and g share 0, 1 and ∞ CM in $\overline{X} = \bigcup_{j=1}^{4}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. Moreover, we can verify that $\rho(f) < \omega$, where $\rho(f) = 1$ and $\omega = \frac{\pi}{\min\limits_{1\leq j\leq 4}\{\beta_j-\alpha_j\}} = \frac{\pi}{2\varepsilon}$, and that for any $a \in \mathbb{C} \setminus \{0, 1\}$ we have $\delta(a, f) = 0$. But f and g do not satisfy one of the three relations (i)-(iii) of Theorem B. This example shows that the assumption "$\delta(a, f) > 0$" in Theorem 1.1 is necessary.

Example 1.2 ([14, Remark A]). For each real number a satisfying $0 \leq a \leq 1$, we let $\sin z = \frac{e^{iz}-e^{-iz}}{2i} = a$, where $z = x + yi$ and $x, y \in \mathbb{R}$. Then we have $e^{-y}\cos x = a$ and $e^{-y}\sin x = \sqrt{1 - a^2}$, and so $e^{-2y} = 1$, which implies $y = 0$. Hence $z = x$ is a real number. Similarly, if $\cos z = \frac{e^{iz}+e^{-iz}}{2} = a$, where

$z = x + yi$ and $x, y \in \mathbb{R}$, then we can deduce $y = 0$ and so $z = x$ is also a real number. Therefore, for each real number a satisfying $0 \le a \le 1$, $\sin z$ and $\cos z$ can take over a only on the real axis, and so all the x_j-points of $\sin z$ and $\cos x$ for $1 \le j \le 3$ such that $\sin z$ and $\cos z$ CM share x_1, x_2, x_3 in the domain $\mathbb{C} \setminus \mathbb{R}$, where x_1, x_2, x_3 are three distinct finite real numbers satisfying $0 \le x_j \le 1$ for $1 \le j \le 3$. Obviously, $\rho(\sin z) = \rho(\cos z) = 1$, $\delta(\infty, \sin z) = \delta(\infty, \cos z) = 1$ and $\rho(f) = \omega = 1$. But $f(z) = \sin z$ and $g(z) = \cos z$ do not satisfy one of the relations (i)-(iii) of Theorem B. This example shows that the assumption "$\rho(f) > \omega$" of Theorem 1.1 is best possible.

Example 1.3 ([14, Remark A])**.** We will give an example to show that $\mu(f) < \infty$ in Theorem 1.1 can not be removed by using the theory of complex dynamics. For the basic knowledge of complex dynamics, I suggest the readers see, for example, Bergweiler [3]. We consider the following function:

$$f(z) = z - (a+1) + \frac{1}{2\pi} \int_L \frac{e^{e^t}}{t - z} dt,$$

where L is the boundary of the region $\{z : \mathrm{Re}z > 0, -\pi < \mathrm{Im}z < \pi\}$ described in a clockwise direction. Then f is an entire function with infinite lower order. From the proof of Theorem 2 in Baker [2] we can find that the Julia set $J(f)$ of f lies in the region $\{z : \mathrm{Re}z > -a, -h < \mathrm{Im}z < h\}$ for suitable a and h. Since $J(f)$ does not contain any isolated Jordan arcs, there exists a horizontal straight line which intersects $J(f)$ at least three points. By a translation, we conjugate g to an entire function $f(z)$ such that the Julia set $J(f)$ of $f(z)$ contains at least three real points c_j $(1 \le j \le 3)$. Then all the roots of $f(z) = c_j$ $(1 \le j \le 3)$ lie in $G = \{z : \mathrm{Re}z > -a, -2h < \mathrm{Im}z < 2h\}$. It is well known that $\tan z = c_j$ $(1 \le j \le 3)$ have only real roots. Thus f and $\tan z$ share three distinct CM shared values in $\mathbb{C} \setminus G \cup \mathbb{R}$, $\delta(\infty, f) = 1$ and $\mu(f) = \infty$. But $f(z)$ and $g(z) = \tan z$ do not satisfy one of the three relations (i)-(iii) of Theorem B. This example shows that the assumption "$\mu(f) < \infty$" in Theorem 1.1 is necessary.

If we remove the assumption "$\mu(f) < \infty$" in Theorem 1.1, we can get the following result:

Theorem 1.2. *Let f and g be two distinct transcendental meromorphic functions and let $a \in \mathbb{C} \setminus \{0, 1\}$ and an integer $\delta = \delta(a, f) > 0$. Assume that for q radii* $\arg z = \alpha_j$ $(1 \le j \le q)$ *satisfying* (1.4)*, f and g share* 0, 1 *and* ∞ *CM in* $X = \mathbb{C} \setminus \bigcup_{j=1}^{q} \{z : \arg z = \alpha_j\}$. *If*

$$\rho(f) > \frac{\pi}{\min\limits_{1 \le j \le q} \{\alpha_{j+1} - \alpha_j\}},$$

then one of the three equations (i)-(iii) *of Theorem B holds.*

2. Preliminaries

In this section, we introduce some important lemmas to prove the main results in this paper. First we introduce Nevanlinna theory on an angular domain, which can be found, for example, in [6, p. 23–26]:

Let f be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\le \arg z\le\beta\}$, where $\alpha,\beta\in[0,2\pi]$ and so $0\le\beta-\alpha<2\pi$. Following Nevanlinna notations (cf. [6, p. 23–26]), we define

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t}, \tag{2.1}$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta, \tag{2.2}$$

and

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_m|<r}\left(\frac{1}{|b_m|^\omega}-\frac{|b_m|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_m-\alpha) \tag{2.3}$$

respectively, where $\omega=\pi/(\beta-\alpha)$, $1\le r<+\infty$ and $b_m=|b_m|e^{i\theta_m}$ are the poles of f on $\overline{\Omega}(\alpha,\beta)$ appearing often according to their multiplicities. $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\overline{X}(\alpha,\beta)$ and the Nevanlinna angular characteristic function is defined as

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Similarly, for any finite value a, we define $A_{\alpha,\beta}(r,f_a)$, $B_{\alpha,\beta}(r,f_a)$, $C_{\alpha,\beta}(r,f_a)$ and $S_{\alpha,\beta}(r,f_a)$, where $f_a=1/(f-a)$. We denote by $\overline{C}_{\alpha,\beta}(r,f)$ and $\overline{C}_{\alpha,\beta}(r,f_a)$ the reduced forms of $C_{\alpha,\beta}(r,f)$ and $C_{\alpha,\beta}(r,f_a)$ respectively. For the sake of simplicity, next we omit the subscript of all the above notations and respectively use the notations $A(r,f_a)$, $B(r,f_a)$, $C(r,f_a)$, $\overline{C}(r,f)$, $\overline{C}(r,f_a)$ and $S(r,f_a)$ instead of $A_{\alpha,\beta}(r,f_a)$, $B_{\alpha,\beta}(r,f_a)$, $C_{\alpha,\beta}(r,f_a)$, $\overline{C}_{\alpha,\beta}(r,f)$, $\overline{C}_{\alpha,\beta}(r,f_a)$ and $S_{\alpha,\beta}(r,f_a)$ for any finite complex value a.

Lemma 2.1 ([6, p. 23–26] and [6, Theorem 3.1])**.** *Let f be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then, for arbitrary complex number $a\in\mathbb{C}$ we have*

$$S\left(r,\frac{1}{f-a}\right)=S(r,f)+O(1),$$

and for an integer $k\ge 0$,

$$S\left(r,f^{(k)}\right)\le 2^kS(r,f)+R(r,f)$$

and

$$A\left(r,\frac{f^{(k)}}{f}\right)+B\left(r,\frac{f^{(k)}}{f}\right)=R(r,f),$$

where and in what follows, $R(r,f)$ is such a quantity that if $\rho(f)<\infty$, then $R(r,f)=O(1)$ as $r\to\infty$, if $\rho(f)=\infty$, then $R(r,f)=O(\log(rT(r,f)))$ as

$r \notin E$ and $r \to \infty$, where and in what follows, E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context.

Lemma 2.2 ([6, p. 112, Theorem 3.3]). *Let f be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary q distinct values $a_j \in \mathbb{C}\cup\{\infty\}$ $(1 \le j \le q)$ we have*

$$(q-2)S(r,f) \le \sum_{j=1}^{q} \overline{C}\left(r, \frac{1}{f-a_j}\right) + R(r,f).$$

The following three results play an important role in proving the main results in this paper:

Lemma 2.3 ([12, Proof of Lemma 1]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM in $\overline{\Omega}(\alpha,\beta)$. Then there exist two meromorphic functions h_1 and h_2 such that*

$$f = \frac{h_1 - 1}{h_2 - 1}, \quad g = \frac{h_1^{-1} - 1}{h_2^{-1} - 1}, \tag{2.4}$$

where h_1 and h_2 are meromorphic functions such that $h_2 \not\equiv 1$, $h_1 \not\equiv 1$, $h_2 h_1^{-1} \not\equiv 1$, and $h_j(z) \notin \{0,\infty\}$ for any $z \in \overline{\Omega}(\alpha,\beta)$, where $j = 1, 2$. Moreover,

$$S(r,g) + S(r,h_1) + S(r,h_2) = O(S(r,f)) + R(r,f). \tag{2.5}$$

Lemma 2.4 ([12, Proof of Lemma 2]). *Let f and g be two nonconstant meromorphic functions on $\overline{\Omega}(\alpha,\beta)$, and let c_1, c_2 and c_3 be three nonzero constants. If $c_1 f + c_2 g = c_3$ on $\overline{\Omega}(\alpha,\beta)$, then*

$$S(r,f) \le \overline{C}\left(r, \frac{1}{f}\right) + \overline{C}\left(r, \frac{1}{g}\right) + \overline{C}\,(r,f) + R(r,f).$$

Lemma 2.5 ([12, Proof of Lemma 4]). *Let f and g be two distinct nonconstant meromorphic functions that share 0, 1, ∞ CM in $\overline{\Omega}(\alpha,\beta)$. Then*

$$C_{(2}\left(r, \frac{1}{f}\right) + C_{(2}\left(r, \frac{1}{f-1}\right) + C_{(2}\,(r,f) = R(r,f),$$

where and in what follows, $C_{(2}\left(r, \frac{1}{f}\right)$ is the angular counting function of those zeros of f in $C\left(r, \frac{1}{f}\right)$, here each such zero of f is of multiplicity ≥ 2, and each such zero of f is counted according to its multiplicity, $C_{(2}\left(r, \frac{1}{f-1}\right)$ and $C_{(2}\,(r,f)$ have the similar meanings.

Proceeding as in Case d) of the proof of Theorem 1 [12], we can get the following result from Lemma 2.2, Lemmas 2.4 and 2.5:

Lemma 2.6 ([12, Proof of Theorem 1]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM in $\overline{\Omega}(\alpha,\beta)$, and let h_1*

and h_2 *be defined as in Lemma 2.3 such that none of* h_1, h_2 *and* $h_2h_1^{-1}$ *is a constant. Then for any* $a \in \mathbb{C} \setminus \{0,1\}$ *we have*

$$C\left(r, \frac{1}{f-a}\right) = S(r,f) + R(r,f). \tag{2.6}$$

The following result was proved by Edrei [5] and Yang [11] independently:

Lemma 2.7 ([5] or [11]). *Let* f *be transcendental and meromorphic in* $\mathbb{C}$ *with the lower order* $0 \le \mu < \infty$ *and the order* $0 < \rho < \infty$. *Then for arbitrary positive number* σ *satisfying* $\mu \le \sigma \le \rho$ *and a set* E *with finite linear measure, there exists a sequence of positive numbers* $\{r_n\}$ *such that* (i) $r_n \notin E$ *and* $\lim_{n\to\infty} \frac{r_n}{n} = \infty$, (ii) $\liminf_{r\to\infty} \frac{\log T(r_n,f)}{\log r_n} \ge \sigma$ *and* (iii) $T(t,f) < (1+o(1))\left(\frac{t}{r_n}\right)^{\sigma} T(r_n,f)$.

A sequence $\{r_n\}$ satisfying (i), (ii) and (iii) in Lemma 2.7 is called a Pólya peak of order σ outside E in this paper. For $r > 0$ and $a \in \mathbb{C}$, we define

(iv)

$$D(r,a) := \left\{\theta \in [-\pi,\pi) : \log^{+} \frac{1}{|f(re^{i\theta}) - a|} > \frac{1}{\log r} T(r,f)\right\}$$

and

$$D(r,\infty) := \left\{\theta \in [-\pi,\pi) : \log^{+} |f(re^{i\theta})| > \frac{1}{\log r} T(r,f)\right\}.$$

The following result is a special version of the main result of Baernstein [1]:

Lemma 2.8 ([1]). *Let* f *be transcendental and meromorphic in* $\mathbb{C}$ *with the finite lower order* μ *and the order* $0 < \rho \le \infty$, *and for some* $a \in \mathbb{C} \cup \{\infty\}$, $\delta(a,f) = \delta > 0$. *Then for arbitrary Pólya peak* $\{r_n\}$ *of order* $\sigma > 0$, $\mu \le \sigma \le \rho$, *we have*

$$\liminf_{n\to\infty} mesD(r_n,a) \ge \min\left\{2\pi, \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\right\}.$$

Remark 2.1. Lemma 2.8 was proved in [1] for the Pólya peak of order μ, the same argument of Baernstein [1] can derive Lemma 2.8 for the Pólya peak of order σ, $\mu \le \sigma \le \rho$.

The following result is due to Edrei [4]:

Lemma 2.9 ([4]). *Let* f *be a meromorphic function with* $\delta(\infty,f) = \delta > 0$. *Then, given* $\varepsilon > 0$, *we have*

$$mesE(r,f) > \frac{1}{(T(r,f))^{\varepsilon}(\log r)^{1+\varepsilon}}, \quad r \notin F,$$

where

$$E(r,f) = \left\{\theta \in [-\pi,\pi) : \log^{+} |f(re^{i\theta})| > \frac{\delta}{4} T(r,f)\right\}$$

and F *is a set of positive real numbers with finite logarithmic measure depending on* ε.

3. Proof of theorems

Proof of Theorem 1.1. First of all, by the assumptions of Theorem 1.1 and Lemma 2.3 we have

$$f=\frac{h_1-1}{h_2-1},\quad g=\frac{h_1^{-1}-1}{h_2^{-1}-1} \tag{3.1}$$

and

$$S_{\alpha_j,\beta_j}(r,g)+S_{\alpha_j,\beta_j}(r,h_1)+S_{\alpha_j,\beta_j}(r,h_2)=O(S_{\alpha_j,\beta_j}(r,f))+R_{\alpha_j,\beta_j}(r,f) \tag{3.2}$$

for $1\le j\le q$, where h_1 and h_2 are meromorphic functions such that $h_2\not\equiv 1$, $h_1\not\equiv 1$, $h_2h_1^{-1}\not\equiv 1$, $h_1(z)\notin\{0,\infty\}$ and $h_2(z)\notin\{0,\infty\}$ for any $z\in\overline{X}=\bigcup_{j=1}^q\{z:\alpha_j\le\arg z\le\beta_j\}$. From (3.1) we have

$$\frac{f-1}{g-1}=h_1,\quad \frac{f}{g}=h_1h_2^{-1}. \tag{3.3}$$

We consider the following four cases:

Case 1. Suppose that none of h_1, h_2 and $h_2h_1^{-1}$ is a constant. Then, from Lemma 2.6 we have

$$C_{\alpha_j,\beta_j}\left(r,\frac{1}{f-a}\right)=S_{\alpha_j,\beta_j}(r,f)+R_{\alpha_j,\beta_j}(r,f),\quad 1\le j\le q. \tag{3.4}$$

From (3.4) and Lemma 2.1 we have for $1\le j\le q$ that

$$\begin{aligned}A_{\alpha_j,\beta_j}\left(r,\frac{1}{f-a}\right)+B_{\alpha_j,\beta_j}\left(r,\frac{1}{f-a}\right)&=R_{\alpha_j,\beta_j}\left(r,\frac{1}{f-a}\right)\\&\le O(\log r+\log T(r,f))\end{aligned} \tag{3.5}$$

as $r\notin E$ and $r\to\infty$. Now we prove

$$\rho(f)\le\omega. \tag{3.6}$$

Suppose that, on the contrary, (3.6) does not hold. Then

$$\rho(f)>\omega. \tag{3.7}$$

Therefore, from (3.7) and the assumptions of Theorem 1.1 we have a contradiction. To do this, we consider the following two cases:

Subcase 1.1. Suppose that $\rho(f)>\mu(f)$. Then, by the fact $\sigma=\max\{\omega,\mu\}$ we have

$$\rho(f)>\sigma\ge\mu(f). \tag{3.8}$$

From (1.3) we can find some sufficiently small positive number ε such that

$$\sum_{j=1}^q(\alpha_{j+1}-\beta_j)+4\varepsilon<\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\frac{\delta}{2}} \tag{3.9}$$

and

$$\rho(f)>\sigma+2\varepsilon>\mu(f). \tag{3.10}$$

Applying Lemma 2.7 to f, we can find that there exists a Pólya peak of order $\sigma + 2\varepsilon$ outside E. Combining this with Lemma 2.8 and

$$\sigma + 2\varepsilon \geq \omega + 2\varepsilon \geq \omega_j + 2\varepsilon \geq 1 + 2\varepsilon, \tag{3.11}$$

we have

$$measD(r_n, a) \geq \frac{4}{\sigma + 2\varepsilon} \arcsin \sqrt{\frac{\delta}{2}} - \varepsilon. \tag{3.12}$$

Without loss of generality, we can assume that (3.12) holds for all the positive integers n. Set

$$K_n = meas \left(D(r_n, a) \cap \bigcup_{j=1}^{q} (\alpha_j + \varepsilon, \beta_j - \varepsilon) \right). \tag{3.13}$$

Then, by (3.9), (3.12) and (3.13) we have

$$\begin{aligned} K_n &\geq measD(r_n, a) - meas \left([0, 2\pi) \setminus \bigcup_{j=1}^{q} (\alpha_j + \varepsilon, \beta_j - \varepsilon) \right) \\ &= measD(r_n, a) - meas \left(\bigcup_{j=1}^{q} (\beta_j - \varepsilon, \alpha_{j+1} + \varepsilon) \right) \\ &= measD(r_n, a) - \sum_{j=1}^{q} (\alpha_{j+1} - \beta_j + 2\varepsilon) \\ &\geq \varepsilon. \end{aligned} \tag{3.14}$$

By (3.14) we can find that there exists some positive integer j_0 satisfying $1 \leq j_0 \leq q$ such that for infinitely many positive integers n, we have

$$meas\left(D(r_n, a) \cap (\alpha_{j_0} + \varepsilon, \beta_{j_0} - \varepsilon)\right) \geq \frac{K_n}{q} > \frac{\varepsilon}{q}. \tag{3.15}$$

Without loss of generality, we can assume that (3.15) holds for all the positive integers n. Next we set $E_n = D(r_n, a) \cap (\alpha_{j_0} + \varepsilon, \beta_{j_0} - \varepsilon)$. Then, by (3.15) and the definition of $D(r, a)$ in (iv) of Lemma 2.7 we have

$$\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon} \log^+ \frac{1}{|f(r_n e^{i\theta}) - a|} d\theta \geq \frac{T(r_n, f)}{\log r_n} measE_n > \frac{\varepsilon}{q} \frac{T(r_n, f)}{\log r_n}. \tag{3.16}$$

On the other hand, by (3.16), Lemma 2.1 and the definition of $B_{\alpha,\beta}(r, f)$ in (2.2) we have

(3.17)

$$\begin{aligned} \int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon} \log^+ \frac{1}{|f(r_n e^{i\theta}) - a|} d\theta &\leq \frac{\pi}{2\omega_{j_0} \sin(\varepsilon \omega_{j_0})} r_n^{\omega_{j_0}} B_{\alpha_{j_0}, \beta_{j_0}} \left(r_n, \frac{1}{f(r_n e^{i\theta}) - a} \right) \\ &\leq K_{j_0,\varepsilon} r_n^{\omega_{j_0}} \log(r_n T(r_n, f)) \\ &= K_{j_0,\varepsilon} r_n^{\omega_{j_0}} (\log r_n + \log T(r_n, f)), \end{aligned}$$

where $r_n \notin E$ and $\omega_{j_0} = \frac{\pi}{\beta_{j_0}-\alpha_{j_0}}$, $K_{j_0,\varepsilon}$ is a positive constant depending only on j_0 and ε. By (3.16) and (3.17) we have

$$\log T(r_n, f) \leq \log\log T(r_n, f) + \omega_{j_0} \log r_n + 3\log\log r_n + O(1), \tag{3.18}$$

where $r_n \notin E$ and $r_n \to \infty$. Noting that $\{r_n\}$ is a Pólya peak of order $\sigma + 2\varepsilon$ of f outside E, we can get by (3.18) that

$$\sigma + 2\varepsilon \leq \lim_{r_n\to\infty} \frac{\log T(r_n, f)}{\log r_n} \leq \omega_{j_0} \leq \omega,$$

which contradicts the assumption $\sigma = \max\{\omega, \mu\}$, and so we have (3.6). By (3.6) and the assumption of Theorem 1.1 we get a contradiction.

Subcase 1.2. Suppose that $\rho(f) = \mu(f)$. By the same argument as in Case 1 with all $\sigma + 2\varepsilon$ replaced with $\sigma = \mu(f) = \rho(f)$, we can derive $\rho(f) = \sigma \leq \omega$, which contradicts (3.7). Therefore, we have (3.6). By (3.6) and the assumption of Theorem 1.1 we get a contradiction.

Case 2. Suppose that $h_2 = c$, where c is a constant such that $c \in \mathbb{C}\backslash\{0, 1\}$. Then, by (3.1) we can see that h_1 is not a constant such that

$$f = \frac{h_1 - 1}{c - 1}, \tag{3.19}$$

and so

$$f - a = \frac{h_1 - 1 - a(c - 1)}{c - 1}. \tag{3.20}$$

By (3.20) and Lemma 2.1 we have

$$S_{\alpha_j,\beta_j}(r, f) = S_{\alpha_j,\beta_j}(r, h_1) + O(1), \quad 1 \leq j \leq q. \tag{3.21}$$

If $a(c-1)+1 \neq 0$, by (3.20), (3.21), Lemmas 2.1 and 2.2 and the assumption that f and g share 0, 1, ∞ CM in $\overline{X} = \bigcup_{j=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}$ we have

(3.22)

$$\begin{aligned} S_{\alpha_j,\beta_j}(r, h_1) &\leq \overline{C}_{\alpha_j,\beta_j}(r, h_1) + \overline{C}_{\alpha_j,\beta_j}\left(r, \frac{1}{h_1}\right) + \overline{C}_{\alpha_j,\beta_j}\left(r, \frac{1}{h_1 - 1 - a(c-1)}\right) \\ &\quad + R_{\alpha_j,\beta_j}(r, h_1) \\ &= \overline{C}_{\alpha_j,\beta_j}\left(r, \frac{1}{f-a}\right) + R_{\alpha_j,\beta_j}(r, h_1) \\ &\leq C_{\alpha_j,\beta_j}\left(r, \frac{1}{f-a}\right) + O(\log r + \log T(r, h_1)) \\ &\leq S_{\alpha_j,\beta_j}(r, f) + O(\log r + \log T(r, f)) \end{aligned}$$

as $r \notin E$ and $r \to \infty$, where j is a positive integer satisfying $1 \leq j \leq q$. Again by (3.21), (3.22) and Lemma 2.1 we have (3.5). Next, in the same manner as in Case 1 we can get a contradiction. Therefore. we have $a(c-1)+1 = 0$, and so $c = (a-1)/a$. Combining this with (3.19) and (3.20), we get the conclusion (i) of Theorem B.

Case 3. Suppose that $h_1 = c$, where c is a constant such that $c \in \mathbb{C}\backslash\{0,1\}$. Then, (3.1) can be rewritten as

$$f = \frac{c-1}{h_2 - 1}, \quad g = \frac{c^{-1}-1}{h_2^{-1}-1}, \tag{3.23}$$

and so

$$f - a = -\frac{a\,(h_2 - (a+c-1)/a)}{h_2 - 1}. \tag{3.24}$$

Noting that f is a nonconstant meromorphic function, we can see by (3.23) that $(a+c-1)/a \neq 1$. Combining this with the assumption that f and g share 0, 1, ∞ CM in $\overline{X} = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$, we have

$$C_{\alpha_j,\beta_j}\left(r, \frac{1}{f-a}\right) = C_{\alpha_j,\beta_j}\left(r, \frac{1}{h_2 - (a+c-1)/a}\right), \quad 1 \leq j \leq q. \tag{3.25}$$

If $(a+c-1)/a \neq 0$, then in the same manner as in the proof of (3.22), we have by (3.25) that

$$S_{\alpha_j,\beta_j}(r, f) = C_{\alpha_j,\beta_j}\left(r, \frac{1}{f-a}\right) + O(\log r + \log T(r, f)) \tag{3.26}$$

for $1 \leq j \leq q$ as $r \notin E$ and $r \to \infty$. By (3.26) we have (3.5). Next, in the same manner as in Case 1 we can get a contradiction. Therefore, we have $(a+c-1)/a = 0$, and so $c = 1-a$. Combining this with (3.23), we have the conclusion (ii) of Theorem B.

Case 4. Suppose that $h_1 h_2^{-1} = c$, where c is a constant such that $c \in \mathbb{C} \setminus \{0,1\}$. Then, (3.1) can be rewritten as

$$f = \frac{ch_2 - 1}{h_2 - 1}, \quad g = \frac{c^{-1}h_2^{-1} - 1}{h_2^{-1} - 1}, \tag{3.27}$$

and so

$$f - a = \frac{(c-a)\,(h_2 - (a-1)/(a-c))}{h_2 - 1}. \tag{3.28}$$

Suppose that $a = c$. Then, by (3.27) we have the conclusion (iii) of Theorem B. Next we suppose that $a \neq c$. Then, by the assumption that f is a nonconstant meromorphic function, we can see by (3.28) that $(a-1)/(a-c) \neq 1$. Combining this with the assumption that f and g share 0, 1, ∞ CM in $\overline{X} = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$, we have

$$C_{\alpha_j,\beta_j}\left(r, \frac{1}{f-a}\right) = C_{\alpha_j,\beta_j}\left(r, \frac{1}{h_2 - (a-1)/(a-c)}\right), \quad 1 \leq j \leq q. \tag{3.29}$$

Next, in the same manner as in the proof of (3.22), we get (3.26) by (3.29) and the fact $(a-1)/(a-c) \neq 0$. By (3.26) we have (3.5). Next, in the same manner as in Case 1 we can get a contradiction. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. First of all, by the assumptions of Theorem 1.2 and Lemma 2.3 we have

$$f=\frac{\hat{h}_1-1}{\hat{h}_2-1},\quad g=\frac{\hat{h}_1^{-1}-1}{\hat{h}_2^{-1}-1} \tag{3.30}$$

and

$$\begin{aligned} &S_{\alpha_j,\alpha_{j+1}}(r,g)+S_{\alpha_j,\alpha_{j+1}}(r,\hat{h}_1)+S_{\alpha_j,\alpha_{j+1}}(r,\hat{h}_2)\\ &=O(S_{\alpha_j,\alpha_{j+1}}(r,f))+R_{\alpha_j,\alpha_{j+1}}(r,f)\end{aligned} \tag{3.31}$$

for $1\le j\le q$, where $\hat{h}_1$ and $\hat{h}_2$ are meromorphic functions such that $\hat{h}_2\not\equiv 1$, $\hat{h}_1\not\equiv 1$, $\hat{h}_2\hat{h}_1^{-1}\not\equiv 1$, $\hat{h}_1(z)\notin\{0,\infty\}$ and $\hat{h}_2(z)\notin\{0,\infty\}$ for any $z\in\overline{X}=\bigcup_{j=1}^{q}\{z:\alpha_j\le\arg z\le\alpha_{j+1}\}$. By (3.30) we have

$$\frac{f-1}{g-1}=\hat{h}_1,\quad \frac{f}{g}=\hat{h}_1\hat{h}_2^{-1}. \tag{3.32}$$

We consider the following four cases:

Case 1. Suppose that none of $\hat{h}_1$, $\hat{h}_2$ and $\hat{h}_2\hat{h}_1^{-1}$ is a constant. Then, by Lemma 2.6 we have

$$C_{\alpha_j,\alpha_{j+1}}\left(r,\frac{1}{f-a}\right)=S_{\alpha_j,\alpha_{j+1}}(r,f)+R_{\alpha_j,\alpha_{j+1}}(r,f),\quad 1\le j\le q. \tag{3.33}$$

By (3.33) and Lemmas 2.1 we have for $1\le j\le q$ that

$$\begin{aligned} A_{\alpha_j,\alpha_{j+1}}\left(r,\frac{1}{f-a}\right)+B_{\alpha_j,\alpha_{j+1}}\left(r,\frac{1}{f-a}\right)&=R_{\alpha_j,\alpha_{j+1}}\left(r,\frac{1}{f-a}\right)\\ &\le O(\log r+\log T(r,f))\end{aligned} \tag{3.34}$$

as $r\notin E$ and $r\to\infty$. Next we prove

$$\mu(f)<\infty. \tag{3.35}$$

Indeed, for the exceptional set F in Lemma 2.9 and the exceptional set E in (3.34),we have $\overline{\log dens}(F\cup E)=0$. Applying this and Lemma 2.9 to f, we can find that there exist a sequence of positive numbers $r_n\notin F\cup E$ such that

$$meas E\left(r_n,\frac{1}{f-a}\right)>\frac{1}{(T(r_n,f))^{\varepsilon}(\log r_n)^{1+\varepsilon}} \tag{3.36}$$

as $r_n\to\infty$. Set

$$\varepsilon_n=\frac{1}{2q+1}\frac{1}{(T(r_n,f))^{\varepsilon}(\log r_n)^{1+\varepsilon}}. \tag{3.37}$$

Then, by (3.36) and (3.37) we have

$$meas\left(E\left(r_n,\frac{1}{f-a}\right)\cap\bigcup_{j=1}^{q}(\alpha_j+\varepsilon_n,\alpha_{j+1}-\varepsilon_n)\right)$$

$$\geq measE\left(r_n, \frac{1}{f-a}\right) - meas\left(\bigcup_{j=1}^{q}(\alpha_j-\varepsilon_n, \alpha_j+\varepsilon_n)\right)$$
$$> (2q+1)\varepsilon_n - 2q\varepsilon_n$$
$$= \varepsilon_n,$$

which implies that there exists some j_0 satisfying $1 \leq j_0 \leq q$ such that

$$meas\left(E\left(r_n, \frac{1}{f-a}\right) \cap (\alpha_{j_0}+\varepsilon_n, \alpha_{j_0+1}-\varepsilon_n)\right) \geq \frac{\varepsilon_n}{q}. \tag{3.38}$$

Without loss of generality, we can assume that (3.38) holds for all the positive integers n. Next we set

$$\tilde{E}_n = E\left(r_n, \frac{1}{f-a}\right) \cap (\alpha_{j_0}+\varepsilon_n, \alpha_{j_0+1}-\varepsilon_n). \tag{3.39}$$

By (3.39) and the definition of $\tilde{E}\left(r_n, \frac{1}{f-a}\right)$ we have

$$\begin{aligned}\int_{\alpha_{j_0}+\varepsilon_n}^{\alpha_{j_0+1}-\varepsilon_n} \log^+ \frac{1}{|f(r_ne^{i\theta})-a|}d\theta &\geq \int_{\tilde{E}_n} \log^+ \frac{1}{|f(r_ne^{i\theta})-a|}d\theta \\ &\geq meas(\tilde{E}_n)\frac{\delta(a,f)}{4}T(r_n,f) \\ &\geq \frac{\varepsilon_n\delta(a,f)}{4q}T(r_n,f).\end{aligned} \tag{3.40}$$

On the other hand, by (3.34), Lemma 2.1 and the definition of $B_{\alpha,\beta}(r,f)$ in (2.2) we have

$$\begin{aligned}&\int_{\alpha_{j_0}+\varepsilon_n}^{\alpha_{j_0+1}-\varepsilon_n} \log^+ \frac{1}{|f(r_ne^{i\theta})-a|}d\theta \\ &\leq \frac{\pi}{2\omega_{j_0}\sin(\varepsilon_n\omega_{j_0})}r_n^{\omega_{j_0}}B_{\alpha_{j_0},\alpha_{j_0+1}}\left(r_n, \frac{1}{f(r_ne^{i\theta})-a}\right) \\ &\leq \tilde{K}_{j_0,\varepsilon}r_n^{\omega_{j_0}}\log(r_nT(r_n,f)) \\ &= \tilde{K}_{j_0,\varepsilon}r_n^{\omega_{j_0}}(\log r_n + \log T(r_n,f))\end{aligned} \tag{3.41}$$

as $r_n \notin F \cup E$ and $r_n \to \infty$, where $\omega_{j_0} = \frac{\pi}{\alpha_{j_0+1}-\alpha_{j_0}}$, $\tilde{K}_{j_0,\varepsilon}$ is a positive constant depending only on j_0 and ε. By (3.40) and (3.41) we have

$$\begin{aligned}&\delta(a,f)(T(r_n,f))^{1-\varepsilon} \\ &\leq 4q(2q+1)\tilde{K}_{j_0,\varepsilon}r_n^{\omega_{j_0}}(\log r_n)^{1+\varepsilon}(\log r_n + \log T(r_n,f)) + O(1)\end{aligned} \tag{3.42}$$

as $r_n \notin F \cup E$ and $r_n \to \infty$. By (3.42) we derive $\mu(f) \leq \omega_{j_0} \leq \omega$, which implies (3.35). Next, by (3.34), (3.35) and in the same manner as in Case 1 of the proof of Theorem 1.1 we can get a contradiction.

Case 2. Suppose that $\hat{h}_2 = c$, where c is a constant such that $c \in \mathbb{C}\backslash\{0, 1\}$. Then, (3.30) can be rewritten as

$$f = \frac{\hat{h}_1 - 1}{c - 1}, \quad g = \frac{\hat{h}_1^{-1} - 1}{c^{-1} - 1}, \tag{3.43}$$

and so

$$f - a = \frac{\hat{h}_1 - 1 - a(c - 1)}{c - 1}. \tag{3.44}$$

By (3.44) and Lemma 2.1 we have

$$S_{\alpha_j,\alpha_{j+1}}(r, f) = S_{\alpha_j,\alpha_{j+1}}(r, h_1) + O(1), \quad 1 \leq j \leq q. \tag{3.45}$$

If $a(c-1)+1 \neq 0$, in the same manner as in Case 2 in the proof of Theorem 1.1, by (3.44), (3.43), Lemmas 2.1, Lemma 2.2 and the assumption that f and g share 0, 1, ∞ CM in $\bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \alpha_{j+1}\}$ we deduce (3.33), and so we have (3.34). Next, in the same manner as in Case 1 we can get a contradiction. Therefore, we have $a(c - 1) + 1 = 0$. Combining this with (3.43), we can get the conclusion (i) of Theorem B.

Case 3. Suppose that $\hat{h}_1 = c$, where $\hat{c}$ is a constant such that $\hat{c} \in \mathbb{C}\backslash\{0, 1\}$. Then, in the same manner as in Case 3 in the proof of Theorem 1.1 we can get the conclusion (ii) of Theorem B.

Case 4. Suppose that $\hat{h}_1\hat{h}_2^{-1} = c$, where c is a constant such that $c \in \mathbb{C} \setminus \{0, 1\}$. Then, in the same manner as in Case 4 in the proof of Theorem 1.1 we can get the conclusion (iii) of Theorem B. This completes the proof of Theorem 1.2. □

4. Concluding remarks

Regarding Theorem 1.1, Theorem 1.2 and Example 1.2, now we pose the following questions:

Question 4.1. What can be said about the conclusion of Theorem 1.1, if we change the assumption " $\rho(f) > \omega$" in Theorem 1.1?

Question 4.2. What can be said about the conclusion of Theorem 1.2, if we change the assumption "$\rho(f) > \frac{\pi}{\min\limits_{1\leq j\leq q}\{\alpha_{j+1}-\alpha_j\}}$" in Theorem 1.2?

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References

[1] A. Baernstein, *Proof of Edrei's spread conjecture*, Proc. London Math. Soc. **26** (1973), 418–434.
[2] I. N. Baker, *Sets of non-normality in iteration theory*, J. London Math. Soc. **40** (1965), 499–502.
[3] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer.Math. Soc. (N.S.) **29** (1993), no. 2 151–188.

[4] A. Edrei, *Meromorphic functions with three radially distributed values*, Trans. Amer. Math. Soc. **78** (1955), 276–293.
[5] ———, *Sums of deficiencies of meromorphic functions*, J. Analyse Math. **14** (1965), 79–107.
[6] A. A. Goldberg and I. V. Ostrovskii, *Value Distribution of Meromorphic Functions*, 2nd Edition (Translations of Mathematical Monographs), Vol. 236, American Mathematical Society, Rhode Island, Providence, 2008.
[7] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.
[8] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin/ New York, 1993.
[9] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions mromorphes*, Gauthier-Villars, Paris, 1929.
[10] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[11] L. Yang, *Borel directions of meromorphic functions in an angular domain*, Sci. Sinica (1979), 149–164.
[12] H. X. Yi, *Unicity theorems for meromorphic functions that share three values*, Kodai Math. J. **18** (1995), no. 2, 300–314.
[13] J. H. Zheng, *On Uniqueness of meromorphic functions with shared values in one angular domain*, Complex Var. Theory Appl. **48** (2003), no. 9, 777–785.
[14] ———, *On uniqueness of meromorphic functions with shared values in some angular domains*, Canad. Math. Bull. **47** (2004), no. 1, 152–160.

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