Commun. Korean Math. Soc. **31** (2016), No. 3, pp. 461–466
http://dx.doi.org/10.4134/CKMS.c150196
pISSN: 1225-1763 / eISSN: 2234-3024

ON THE RESIDUAL FINITENESS OF CERTAIN POLYGONAL PRODUCTS OF FREE GROUPS

Goansu Kim

Abstract. In general, polygonal products of free groups are not residually finite. Using the residual finiteness of polygonal products of nilpotent groups, we show that certain polygonal products of free groups are residually finite.

1. Introduction

A group G is called *residually finite* ($\mathcal{RF}$) if and only if, for each element $1 \neq x \in G$, there exists a normal subgroup N of finite index in G such that $x \notin N$. Since Mostowski [11] solved the word problem for finitely presented residually finite groups, it is interesting to find residually finite groups.

Polygonal products of groups were introduced by A. Karrass et al. [4] in the study of the subgroup structure of the Picard group $PSL(2, \mathbb{Z}[i])$, which is a polygonal product of four finite groups amalgamating cyclic subgroups, with trivial intersections. Since a polygonal product can appear as a subgroup of a group, and then the residual properties of the polygonal product determine the residual properties of the whole group [6, Example 1.1], we are interested in the residual properties of polygonal products. In [3], Allenby and Tang proved that polygonal products of four finitely generated free abelian groups, amalgamating cyclic subgroups with trivial intersections, are residually finite ($\mathcal{RF}$). And they gave an example of a polygonal product of finitely generated nilpotent -or free- groups which is not $\mathcal{RF}$. However, in [6, 8], Tang and Kim showed that certain polygonal products of finitely generated nilpotent groups are $\mathcal{RF}$ or π_c. Then, Allenby [1] constructed polygonal products of nilpotent groups which are not $\mathcal{RF}$, hence untidy conditions in [8] can not be removed. In [5, 7], Kim proved that polygonal products of more than three polycyclic-by-finite groups amalgamating central subgroups with trivial intersections are π_c and conjugacy separable, hence they are $\mathcal{RF}$. Allenby [2] showed, using the criterion in [6], that polygonal products of four polycyclic-by-finite groups, amalgamating normal subgroups, are π_c. Subgroup separability of polygonal

Received October 27, 2015; Revised December 7, 2015.
2010 *Mathematics Subject Classification.* Primary 20E26, 20E06; Secondary 20F10.
Key words and phrases. polygonal products of groups, free groups, residually finite.

products is also considered in [5]. Hence, for polygonal products of nilpotent groups, most of important residual properties are known. Since free groups are residually finitely generated nilpotent, it is of interest to study the residual properties of polygonal products of free groups. In this paper we show that certain polygonal products of free groups amalgamating cyclic subgroups are residually finite.

2. Preliminaries

Briefly, polygonal products of groups can be described as follows [3]: Let P be a polygon. Assign a group G_v for each vertex v and a group G_e for each edge e of P. Let α_e and β_e be monomorphisms which embed G_e as a subgroup of the two vertex groups at the ends of the edge e. Then the *polygonal product* G is defined to be the group presented by the generators and relations of the vertex groups together with the extra relations obtained by identifying $g_e\alpha_e$ and $g_e\beta_e$ for each $g_e \in G_e$. By abuse of language, we say that G is the polygonal product of the (vertex) groups $G_1, G_2, \ldots, G_n$, amalgamating the (edge) subgroups $H_1, H_2, \ldots, H_n$ with *trivial intersections*, if $G_i \cap G_{i+1} = H_i$ and $H_i \cap H_{i+1} = 1$, where $1 \leq i \leq n$ and the subscripts i are taken modulo n. We only consider the case $n \geq 4$ (see [3]).

The proofs of next two results are very similar to Lemma 4.5 and Theorem 4.6 in [8]. Here we use $Z_i(G)$ to denote the i-th term of the upper central series of G with $Z_1(G)$, the center of G.

Lemma 2.1. *Let G be a finitely generated torsion-free nilpotent group and let $a, b \in G$ be such that $\langle a\rangle \cap \langle b\rangle = 1$. If Δ is an infinite set of primes, and if $A = \langle a, b\rangle$ is nilpotent of class γ, and $a \in Z_\gamma(G)$, then we have*:

(1) $\cap_{p\in\Delta} \langle x^p\rangle^G \langle x\rangle = \langle x\rangle$ *for every* $x \in G$;
(2) $\cap_{p\in\Delta} \langle a^p\rangle^G \langle a\rangle\langle b\rangle = \langle a\rangle\langle b\rangle$;
(3) $\cap_{p\in\Delta} \langle a^p\rangle^G \langle b\rangle = \langle b\rangle$.

Theorem 2.2. *Let P_0 be the polygonal product of the four finitely generated torsion-free nilpotent groups A_0, B_0, C_0, D_0, amalgamating the subgroups $\langle b\rangle, \langle c\rangle, \langle d\rangle, \langle a\rangle$, with trivial intersections. Assume that $A = \langle a, b\rangle$, $B = \langle b, c\rangle$, $C = \langle c, d\rangle$, $D = \langle d, a\rangle$ have nilpotent classes k_1, k_2, k_3, k_4, respectively, and that $a \in Z_{k_1}(A_0) \cap Z_{k_4}(D_0)$, $c \in Z_{k_2}(B_0) \cap Z_{k_3}(C_0)$. Then P_0 is $\mathcal{RF}$.*

Corollary 2.3 ([8, Theorem 4.6])**.** *Let P_0 be the polygonal product of the finitely generated torsion-free nilpotent groups A_0, B_0, C_0, D_0, amalgamating $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$, with trivial intersections. If $A = \langle a, b\rangle$, $B = \langle b, c\rangle$, $C = \langle c, d\rangle$ and $D = \langle d, a\rangle$ have the same nilpotent class as A_0, B_0, C_0, and D_0, respectively, then P_0 is $\mathcal{RF}$.*

Since abelian groups and their subgroups have the same nilpotent class 1, we directly have the following result.

Corollary 2.4 ([3, Theorem 3.4]). *Let P_0 be the polygonal product of the finitely generated free abelian groups A_0, B_0, C_0, D_0, amalgamating $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$, with trivial intersections. Then P_0 is $\mathcal{RF}$.*

Lemma 2.5 (Magnus [9] or [12, Theorem 6.1.10]). *If F is a free group, then the intersection of all the terms of the lower central series of F is trivial, that is, F is residually nilpotent.*

3. Main results

Since a free group is residually a finitely generated torsion-free nilpotent group (Lemma 2.5), using Theorem 2.2 and Corollary 2.3, we can study the residual finiteness of polygonal products of free groups. As mentioned by Allenby and Tang, Example 3.1 in [3] can be extended to a polygonal product of free groups, hence not all polygonal products of free groups are residually finite (Example 3.5 below). We denote by $\Gamma_i(G)$ the i-th term of the lower central series of G, that is $\Gamma_1(G) = G$ and $\Gamma_{i+1}(G) = [\Gamma_i(G), G]$ for $i \geq 1$, where $[A, B] = \langle a^{-1}b^{-1}ab \mid a \in A, b \in B\rangle$.

Lemma 3.1. *Let G be a finitely generated free group. Let $a, b \in G$ be such that $\langle a\rangle \cap \langle b\rangle = 1$. If Δ is an infinite set of integers and if $\Gamma_\ell(G) \cap \langle a\rangle\langle b\rangle = 1$, for some integer ℓ, then we have*:

(1) $\cap_{k\in\Delta}\Gamma_k(G)\langle a\rangle = \langle a\rangle$ *and* $\cap_{k\in\Delta}\Gamma_k(G)\langle b\rangle = \langle b\rangle$,

(2) $\cap_{k\in\Delta}\Gamma_k(G)\langle a\rangle\langle b\rangle = \langle a\rangle\langle b\rangle$.

Proof. To prove (1), we let $n > m \geq \ell$. If $y \in \Gamma_n(G)\langle a\rangle$, then $y = w_n a^{i_n}$ for some integer i_n, where $w_n \in \Gamma_n(G)$. Also, if $y \in \Gamma_m(G)\langle a\rangle$, then $y = w_m a^{i_m}$ for some integer i_m, where $w_m \in \Gamma_m(G)$. It follows that

$$w_n^{-1}w_m = a^{i_n}y^{-1}ya^{-i_m} \in \Gamma_m(G) \subset \Gamma_\ell(G),$$

since $\ell \leq m < n$. Thus, we have $a^{i_n-i_m} \in \Gamma_\ell(G) \cap \langle a\rangle\langle b\rangle = 1$. Hence $i_n = i_m$. Therefore, $\alpha = i_n$ is independent of $n \geq \ell$. Now, if $y \in \cap_{k\in\Delta}\Gamma_k(G)\langle a\rangle$, then

$$y \in \bigcap_{\ell<k\in\Delta} \Gamma_k(G)\langle a\rangle;$$

hence,

$$ya^{-\alpha} \in \bigcap_{\ell<k\in\Delta} \Gamma_k(G) = 1,$$

by Lemma 2.5. Thus we have $y = a^\alpha \in \langle a\rangle$. This proves that $\cap_{k\in\Delta}\Gamma_k(G)\langle a\rangle = \langle a\rangle$. Similarly $\cap_{k\in\Delta}\Gamma_k(G)\langle b\rangle = \langle b\rangle$.

To prove (2), we let $n > m \geq \ell$. If $y \in \Gamma_n(G)\langle a\rangle\langle b\rangle$, then $y = w_n a^{i_n} b^{j_n}$ for some integers i_n and j_n, where $w_n \in \Gamma_n(G)$. Also, if $y \in \Gamma_m(G)\langle a\rangle\langle b\rangle$, then $y = w_m a^{i_m} b^{j_m}$ for some integers i_m and j_m, where $w_m \in \Gamma_m(G)$. Thus, we have

$$w_n^{-1}w_m = a^{i_n}b^{j_n}y^{-1}yb^{-j_m}a^{-i_m} \in \Gamma_m(G) \subset \Gamma_\ell(G),$$

since $\ell \leq m < n$. Thus $a^{i_n} b^{j_n - j_m} a^{-i_m} \in \Gamma_\ell(G)$, whence, $a^{i_n - i_m} b^{j_n - j_m} \in \Gamma_\ell(G)$. It follows, by assumption, that $i_n = i_m$ and $j_n = j_m$. Therefore, $i_n = i_m = \alpha$ and $j_n = j_m = \beta$ are independent of $n, m \geq \ell$. Thus, if $y \in \cap_{k \in \Delta} \Gamma_k(G)\langle a\rangle\langle b\rangle$, then

$$y \in \bigcap_{\ell < k \in \Delta} \Gamma_k(G)\langle a\rangle\langle b\rangle;$$

whence,

$$yb^{-\beta} a^{-\alpha} \in \bigcap_{\ell < k \in \Delta} \Gamma_k(G) = 1.$$

Hence, $y = a^\alpha b^\beta \in \langle a\rangle\langle b\rangle$, proving (2). □

If $\overline{G}$ is a homomorphic image of G, then we use $\overline{x}$ to denote the image of $x \in G$ in $\overline{G}$.

Theorem 3.2. *Let P_0 be the polygonal product of the free groups A_0, B_0, C_0, D_0, amalgamating the cyclic subgroups $\langle b\rangle, \langle c\rangle, \langle d\rangle, \langle a\rangle$, with trivial intersections, where $A_0 \cap B_0 = \langle b\rangle$, $B_0 \cap C_0 = \langle c\rangle$, $C_0 \cap D_0 = \langle d\rangle$, $D_0 \cap A_0 = \langle a\rangle$. Let $A = \langle a, b\rangle, B = \langle b, c\rangle, C = \langle c, d\rangle, D = \langle d, a\rangle$. Assume that there exist integers r_1, r_2, r_3, r_4 such that, for all positive integers n, $\Gamma_n(A) = \Gamma_{n+r_1}(A_0) \cap A$, $\Gamma_n(B) = \Gamma_{n+r_2}(B_0) \cap B$, $\Gamma_n(C) = \Gamma_{n+r_3}(C_0) \cap C$ and $\Gamma_n(D) = \Gamma_{n+r_4}(D_0) \cap D$. If $a \in \Gamma_{r_1+1}(A_0) \cap \Gamma_{r_4+1}(D_0)$ and $c \in \Gamma_{r_2+1}(B_0) \cap \Gamma_{r_3+1}(C_0)$, then P_0 is $\mathcal{RF}$.*

Proof. Case 1. Suppose that A_0, B_0, C_0, D_0 are finitely generated. Let $r = \max\{r_1, r_2, r_3, r_4\}$ and let $s \geq r + 2$. Note that $\Gamma_2(A) \cap \langle a\rangle\langle b\rangle = 1$, where $\Gamma_2(A) = [A, A]$. Since $\Gamma_s(A_0) \subset \Gamma_{r_1+2}(A_0)$ and $\Gamma_{r_1+2}(A_0) \cap A = \Gamma_2(A)$ by assumption, we have $\Gamma_s(A_0) \cap \langle a\rangle\langle b\rangle = 1$. Similarly, we have $\Gamma_s(B_0) \cap \langle b\rangle\langle c\rangle = 1$, $\Gamma_s(C_0) \cap \langle c\rangle\langle d\rangle = 1$ and $\Gamma_s(D_0) \cap \langle d\rangle\langle a\rangle = 1$. Thus, we can form the polygonal product $\overline{P_0}$ of the finitely generated nilpotent groups $\overline{A_0}$, $\overline{B_0}$, $\overline{C_0}$ and $\overline{D_0}$, amalgamating $\langle\overline{b}\rangle$, $\langle\overline{c}\rangle$, $\langle\overline{d}\rangle$ and $\langle\overline{a}\rangle$, with trivial intersections, where $\overline{A_0} = A_0/\Gamma_s(A_0)$, $\overline{B_0} = B_0/\Gamma_s(B_0)$, $\overline{C_0} = C_0/\Gamma_s(C_0)$ and $\overline{D_0} = D_0/\Gamma_s(D_0)$. Let ϕ_s be the canonical homomorphism of P_0 onto $\overline{P_0}$. Note that $A\Gamma_s(A_0)/\Gamma_s(A_0) \cong A/\Gamma_{s-r_1}(A)$. Hence, $\overline{A} = A\phi_s = A\Gamma_s(A_0)/\Gamma_s(A_0)$ has nilpotent class $s - r_1 - 1$. Similarly, $B\phi_s$, $C\phi_s$ and $D\phi_s$ have the nilpotent classes $s - r_2 - 1$, $s - r_3 - 1$ and $s - r_4 - 1$, respectively. Note that $\Gamma_{r_1+1}(A_0)/\Gamma_s(A_0) \leq Z_{s-r_1-1}(A_0/\Gamma_s(A_0))$ (see Theorem 7.54 in [13]). Since $a \in \Gamma_{r_1+1}(A_0)$ by assumption, we have $\overline{a} = a\phi_s \in Z_{s-r_1-1}(\overline{A_0})$. Similarly, $\overline{a} \in Z_{s-r_4-1}(\overline{D_0})$, $\overline{c} \in Z_{s-r_2-1}(\overline{B_0})$ and $\overline{c} \in Z_{s-r_3-1}(\overline{C_0})$. It follows from Theorem 2.2 that $P_0\phi_s$ is $\mathcal{RF}$. Thus, if for each $1 \neq g \in P_0$, we can find an integer $s \geq r + 2$ such that $g\phi_s \neq 1$, then we will have proved that P_0 is $\mathcal{RF}$. The method to find such s is similar to that of Theorem 2.2, but we use Lemma 3.1 instead of Lemma 2.1. We omit the details.

Case 2. A_0, B_0, C_0, D_0 are not necessarily finitely generated. For each $1 \neq g \in P_0$, we can find a canonical homomorphism $\pi : P_0 \to P_0'$ such that

(1) P_0' is a polygonal product of finitely generated subgroups A_0', B_0', C_0', D_0' of A_0, B_0, C_0, D_0, amalgamating $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$,
(2) $\pi|_{P_0'}$ is the identity map on P_0',
(3) $g \in P_0'$.

Thus, $g\pi \neq 1$, and $P_0\pi = P_0'$ is $\mathcal{RF}$ by Case 1. There exists $N' \lhd_f P_0'$ such that $g\pi \notin N'$. Let N be the preimage of N' in P_0. It follows that $N \lhd_f P_0$ and $g \notin N$. This completes the proof. □

The following result follows directly from the above theorem.

Corollary 3.3. *Let P_0 be the polygonal product of the free groups A_0, B_0, C_0, D_0, amalgamating the cyclic subgroups $\langle b\rangle, \langle c\rangle, \langle d\rangle, \langle a\rangle$, with trivial intersections, where $A_0 \cap B_0 = \langle b\rangle$, $B_0 \cap C_0 = \langle c\rangle$, $C_0 \cap D_0 = \langle d\rangle$, $D_0 \cap A_0 = \langle a\rangle$. Let $A = \langle a, b\rangle$, $B = \langle b, c\rangle$, $C = \langle c, d\rangle$, $D = \langle d, a\rangle$. Assume that $\Gamma_n(A) = \Gamma_n(A_0) \cap A$, $\Gamma_n(B) = \Gamma_n(B_0) \cap B$, $\Gamma_n(C) = \Gamma_n(C_0) \cap C$ and $\Gamma_n(D) = \Gamma_n(D_0) \cap D$ for all n. Then P_0 is $\mathcal{RF}$.*

Remark 3.4. The condition "$\Gamma_n(A) = \Gamma_n(A_0) \cap A$ for all n" looks quite strong but, since $\Gamma_i(A) = \Gamma_i(A_0) \cap A$ implies $\Gamma_{i-1}(A) = \Gamma_{i-1}(A_0) \cap A$, we have only two possibilities:

(1) $\Gamma_n(A) = \Gamma_n(A_0) \cap A$ for all n; or
(2) there exists an integer k such that $\Gamma_i(A) = \Gamma_i(A_0) \cap A$ for all $i \leq k$, and $\Gamma_j(A) \neq \Gamma_j(A_0) \cap A$ for all $j > k$.

Moreover, if (2) occurs, then the following example shows that the polygonal product may not be $\mathcal{RF}$.

Example 3.5. Let $A_0 = \langle a_1, a_2\rangle$, $B_0 = \langle b_1, b_2\rangle$, $C_0 = \langle c_1, c_2\rangle$, $D_0 = \langle d_1, d_2\rangle$ be free. Let G be the polygonal product of A_0, B_0, C_0, D_0, obtained by setting $a_1^{-1}a_2^3a_1 = b_2^2$, $b_1^{-1}b_2b_1 = c_2$, $c_1^{-1}c_2c_1 = d_2$, and $d_1^{-1}d_2d_1 = a_2$. Then G has a presentation

$$G = \langle a_1, b_1, b_2, c_1, d_1 \ : \ a_1^{-1}d_1^{-1}c_1^{-1}b_1^{-1}b_2^3b_1c_1d_1a_1 = b_2^2\rangle.$$

This implies that G contains the Baumslag-Solitar group $\langle a, t \ : \ t^{-1}a^3t = a^2\rangle$ which is not $\mathcal{RF}$ [10, p. 307], hence G is not $\mathcal{RF}$. Note that $a_2^{-3} \cdot a_1^{-1}a_2^3a_1 \in A \cap \Gamma_2(A_0) \backslash \Gamma_2(A)$, where $A = \langle a_2, a_1^{-1}a_2^3a_1\rangle$. Hence $\Gamma_2(A) \neq \Gamma_2(A_0) \cap A$, hence $\Gamma_i(A) \neq \Gamma_i(A_0) \cap A$ for all $i \geq 2$. Thus (2) above holds and the condition in Corollary 3.3 fails.

Acknowledgement. The author would like to thank the referee for the careful reading of this paper and giving valuable comments.

References

[1] R. B. J. T. Allenby, *The residual finiteness of polygonal products – Two counterexamples*, Canad. Math. Bull. **37** (1994), no. 4, 433–436.
[2] ———, *Polygonal products of polycyclic by finite groups*, Bull. Austral. Math. Soc. **54** (1996), no. 3, 369–372.

[3] R. B. J. T. Allenby and C. Y. Tang, *On the residual finiteness of certain polygonal products*, Canad. Math. Bull. **32** (1989), no. 1, 11–17.
[4] A. Karrass, A. Pietrowski and D. Solitar, *The subgroups of polygonal products of groups*, unpublished manuscript.
[5] G. Kim, *On polygonal products of finitely generated abelian groups*, Bull. Austral. Math. Soc. **45** (1992), no. 3, 453–462.
[6] ______, *Cyclic subgroup separability of generalized free products*, Canad. Math. Bull. **36** (1993), no. 3, 296–302.
[7] ______, *Conjugacy separability of certain polygonal products*, Canad. Math. Bull. **39** (1996), no. 3, 294–307.
[8] G. Kim and C. Y. Tang, *On the residual finiteness of polygonal products of nilpotent groups*, Canad. Math. Bull. **35** (1992), no. 3, 390–399.
[9] W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen ring*, Math. Ann. **111** (1935), 259–280.
[10] ______, *Residually finite groups*, Bull. Amer. Math. Soc. **75** (1969), 305–316.
[11] A. W. Mostowski, *On the decidability of some problems in special classes of groups*, Fund. Math. **59** (1966), 123–135.
[12] D. J. S. Robinson, *A course in the theory of groups*, G. T. M. 80. Springer-Verlag, New York–Heidelberg–Berlin, 1982.
[13] J. S. Rose, *A Course on Group Theory*, Dover Publ., New York, 1994.

Goansu Kim
Department of Mathematics
Yeungnam University
Kyongsan 712-749, Korea
E-mail address: gskim@yu.ac.kr