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A COMPLETE FORMULA FOR THE ORDER OF APPEARANCE OF THE POWERS OF LUCAS NUMBERS

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Abstract. Let F_n and L_n be the nth Fibonacci number and Lucas number, respectively. The order of appearance of m in the Fibonacci sequence, denoted by $z(m)$, is the smallest positive integer k such that m divides F_k. Marques obtained the formula of $z(L_n^k)$ in some cases. In this article, we obtain the formula of $z(L_n^k)$ for all $n, k \geq 1$.

1. Introduction

Let $(F_n)_{n\geq 1}$ and $(L_n)_{n\geq 1}$ be, respectively, the Fibonacci sequence and Lucas sequence given by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$. For a positive integer m, the order of appearance of m in the Fibonacci sequence, denoted by $z(m)$, is the smallest positive integer k such that m divides F_k. Recently, Marques [5] has obtained the formula of $z(L_n^k)$ in some cases as follows.

Theorem 1.1 (Marques [5, Theorem 1.2]). *We have*

(i) *if* $k \geq 1$ *and* $n \equiv 3 \pmod 6$, *then* $z(L_n^{k+1}) = nL_n^k$,
(ii) *if* $n \equiv 6 \pmod{12}$, *then* $z(L_n^2) = nL_n$, $z(L_n^3) = nL_n^2/2$, *and* $z(L_n^{k+1}) = nL_n^k/4$ *for* $k \geq 4$,
(iii) *if* $n \equiv 0 \pmod{12}$ *and* $k \geq v_2(n) + 2$, *then* $z(L_n^{k+1}) = \frac{nL_n^k}{2^{v_2(n)+1}}$.

Notice that Theorem 1.1 does not include a formula for $z(L_n^{k+1})$ when $n = 12 \cdot 2^\ell$ and $k \leq \ell + 3$ and does not give a formula for $z(L_n^k)$ when $n \equiv 1, 2 \pmod 3$. The purpose of this article is to give a formula for $z(L_n^k)$ in all cases. Our result is as follows.

Theorem 1.2. *Let* $n \geq 2$. *Then the following statements hold.*

(i) $z(L_n) = 2n$.
(ii) *If* $k \geq 2$ *and* $n \equiv 1, 2 \pmod 3$, *then* $z(L_n^k) = 2nL_n^{k-1}$.
(iii) *If* $k \geq 2$ *and* $n \equiv 3 \pmod 6$, *then* $z(L_n^k) = nL_n^{k-1}$.

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(iv) *If* $k \geq 2$ *and* $n \equiv 0 \pmod 6$, *then*

$$z(L_n^k) = \begin{cases} \dfrac{nL_n^{k-1}}{2^{v_2(n)+1}}, & \text{if } k \geq v_2(n)+3; \\ \dfrac{nL_n^{k-1}}{2^{k-2}}, & \text{if } k < v_2(n)+3. \end{cases}$$

Note that Theorem 1.2(i) is already given in [6, Proposition 4.1] and Theorem 1.2(iii) is the same as Theorem 1.1(i) but we include them here for completeness. Theorem 1.2(iv) extends (ii) and (iii) of Theorem 1.1. Finally, Theorem 1.2(ii) is new.

2. Auxiliary results

We first recall some results which will be used in the proof of main theorems.

Lemma 2.1. *We have*

(i) *if* $n \geq 2$, *then* $L_n \mid F_m$ *if and only if* $2n \mid m$,
(ii) $n \mid F_m$ *if and only if* $z(n) \mid m$,
(iii) $5 \nmid L_n$ *for any* n.

Proof. These are well-known results but we will give some references for the reader's convenience. The statement (i) can be found, for example, in [3, Theorem 16.5, p. 200], and (ii) is given by Halton in [2, Lemma 8, p. 222]. Note that Halton [2] used $\alpha(n)$ instead of $z(n)$ to denote the order of appearance of n and called it by the old name: the rank of apparition. Here we follow the notation used by Marques [5], and Fibonacci Association (see [1, Tables of Fibonacci Entry Points]). Next the identity $5F_n^2 - L_n^2 = 4(-1)^{n+1}$ can be proved by induction, or by using Binet's formula, and can also be found in [8, p. 177]. Then (iii) follows immediately from this identity. □

Lemma 2.2 (Lengyel [4])**.** *For each* $n \geq 1$, *let* $v_p(n)$ *be the* p*-adic order of* n. *Then*

$$v_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ v_2(n)+2, & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$$v_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 3; \\ 2, & \text{if } n \equiv 3 \pmod 6; \\ 1, & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$v_5(F_n) = v_5(n)$, $v_5(L_n) = 0$, *and if* p *is a prime*, $p \neq 2$, *and* $p \neq 5$, *then*

$$v_p(F_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}, \end{cases}$$

$$v_p(L_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } z(p) \text{ is even and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.3 (Pongsriiam [7, Theorem 4])**.** *Assume that k, m, n are positive integers, m is even, and $n \geq 2$. If $L_n^k \mid m$, then $L_n^{k+1} \mid F_{nm}$.*

3. Main results

Theorem 3.1. *Let $n, k \geq 2$. Then $z(L_n^k) = \frac{2nL_n^{k-1}}{j}$ where j satisfies the following conditions:*

(i) $j \mid L_n^{k-1}$.

(ii) $j = 2^a$ *for some* $a \geq 0$.

(iii) j *is the largest integer such that* $v_2(L_n^k) \leq v_2\Big(F_{\frac{2nL_n^{k-1}}{j}}\Big)$.

Proof. Since $L_n^{k-1} \mid 2L_n^{k-1}$, we obtain by Lemma 2.3 that $L_n^k \mid F_{2nL_n^{k-1}}$. By Lemma 2.1(ii), $z(L_n^k) \mid 2nL_n^{k-1}$ and therefore

$$2nL_n^{k-1} = z(L_n^k)j \quad \text{for some } j \geq 1. \tag{1}$$

By the definition of $z(L_n^k)$, we obtain $L_n \mid L_n^k \mid F_{z(L_n^k)}$. Then by Lemma 2.1(i), we have

$$2n \mid z(L_n^k). \tag{2}$$

From (1) and (2), we see that $z(L_n^k) = \frac{2nL_n^{k-1}}{j}$ and $j \mid L_n^{k-1}$. This proves (i). In addition, by the definition of $z(L_n^k)$ and Lemma 2.1(ii),

$$j \text{ is the largest positive integer such that } L_n^k \mid F_{\frac{2nL_n^{k-1}}{j}}. \tag{3}$$

Next let p be an odd prime dividing L_n. Then $v_p(L_n) > 0$. By Lemma 2.1(iii) (or by Lemma 2.2), we see that $p \neq 5$. By Lemma 2.2, we obtain $2n \equiv 0 \pmod{z(p)}$, so $\frac{2nL_n^{k-1}}{j} \equiv 0 \pmod{z(p)}$ and therefore

$$\begin{aligned} v_p\Big(F_{\frac{2nL_n^{k-1}}{j}}\Big) &= v_p(n) + (k-1)v_p(L_n) - v_p(j) + v_p(F_{z(p)}) \\ &= v_p(L_n) + (k-1)v_p(L_n) - v_p(j) \\ &= v_p(L_n^k) - v_p(j). \end{aligned} \tag{4}$$

Then (3) and (4) imply that $v_p(j) = 0$ for every odd prime p. This proves (ii). We also see that (3) and (4) imply (iii). This completes the proof. □

Next we give a proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.1(i), the smallest k such that $L_n \mid F_k$ is $k = 2n$. So $z(L_n) = 2n$. Next we will use Theorem 3.1 to prove (ii), (iii), and (iv) of Theorem 1.2. So assume that j is the integer given in Theorem 3.1.

Case 1: $k \geq 2$ and $n \equiv 1, 2 \pmod 3$. Then $2 \nmid L_n^{k-1}$. Since $j \mid L_n^{k-1}$ and $j = 2^a$ for some $a \geq 0$, we obtain that $j = 1$. So $z(L_n^k) = 2nL_n^{k-1}$.

Case 2: $k \geq 2$ and $n \equiv 3 \pmod 6$. Then by Lemma 2.2,

$$v_2\Big(F_{\frac{2nL_n^{k-1}}{j}}\Big) = v_2\Big(\frac{2nL_n^{k-1}}{j}\Big) + 2 = 1 + v_2(n) + (k-1)v_2(L_n) - v_2(j) + 2$$

$$= 1 + 0 + 2(k-1) - v_2(j) + 2 = 1 - v_2(j) + 2k$$
$$= 1 - v_2(j) + v_2(L_n^k).$$

By Theorem 3.1(iii), j is the largest integer satisfying $v_2(j) \le 1$. Then by Theorem 3.1(ii), $j = 2$ and $z(L_n^k) = nL_n^{k-1}$.

Case 3: $k \ge 2$ and $n \equiv 0 \pmod 6$. Similar to Case 2, we obtain by Lemma 2.2 that $v_2(L_n) = 1$ and $v_2\left(F_{\frac{2nL_n^{k-1}}{j}}\right) = v_2(L_n^k) + v_2(n) + 2 - v_2(j)$. By Theorem 3.1, j is the largest integer satisfying $v_2(j) \le v_2(n) + 2$, $j \mid L_n^{k-1}$, and $j = 2^\ell$ for some $\ell \ge 0$. Therefore $\ell = \min\{v_2(n) + 2, k - 1\}$. So if $k \ge v_2(n) + 3$, then $\ell = v_2(n) + 2$ and

$$z(L_n^k) = \frac{2nL_n^{k-1}}{2^\ell} = \frac{nL_n^{k-1}}{2^{v_2(n)+1}},$$

and if $k < v_2(n) + 3$, then $\ell = k - 1$ and

$$z(L_n^k) = \frac{2nL_n^{k-1}}{2^\ell} = \frac{nL_n^{k-1}}{2^{k-2}}.$$

This completes the proof. □

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