Commun. Korean Math. Soc. **31** (2016), No. 3, pp. 451–460
http://dx.doi.org/10.4134/CKMS.c150168
pISSN: 1225-1763 / eISSN: 2234-3024

ROUGH SET THEORY APPLIED TO FUZZY FILTERS IN BE-ALGEBRAS

Sun Shin Ahn and Chol Kim

Abstract. The notion of a rough fuzzy filter in a BE-algebra is introduced and some properties of such a filter are investigated. The relations between the upper (lower) rough filters and the upper (lower) approximations of their homomorphism images are discussed.

1. Introduction

The notion of rough sets was introduced by Pawlark [15]. The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of and reasoning with vague and/or imprecise knowledge, data analysis, machine learning, and knowledge discovery [15, 16]. The algebraic approach to rough sets was studied in [8]. Biswas and Nanda [4] introduced the notion of rough subgroups, and Kuroki and Morderson [14] discussed the structure of rough sets and rough groups. Kuroki and Wang [11] gave some properties of lower and upper approximations with respect to the normal subgroups and the fuzzy normal subgroups, and Kuroki [9] introduced the notion of a rough ideal in a semigroup, which is an extended notion of an ideal in a semigroup, and gave some properties of such an ideal. Q. M. Xiao and Z. L. Zhang [17] established the notion of a rough prime ideal and a rough fuzzy prime ideal in a semigroup. J. Zhan et al. [19] introduced the notion of a rough soft hemiring, which is an extended notion of a rough hemiring and a soft hemiring. Y. Imai and K. Iséki [5] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. C. R. Lim and H. S. Kim [12] introduced the notion of a rough set in a BCK/BCI-algebra. By introducing the notion of a quick ideal in a BCK/BCI-algebra, they obtained some relations between a quick ideal and an upper (a lower) rough quick ideal in a BCK/BCI-algebra. In [8], H.

Received September 9, 2015; Revised October 26, 2015.
2010 *Mathematics Subject Classification.* 06F35, 03G25.
Key words and phrases. rough sets, rough filters, rough fuzzy filters.

S. Kim and Y. H. Kim defined the notion of a BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. In [2, 3], S. S. Ahn and K. S. So considered the notion of an ideal in a BE-algebra, and then stated and proved several characterizations of such an ideal. Also they generalized the notion of an upper set in a BE-algebra, and discussed properties of the characterizations of generalized upper sets $A_n(u,v)$ while relating them to the structure of an ideal in a transitive and a self distributive BE-algebra. In [7], rough sets and rough filter in BE-algebras were established. Some related their properties were discussed.

In this paper, we introduce the notion of a rough fuzzy filter in a BE-algebra, and give some properties of such a filter. Also, we discuss the relations between the upper (lower) rough filters and the upper (lower) approximations of their homomorphism images.

2. Preliminaries

We recall some definitions and results discussed in [1, 2, 3, 8].

An algebra $(X;*,1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$. (*exchange*)

We introduce a relation "$\le$" on a BE-algebra X by $x\le y$ if and only if $x*y=1$. A non-empty subset A of a BE-algebra X is said to be a *subalgebra* of X if for any $x,y\in A$, $x*y\in A$. Noticing that $x*x=1$ for all $x\in X$, it is clear that $1\in A$. A BE-algebra $(X;*,1)$ is said to be *transitive* if for any $x,y,z\in X$, $y*z\le(x*y)*(x*z)$.

Definition 2.1. Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1\in F$;
(F2) $x*y\in F$ and $x\in F$ imply $y\in F$ for all $x,y\in X$.

Proposition 2.2. *Let $X=(X;*,1)$ be a BE-algebra and F be a filter of X. If $x\le y$ and $x\in F$ for any $y\in F$, then $y\in F$.*

Proposition 2.3 ([6])**.** *If $(X;*,1)$ is a transitive BE-algebra, then for any $x,y,z\in X$,*

(1) *if $x\le y$, then $z*x\le z*y$ and $y*z\le x*z$;*
(2) *if $1\le x$, then $x=1$.*

Proposition 2.4. *Let F be a filter of a BE-algebra X. Then F is a subalgebra of X.*

Let F be a filter of a transitive BE-algebra X. Define a relation ρ on X by $(x,y)\in\rho$ if and only if $x*y\in F$ and $y*x\in F$. Then ρ is a congruence relation on X (see [6]). Denote $X/\rho := \{[x]_\rho \mid x \in X\}$, where $[x]_\rho := \{y \in X \mid (x, y) \in \rho\}$. We define a binary operation $*'$ on X/ρ by $[x]_\rho *' [y]_\rho := [x * y]_\rho$. This definition is well defined since ρ is a congruence relation. Also $[1]_\rho = F$.

Theorem 2.5. *$(X/\rho; *', [1]_\rho)$ is a transitive BE-algebra.*

Let X be a transitive BE-algebra and ρ be a congruence relation on X and let $\mathscr{P}(X)$ denote the power set of X. For all $x \in X$, let $[x]_\rho$ denote the ρ-congruence class of x. Define the functions $\rho_-,\ \rho^- : \mathscr{P}(X) \to \mathscr{P}(X)$ as follows: for any $\emptyset \neq A \in \mathscr{P}(X)$,

$$\rho_-(A) := \{x \in X | [x]_\rho \subseteq A\}$$

and

$$\rho^-(A) := \{x \in X | [x]_\rho \cap A \neq \emptyset\}.$$

$\rho_-(A)$ is called the *ρ-lower approximation* of A while $\rho^-(A)$ is called the *ρ-upper approximation* of A. For a non-empty subset A of X,

$$\rho(A) = (\rho_-(A), \rho^-(A))$$

is called a *rough set* with respect to ρ of $\mathscr{P}(X) \times \mathscr{P}(X)$ if $\rho_-(A) \neq \rho^-(A)$. A subset A of X is said to be *definable* if $\rho_-(A) = \rho^-(A)$. The pair (X, ρ) is called an *approximation space*. A congruence relation ρ on a set X is called *complete* if $[x]_\rho * [y]_\rho = [x * y]_\rho$ for any $x, y \in X$.

Proposition 2.6 ([7])**.** *Let X be a transitive BE-algebra and $\emptyset \neq A, B \subseteq X$. Let ρ and λ be congruence relations on X. Then the following hold*:
(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$,
(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$,
(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,
(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$,
(5) $A \subseteq B$ *implies* $\rho^-(A) \subseteq \rho^-(B)$,
(6) $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$,
(7) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$,
(8) $\rho \subseteq \lambda$ *implies* $\rho_-(A) \supseteq \lambda_-(A)$,
(9) $\rho \subseteq \lambda$ *implies* $\rho^-(A) \subseteq \lambda^-(A)$.

Let X be a transitive BE-algebra and let $\emptyset \neq A \subseteq X$. Let ρ be a congruence relation on X. Then A is called an *upper* (a *lower*, respectively) *rough filter* of X if $\rho^-(A)$ ($\rho_-(A)$, respectively) is a filter of X.

Theorem 2.7 ([7])**.** *Let ρ be a congruence relation on a transitive BE-algebra X. If A is a filter of X, then it is an upper rough filter of X.*

Example 2.8. Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then X is a BE-algebra [8]. Let $F := \{1, a\}$ be a filter of X. Let ρ be a congruence relation on X such that $\{1, a\}, \{b\}, \{c\}$ and $\{d\}$ are all ρ-congruences of X. Let $A := \{a, c\}$. Then A is not a filter of X. But $\rho^-(A) = \{1, a, c\}$ is a filter of X.

The notion of an upper rough filter is an extended notion of a filter in a BE-algebra.

Theorem 2.9 ([7])**.** *Let ρ be a congruence relation on a transitive BE-algebra X and let A be a filter of X. If $\rho_-(A)$ is non-empty, then A is a lower rough filter of X, i.e., $\rho_-(A)$ is a filter of X.*

Let ρ be a congruence relation on a transitive BE-algebra X and let $\emptyset \neq A \subseteq X$. The lower and upper approximations can be presented in an equivalent form as shown below:

$$\rho_-(A)/\rho = \{[x]_\rho \in X/\rho | [x]_\rho \subseteq A\},$$
$$\rho^-(A)/\rho = \{[x]_\rho \in X/\rho | [x]_\rho \cap A \neq \emptyset\}.$$

Theorem 2.10 ([7])**.** *Let ρ be a congruence relation on a transitive BE-algebra X. If A is an upper rough filter of X, then $\rho^-(A)/\rho$ is a filter of X/ρ.*

Theorem 2.11 ([7])**.** *Let ρ be a congruence relation on a transitive BE-algebra X. If A is a lower rough filter of X, then $\rho_-(A)/\rho$ is, if it is non-empty, a filter of the quotient transitive BE-algebra X/ρ.*

3. The lower and upper approximation in fuzzy sets

Let μ and λ be two fuzzy subsets of a BE-algebra X. The inclusion $\lambda \subseteq \mu$ is denoted by $\lambda(x) \leq \mu(x)$ for all $x \in X$, and $\mu \cap \lambda$ is defined by $(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x)$ for all $x \in X$.

Definition 3.1. Let ρ be a congruence relation on a transitive BE-algebra X and μ a fuzzy subset of X. Then we define the fuzzy sets $\rho_-(\mu)$ and $\rho^-(\mu)$ as follows:

$$\rho_-(\mu)(x) := \wedge_{a \in [x]_\rho} \mu(a) \text{ and } \rho^-(\mu)(x) := \vee_{a \in [x]_\rho} \mu(a).$$

The fuzzy sets $\rho_-(\mu)$ and $\rho^-(\mu)$ are called respectively the *ρ-lower* and *ρ-upper approximations* of the fuzzy set μ. $\rho(\mu) = (\rho_-(\mu), \rho^-(\mu))$ is called a *rough fuzzy set* with respect to ρ if $\rho_-(\mu) \neq \rho^-(\mu)$.

Definition 3.2 ([6])**.** A fuzzy subset μ of a BE-algebra X is called a *fuzzy filter* of X if

(FF_1) $\mu(1) \geq \mu(x)$ for all $x \in X$,
(FF_2) $\mu(y) \geq \min\{\mu(x * y), \mu(x)\}$ for all $x, y \in X$.

Let μ and ν be fuzzy filters of a BE-algebra. Then $\mu \cap \nu$ is also a fuzzy filter of X.

A fuzzy subset μ of a transitive BE-algebra X is called an *upper* (a *lower*, respectively) *rough fuzzy filter* of X if $\rho^-(\mu)$ ($\rho_-(\mu)$, respectively) is a fuzzy filter of X.

Theorem 3.3. *Let ρ be a congruence relation on a transitive BE-algebra X. If μ is a fuzzy filter of X, then $\rho^-(\mu)$ is a fuzzy filter of X.*

Proof. Since μ is a fuzzy filter of X, $\mu(1) \geq \mu(x)$ for all $x \in X$. Hence we obtain

$$\rho^-(\mu)(1) = \vee_{z\in[1]_\rho}\mu(z) \geq \vee_{x'\in[x]_\rho}\mu(x') = \rho^-(\mu)(x).$$

For any $x, y \in X$, we have

$$\begin{aligned}\rho^-(\mu)(y) = \vee_{y'\in[y]_\rho}\mu(y') &\geq \vee_{x'*y'\in[x]_\rho*'[y]_\rho, x'\in[x]_\rho} \min\{\mu(x'*y'), \mu(x')\}\\ &= \vee_{x'*y'\in[x*y]_\rho, x'\in[x]_\rho} \min\{\mu(x'*y'), \mu(x')\}\\ &= \min\{\vee_{x'*y'\in[x*y]_\rho}\mu(x'*y'), \vee_{x'\in[x]_\rho}\mu(x')\}\\ &= \min\{\rho^-(\mu)(x*y), \rho^-(\mu)(x)\}.\end{aligned}$$

Thus $\rho^-(\mu)$ is a fuzzy filter of X. □

Theorem 3.4. *Let ρ be a congruence relation on a transitive BE-algebra X. If μ is a fuzzy filter of X, then $\rho_-(\mu)$ is, if it is non-empty, a fuzzy filter of X.*

Proof. Since μ is a fuzzy filter of X, $\mu(1) \geq \mu(x)$ for all $x \in X$. Hence for all $x \in X$, we have

$$\rho_-(\mu)(1) = \wedge_{z\in[1]_\rho}\mu(z) \geq \wedge_{z'\in[x]_\rho}\mu(z') = \rho_-(\mu)(x).$$

For any $x, y \in X$, we obtain

$$\begin{aligned}\rho_-(\mu)(y) = \wedge_{y'\in[y]_\rho}\mu(y') &\geq \wedge_{x'*y'\in[x]_\rho*'[y]_\rho, x'\in[x]_\rho} \min\{\mu(x'*y'), \mu(x')\}\\ &= \wedge_{x'*y'\in[x*y]_\rho, x'\in[x]_\rho} \min\{\mu(x'*y'), \mu(x')\}\\ &= \min\{\wedge_{x'*y'\in[x*y]_\rho}\mu(x'*y'), \wedge_{x'\in[x]_\rho}\mu(x')\}\\ &= \min\{\rho_-(\mu)(x*y), \rho_-(\mu)(x)\}.\end{aligned}$$

Thus $\rho_-(\mu)$ is a fuzzy filter of X. □

Let μ be a fuzzy subset of a transitive BE-algebra X and let $(\rho_-(\mu), \rho^-(\mu))$ be a rough fuzzy set. If $\rho_-(\mu)$ and $\rho^-(\mu)$ are fuzzy filters of X, then we call $(\rho_-(\mu), \rho^-(\mu))$ a *rough fuzzy filter* of X. Therefore we have the corollary.

Corollary 3.5. *If μ is a fuzzy filter of a transitive BE-algebra X, then $(\rho_-(\mu), \rho^-(\mu))$ is a rough fuzzy filter of X. If μ, λ are fuzzy filters of a transitive BE-algebra X, then $(\rho_-(\mu \cap \lambda), \rho^-(\mu \cap \lambda))$ is a rough fuzzy filter of X.*

Let μ be a fuzzy subset of a BE-algebra X. Then the sets

$$\mu_t := \{x \in X \mid \mu(x) \geq t\},\ \mu_t^X := \{x \in X \mid \mu(x) > t\},$$

where $t \in [0, 1]$, are called respectively, *t-level subset* and *t-strong level subset* of μ.

Theorem 3.6. *Let μ be a fuzzy subset of a BE-algebra X. Then μ is a fuzzy filter of X if and only if μ_t and μ_t^X are if they are non-empty, filters of X for every $t \in [0, 1]$.*

Proof. Straightforward. □

Lemma 3.7. *Let ρ be a congruence relation on a transitive BE-algebra X. If μ is a fuzzy subset of X and $t \in [0, 1]$, then*

(1) $(\rho_-(\mu))_t = \rho_-(\mu_t)$;
(2) $(\rho^-(\mu))_t^X = \rho^-(\mu_t^X)$.

Proof. (1) We have

$$\begin{aligned} x \in (\rho_-(\mu))_t &\Leftrightarrow \rho_-(\mu)(x) \geq t \Leftrightarrow \wedge_{a \in [x]_\rho} \mu(a) \geq t \\ &\Leftrightarrow \forall a \in [x]_\rho, \mu(a) \geq t \Leftrightarrow [x]_\rho \subseteq \mu_t \Leftrightarrow x \in \rho_-(\mu_t). \end{aligned}$$

(2) Also we have

$$\begin{aligned} x \in (\rho^-(\mu))_t^X &\Leftrightarrow \rho^-(\mu)(x) > t \Leftrightarrow \vee_{a \in [x]_\rho} \mu(a) > t \\ &\Leftrightarrow \exists a \in [x]_\rho, \mu(a) > t \Leftrightarrow [x]_\rho \cap \mu_t^X \neq \emptyset \Leftrightarrow x \in \rho^-(\mu_t^X). \end{aligned}$$ □

Theorem 3.8. *Let ρ be a congruence relation on a transitive BE-algebra X. Then μ is a lower (an upper) rough fuzzy filter of X if and only if μ_t, μ_t^X are, if they are non-empty, lower (upper) rough filters of X for every $t \in [0, 1]$.*

Proof. By Theorem 3.6 and Lemma 3.7, we can obtain the conclusion easily. □

Lemma 3.9. *Let f be a surjective homomorphism of a transitive BE-algebra X to a transitive BE-algebra Y and let A be any subset of X. Let ρ_2 be a congruence relation on Y, and $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$. Then*

(1) *ρ_1 is a congruence relation on X.*
(2) *If ρ_2 is complete and f is single-valued, then ρ_1 is complete.*
(3) $f(\rho_1^-(A)) = \rho_2^-(f(A))$.
(4) $f(\rho_{1-}(A)) \subseteq \rho_{2-}(f(A))$. *If f is single-valued, then* $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$.

Proof. (1) It is clear that ρ_1 is a congruence relation on X.

(2) Let x' be any element of $[x_1 * x_2]_{\rho_1}$. Since ρ_2 is complete, by the definition of ρ_1, we know that $f(x') \in [f(x_1 * x_2)]_{\rho_2} = [f(x_1)]_{\rho_2} * [f(x_2)]_{\rho_2}$. Since f is surjective, there exist $x_1', x_2' \in X$ such that $f(x_1') \in [f(x_1)]_{\rho_2}, f(x_2') \in [f(x_2)]_{\rho_2}$, and $f(x') = f(x_1') * f(x_2') = f(x_1' * x_2')$. Since f is single-valued, by the definition of ρ_1, we have $x_1' \in [x_1]_{\rho_1}, x_2' \in [x_2]_{\rho_1}$, and $x' = x_1' * x_2'$. Thus $x' \in [x_1]_{\rho_1} * [x_2]_{\rho_1}$. This means that $[x_1 * x_2]_{\rho_1} \subseteq [x_1]_{\rho_1} * [x_2]_{\rho_1}$. On the other hand we have $[x_1]_{\rho_1} * [x_2]_{\rho_1} \subseteq [x_1 * x_2]_{\rho_1}$. Therefore ρ_1 is complete.

(3) Let y be any element of $f({\rho_1}^-(A))$. Then there exists $x \in {\rho_1}^-(A)$ such that $f(x) = y$. Hence $[x]_{\rho_1} \cap A \neq \emptyset$. Then there exists $x' \in [x]_{\rho_1} \cap A$. Then $f(x') \in f(A)$ and by the definition of ρ_1, we have $f(x') \in [f(x)]_{\rho_2}$. So $[f(x)]\rho_2 \cap f(A) \neq \emptyset$, which implies $y = f(x) \in {\rho_2}^-(f(A))$. Thus $f({\rho_1}^-(A)) \subseteq {\rho_2}^-(A))$.

Conversely, let $y \in {\rho_2}^-(f(A))$. Then there exists $x \in X$ such that $f(x) = y$. Hence $[f(x)]_{\rho_2} \cap f(A) \neq \emptyset$. So there exists $x' \in A$ such that $f(x') \in f(A)$ and $f(x') \in [f(x)]_{\rho_2}$. Then by the definition of ρ_1, we have $x' \in [x]_{\rho_1}$. Thus $[x]_{\rho_1} \cap A \neq \emptyset$ which implies $x \in {\rho_1}^-(A)$. So $y = f(x) \in f({\rho_1}^-(A))$. It means that ${\rho_2}^-(f(A)) \subseteq f({\rho_1}^-(A))$. From the above, we have $f({\rho_1}^-(A)) = {\rho_2}^-(f(A))$.

(4) Let y be any element of $f({\rho_1}_-(A))$. Then there exists $x \in {\rho_1}_-(A)$ such that $f(x) = y$, so we have $[x]_{\rho_1} \subseteq A$. Let $y' \in [y]_{\rho_2}$. Then there exists $x' \in X$ such that $f(x') = y'$ and $f(x') \in [f(x)]_{\rho_2}$. Hence $x' \in [x]_{\rho_1} \subseteq A$, and so $y' = f(x') \in f(A)$. Thus $[y]_{\rho_2} \subseteq f(A)$ which yields that $y \in {\rho_2}_-(f(A))$. So we have $f({\rho_1}_-(A)) \subseteq {\rho_2}_-(f(A))$.

Assume that f is single-valued and suppose $y \in {\rho_2}_-(f(A))$. Then there exist $x \in X$ such that $f(x) = y$ and $[f(x)]_{\rho_2} \subseteq f(A)$. Let $x' \in [x]_{\rho_1}$. Then $f(x') \in [f(x)]_{\rho_2} \subseteq f(A)$, and so $x' \in A$. Thus $[x]_{\rho_1} \subseteq A$ which yields $x \in {\rho_1}_-(A)$. Then $y = f(x) \in f(\rho_1(A))$, and so ${\rho_2}_-(f(A)) \subseteq f({\rho_1}_-(A))$. From the above, we have $f({\rho_1}_-(A)) = {\rho_2}_-(f(A))$. □

Proposition 3.10. *Let f be a homomorphism of a transitive BE-algebra X to a BE-algebra Y. The set $\beta := \{(a, b) \in X \times X \mid f(a) = f(b)\}$. Let A be a non-empty subset of X. Then $f(\beta^-(A)) = f(A)$.*

Proof. It is clear that β is a congruence relation on X. Since $A \subseteq \beta^-(A)$, we have $f(A) \subseteq f(\beta^-(A))$. Let $y \in f(\beta^-(A))$. Then there exists $x \in \beta^-(A)$ such that $y = f(x)$. Hence $a \in [x]_\beta \cap A$ and so $(a, x) \in \beta, a \in A$. Therefore $y = f(x) = f(a) \in f(A)$. This completes the proof. □

Theorem 3.11. *Let f be a surjective homomorphism of a transitive BE-algebra X to a transitive BE-algebra Y. Let ρ_2 be a congruence relation on Y and A be a subset of X. If $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$, then ${\rho_1}^-(A)$ is a filter of X if and only if ${\rho_2}^-(f(A))$ is a filter of Y.*

Proof. Assume that ${\rho_1}^-(A)$ is a filter of X. Since $1 \in {\rho_1}^-(A)$, $[1]_{\rho_1} \cap A \neq \emptyset$. Hence there exists $x' \in [1]_{\rho_1} \cap A$. Then $f(x') \in f(A)$, and by the definition of ρ_1, we have $f(x') \in [f(1)]_{\rho_2}$. So $[f(1)]_{\rho_2} \cap f(A) \neq \emptyset$ which means $f(1) \in {\rho_2}^-(f(A))$. Let $x', y' \in Y$ with $x' * y', x' \in {\rho_2}^-(f(A))$. Then there exist $x, z \in A$ such that $f(x) = x'$ and $f(z) = x' * y'$. Hence $[f(x)]_{\rho_2} \cap f(A) \neq \emptyset$ and $[f(z)]_{\rho_2} \cap f(A) \neq \emptyset$. Therefore there exists $b \in A$ such that $f(b) \in [f(x)]_{\rho_2}$. By the definition of ρ_1, $b \in [x]_{\rho_1}$ and so $b \in [x]_{\rho_1} \cap A$. Hence $[x]_{\rho_1} \cap A \neq \emptyset$. Thus $x \in {\rho_1}^-(A)$. Since f is surjective, there exists $y \in X$ such that $f(y) = y'$. Put $u := ((x * y) * z) * y$. Then $u \in X$. Since

$$\begin{aligned} f((x*y)*z) =& f(x*y)*f(z) \\ =& f(x*y)*(x'*y') (\because f(z) = x'*y') \\ =& (f(x)*f(y))*(x'*y') \\ =& (x'*y')*(x'*y') = 1', \end{aligned}$$

we have $f(u) = f(((x*y)*z)*y) = f((x*y)*z)*f(y) = 1'*f(y) = f(y) = y'$. Since $[f(z)]_{\rho_2} \cap f(A) \neq \emptyset$, we obtain

$$\begin{aligned} [x'*y']_{\rho_2} \cap f(A) =& ([x']_{\rho_2} * [y']_{\rho_2}) \cap f(A) \\ =& ([f(x)]_{\rho_2} * [f(u)]_{\rho_2}) \cap f(A) \\ =& [f(x*u)]_{\rho_2} \cap f(A) \neq \emptyset. \end{aligned}$$

Then there exists $a \in A$ such that $f(a) \in f(A)$ and $f(a) \in [f(x*u)]_{\rho_2}$. By the definition of ρ_1, we have $a \in [x*u]_{\rho_1}$. Hence $[x*u]_{\rho_1} \cap A \neq \emptyset$ and so $x*u \in {\rho_1}^-(A)$. Since ${\rho_1}^-(A)$ is a filter of X and $x \in {\rho_1}^-(A)$, we get $u \in {\rho_1}^-(A)$. Therefore $f(u) = y' \in f({\rho_1}^-(A)) = {\rho_2}^-(f(A))$. Thus ${\rho_2}^-(f(A))$ is a filter of Y.

Conversely, suppose that ${\rho_2}^-(f(A))$ is a filter of Y. Since $f(1) = 1' \in {\rho_2}^-(f(A))$, $[f(1)]_{\rho_2} \cap f(A) \neq \emptyset$. Hence there exists $y' \in [f(1)]_{\rho_2} \cap f(A)$. Since f is surjective, there exists $x' \in X$ such that $f(x') = y'$. Hence $f(x') \in [f(1)]_{\rho_2} \cap f(A)$. Therefore $f(x') \in f(A)$. By the definition of ρ_1, $x' \in [1]_{\rho_1}$ and $x' \in A$. Hence $[1]_{\rho_1} \cap A \neq \emptyset$, which means $1 \in {\rho_1}^-(A)$.

Let $x_1, x_2 \in X$ with $x_1 * x_2, x_1 \in {\rho_1}^-(A)$. By Lemma 3.9, we obtain that $f(x_1*x_2) = f(x_1)*f(x_2), f(x_1) \in f({\rho_1}^-(A)) = {\rho_2}^-(f(A))$. Since ${\rho_2}^-(f(A))$ is a filter of Y, we have $f(x_2) \in {\rho_2}^-(f(A))$. Hence $[f(x_2)]_{\rho_2} \cap f(A) \neq \emptyset$. Therefore $y' \in [f(x_2)]_{\rho_2} \cap f(A)$. Since f is surjective, there exists $x' \in X$ such that $f(x') = y'$. Hence $f(x') = y' \in [f(x_2)]_{\rho_2} \cap f(A)$. Therefore $f(x') \in f(A)$. By the definition of ρ_1, there exists $x' \in [x_2]_{\rho_1}$ and $x' \in A$. Therefore $[x_2]_{\rho_1} \cap A \neq \emptyset$, which means $x_2 \in {\rho_1}^-(A)$. Thus ${\rho_1}^-(A)$ is a filter of X. □

Theorem 3.12. *Let f be an isomorphism of a transitive BE-algebra X to a transitive BE-algebra Y. Let ρ_2 be a complete congruence relation on Y and A a subset of X. If $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$, then ${\rho_1}_-(A)$ is a filter of X if and only if ${\rho_2}_-(f(A))$ is a filter of Y.*

Proof. By Lemma 3.9, we have $f({\rho_1}_-(A)) = {\rho_2}_-(f(A))$. The proof is similar to the proof of Theorem 3.11. □

By Theorem 3.11 and Theorem 3.12, we can obtain the following conclusion easily in a quotient BE-algebra.

Corollary 3.13. *Let f be an isomorphism of a transitive BE-algebra X to a transitive BE-algebra Y. Let ρ_2 be a complete congruence relation on Y and A a subset of X. If $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$, then $\rho_{1-}(A)/\rho_1[\rho_1{}^{-}(A)/\rho_1]$ is a filter of X/ρ_1 if and only if $\rho_{2-}(f(A))/\rho_2$ $[\rho_2{}^{-}(f(A))/\rho_2]$ is a filter of Y/ρ_2.*

Theorem 3.14. *Let f be a surjective homomorphism of a transitive BE-algebra X to a transitive BE-algebra Y. Let ρ_2 be a complete congruence relation on Y and A a fuzzy subset of X. If $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$, then*

(1) *$\rho_1{}^{-}(A)$ is a fuzzy filter of X if and only if $\rho_2{}^{-}(f(A))$ is a fuzzy filter of Y.*
(2) *If f is single-valued, then $\rho_{1-}(A)$ is a fuzzy filter of X if and only if $\rho_{2-}(f(A))$ is a fuzzy filter of Y.*

Proof. (1) By Theorem 3.6, we obtain that $\rho_1{}^{-}(A)$ is a fuzzy filter of X if and only if $(\rho_1{}^{-}(A))_t^X$ is, if it is non-empty, a filter of X for every $t \in [0, 1]$. By Lemma 3.7, we have $(\rho_1{}^{-}(A))_t^X = \rho_1{}^{-}(A_t^X)$. By Theorem 3.11, we obtain that $\rho_1{}^{-}(A_t^X)$ is a filter of X if and only if $\rho_2{}^{-}(f(A_t^X))$ is a filter of Y. It is clear that $f(A_t^X) = (f(A))_t^X$. From this and Lemma 3.7, we have

$$\rho_2{}^{-}(f(A_t^X)) = \rho_2{}^{-}(f(A)_t^X) = (\rho_2{}^{-}(f(A)))_t^X.$$

By Theorem 3.6, we obtain that $(\rho_2{}^{-}(f(A)))_t^X$ is a filter of Y for every $t \in [0, 1]$ if and only if $\rho_2{}^{-}(f(A))$ is a fuzzy filter of Y. Thus the conclusion is hold.

(2) Since f is single valued, by Lemma 3.9, we have $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$. The proof is similar to that of (1). $\square$

Corollary 3.15. *Let f be an isomorphism of a transitive BE-algebra X to a transitive BE-algebra Y. Let ρ_2 be a complete congruence relation on Y and A a fuzzy subset of X. If $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$, then $\rho_{1-}(A_t)/\rho_1[\rho_1{}^{-}(A_t^X)/\rho_1]$ is a filter of X/ρ_1 if and only if*

$$\rho_{2-}(f(A_t))/\rho_2[\rho_2{}^{-}(f(A_t^X))/\rho_2]$$

is a filter of Y/ρ_2.

Acknowledgments. The authors wish to thank the anonymous reviewers for their valuable suggestions.

References

[1] S. S. Ahn and J. M. Ko, *Filters in commutative BE-algebras*, Commun. Korean Math. Soc. **27** (2012), no. 2, 233–242.
[2] S. S. Ahn and K. K. So, *On ideals and upper sets in BE-algebras*, Sci. Math. Jpn. **68** (2008), no. 2, 279–285.

[3] ______, *On generalized upper sets in BE-algebras*, Bull. Korean Math. Soc. **46** (2009), no. 2, 281–287.
[4] R. Biswas and S. Nanda, *Rough groups and rough subgroups*, Bull. Pol. Ac. Math. **42** (1994), no. 3, 251–254.
[5] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japo. **23** (1978), no. 1, 1–26.
[6] Y. B. Jun and S. S. Ahn, *Fuzzy implicative filters with degrees in the interval* $(0, 1]$, J. Computational Anal. Appl. **15** (2013), 1456–1466.
[7] S. E. Kang and S. S. Ahn, *Rough filters of BE-algebras*, J. Appl. Math. Inform. **30** (2012), no. 5-6, 1023–1030.
[8] H. S. Kim and Y. H. Kim, *On BE-algebras,* Sci. Math. Jpn. **66** (2007), no. 1, 113–116.
[9] N. Kuroki, *Rough ideals in semigroups*, Inform. Sci. **100** (1995), 139–163.
[10] N. Kuroki and J. N. Mordeson, *Structure of rough sets and rough groups*, J. Fuzzy Math. **5** (1997), no. 1, 183–191.
[11] N. Kuroki and P. P. Wang, *The lower and upper approximations in a fuzzy group*, Infrom. Sci. **90** (1996), no. 1-4, 203–220.
[12] C. R. Lim and H. S. Kim, *Rough ideals in BCK/BCI-algebras*, Bull. Polish Acad. Sci. Math. **51** (2003), no. 1, 59–67.
[13] B. J. Meng, *On filters in BE-algebras*, Sci. Math. Jpn. **71** (2010), no. 2, 201–207.
[14] J. N. Mordeson, *Rough set theory applied to* (*fuzzy*) *ideals theory*, Fuzzy Sets and Systems **121** (2001), no. 2, 315–324.
[15] Z. Pawlak, *Rough sets*, Internat. J. Comput. Inform. Sci. **11** (1982), no. 5, 341–356.
[16] ______, *Rough sets and fuzzy sets*, Fuzzy Sets and Systems **17** (1985), no. 1, 99–102.
[17] Q. M. Xiao and Z. L. Zhang, *Rough prime ideals and rough fuzzy prime ideals in semigroups*, Inform. Sci. **176** (2006), no. 6, 725–733.
[18] J. Zhan, *The Uncertainties of Ideal Theory on Hemirings*, Science Press, 2015.
[19] J. Zhan, Q. Liu, and B. Davvaz, *A new rough set theory*: *rough soft hemirings*, J. Intell. Fuzzy Systems **28** (2015), no. 4, 1687–1697.

Sun Shin Ahn
Department of Mathematics Education
Dongguk University
Seoul 04620, Korea
E-mail address: sunshine@dongguk.edu

Chol Kim
Department of Mathematics Education
Dongguk University
Seoul 04620, Korea
E-mail address: lamontck@hanmail.net