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ON SEMIDERIVATIONS IN 3-PRIME NEAR-RINGS

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Abstract. In the present paper, we expand the domain of work on the concept of semiderivations in 3-prime near-rings through the study of structure and commutativity of near-rings admitting semiderivations satisfying certain differential identities. Moreover, several examples have been provided at places which show that the assumptions in the hypotheses of various theorems are not altogether superfluous.

1. Introduction

Throughout this paper, $\mathcal{N}$ is a zero-symmetric left near ring. A near ring $\mathcal{N}$ is called zero symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that in a left near ring $x0 = 0$ for all $x \in \mathcal{N}$). $\mathcal{N}$ is called 3-prime if $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. The symbol $Z(\mathcal{N})$ will represent the multiplicative center of $\mathcal{N}$, that is, $Z(\mathcal{N}) = \{x \in \mathcal{N} \mid xy = yx \text{ for all } y \in \mathcal{N}\}$. For any $x, y \in \mathcal{N}$; as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product, respectively. Recall that $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. For terminologies concerning near-rings we refer to G. Pilz [7].

An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [8], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called semiderivation if there exists a function $g : \mathcal{N} \to \mathcal{N}$ such that $d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y$ and $d(g(x)) = g(d(x))$ for all $x, y \in \mathcal{N}$. Obviously, any derivation is a semiderivation, but the converse is not true in general (see [6]). There has been a great deal of work concerning derivations in near-rings (see [1, 2, 4, 5] where further references can be found). In this paper, we study the commutativity of addition and multiplication of near-rings. Two well-known results for derivations in near-rings have been generalized for semiderivation. In fact, our results generalize some theorems obtained by the authors together with Raji in [1].

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2. Some preliminaries

We begin with the following lemmas which are essential for developing the proof of our main result.

Lemma 2.1 ([3, Lemma 1.2(iii)])**.** *Let $\mathcal{N}$ be a 3-prime near-ring. If $z \in Z(\mathcal{N}) - \{0\}$ and $xz \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

Lemma 2.2 ([6, Lemma 2.4])**.** *Let $\mathcal{N}$ be a near-ring and d a semiderivation of $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive law*

$$\Big(xd(y) + d(x)g(y)\Big)g(z) = xd(y)g(z) + d(x)g(yz) \quad \text{for all } x, y, z \in \mathcal{N}.$$

Lemma 2.3 ([6, Theorem 2.1])**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero semiderivation d such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.4 ([6, Theorem 2.2])**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero semiderivation d such that $d([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.5 ([4, Theorem 2.9])**.** *Let $\mathcal{N}$ be a 3-prime near-ring. Then the following assertions are equivalent*:

(i) *$[x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.*
(ii) *$\mathcal{N}$ is a commutative ring.*

Lemma 2.6 ([6, Lemma 2.3])**.** *Let $\mathcal{N}$ be a near-ring. If $\mathcal{N}$ admits an additive mapping d, then the following statements are equivalent*:

(i) *d is a semiderivation associated with an additive mapping g.*
(ii) *$d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in \mathcal{N}$.*

Lemma 2.7 ([3, Lemma 1.5])**.** *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.8. *Let $\mathcal{N}$ be a 3-prime near-ring. If d is a semiderivation associated with an onto map g, then $d(Z(\mathcal{N})) \subseteq Z(\mathcal{N})$.*

Proof. Let $z \in Z(\mathcal{N})$. Then $d(zx) = d(xz)$ for all $x \in \mathcal{N}$. Using the definition of d and Lemma 2.6, we obtain $zd(x) + d(z)g(x) = d(x)z + g(x)d(z)$ for all $x \in \mathcal{N}$. Since $z \in Z(\mathcal{N})$, then the last expression implies $d(z)g(x) = g(x)d(z)$ for all $x \in \mathcal{N}$. Since g is onto, we find that $d(z)x = xd(z)$ for all $x \in \mathcal{N}$, i.e., $d(Z(\mathcal{N})) \subseteq Z(\mathcal{N})$. □

Lemma 2.9 ([4, Theorem 2.10])**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $u \circ v \in Z(\mathcal{N})$ for all $u, v \in Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

3. Main results

We shall start our investigation for semiderivation with the following result:

Theorem 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring which admits a nonzero semiderivation d associated with an onto map g. Then the following assertions are equivalent*:

(i) $d([x,y]) \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$.

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof. It is clear that (ii)⇒(i).

(i)⇒(ii). We are given that

$$d([x,y]) \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.1}$$

Replacing y by xy in (3.1), we get

$$xd([x,y]) + d(x)g([x,y]) \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.2}$$

In view of Lemma 2.2, (3.2) becomes

$$\begin{aligned} &xd([x,y])g(z) + d(x)g([x,y]z) \\ &= g(z)xd([x,y]) + g(z)d(x)g([x,y]) \quad \text{for all } x, y, z \in \mathcal{N}. \end{aligned} \tag{3.3}$$

Putting $xd([u,v])$ instead of x in (3.1) and using (3.1), we have

$$\begin{aligned} d\Big([xd([u,v]), y]\Big) &= d\Big(d([u,v])[x,y]\Big) \\ &= d([u,v])d([x,y]) + d^2([u,v])g([x,y]) \quad \text{for all } u, v, x, y \in \mathcal{N}. \end{aligned}$$

The above relation reduces to

$$d([u,v])d([x,y]) + d^2([u,v])g([x,y]) \in Z(\mathcal{N}) \quad \text{for all } u, v, x, y \in \mathcal{N}.$$

Applying Lemmas 2.2 & 2.8, we arrive at

$$d^2([u,v])\mathcal{N}\Big(g\big([x,y]z\big) - g(z)g\big([x,y]\big)\Big) = \{0\} \quad \text{for all } u, v, x, y, z \in \mathcal{N}. \tag{3.4}$$

Since $\mathcal{N}$ is 3-prime, the above relation yields that

$$\text{either } d^2([u,v]) = 0 \text{ or } g\big([x,y]z\big) = g(z)g\big([x,y]\big) \text{ for all } u, v, x, y, z \in \mathcal{N}. \tag{3.5}$$

Suppose that

$$g\big([x,y]z\big) = g(z)g\big([x,y]\big) \quad \text{for all } x, y, z \in \mathcal{N}.$$

Taking $[r,s]$ instead of x in (3.3) and invoking the last equation, we obtain

$$[r,s]d\Big([[r,s],y]\Big)g(z) = g(z)[r,s]d\Big([[r,s],y]\Big) \quad \text{for all } r, s, y, z \in \mathcal{N}.$$

This implies that

$$d\Big([[r,s],y]\Big)\mathcal{N}\Big[g(z), [r,s]\Big] = \{0\} \quad \text{for all } r, s, y, z \in \mathcal{N}. \tag{3.6}$$

By using 3-primeness of $\mathcal{N}$, the above relation (3.6) yields that

$$d\Big([[r,s],y]\Big)=0 \ \text{ or } \ g(z)[r,s]=[r,s]g(z) \quad \text{for all } \ r,s,y,z\in\mathcal{N}.$$

Suppose there exist two elements r_0, s_0 in $\mathcal{N}$ such that

$$d\Big([[r_0,s_0],y]\Big)=0 \quad \text{for all } \ y\in\mathcal{N}. \tag{3.7}$$

Substituting $[r_0,s_0]y$ for y in (3.7), we obtain

$$d\big([r_0,s_0]\big)\Big[[r_0,s_0],y\Big]+g([r_0,s_0])d\Big([[r_0,s_0],y]\Big)=0 \quad \text{for all } \ y\in\mathcal{N}$$

which yields that,

$$d\big([r_0,s_0]\big)\mathcal{N}\Big[[r_0,s_0],y\Big]=\{0\} \quad \text{for all } \ y\in\mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, the above relation implies that

$$d\big([r_0,s_0]\big)=0 \ \text{ or } \ [r_0,s_0]\in Z(\mathcal{N}).$$

If there exist two elements r_1, s_1 of $\mathcal{N}$ such that $g(z)[r_1,s_1]=[r_1,s_1]g(z)$ for all $z\in\mathcal{N}$, then since g is onto we arrive at $z[r_1,s_1]=[r_1,s_1]z$ for all $z\in\mathcal{N}$. This implies that $[r_1,s_1]\in Z(\mathcal{N})$. Hence in all in all cases, (3.5) becomes

$$d^2([u,v])=0 \text{ or } \ [u,v]\in Z(\mathcal{N}) \quad \text{for all } \ u,v\in\mathcal{N}. \tag{3.8}$$

If there exist two elements u_1, v_1 such that $[u_1,v_1]\in Z(\mathcal{N})$, then by the simple calculation of $d(x[u_1,v_1])=d([u_1,v_1]x)$ and using (3.1), we can easily arrive at

$$g([u_1,v_1])d(x)=d(x)g([u_1,v_1]) \quad \text{for all } \ x\in\mathcal{N}. \tag{3.9}$$

Putting $[u_1,v_1]x$ instead of x in (3.9), we find that for all $x\in\mathcal{N}$,

$$\begin{aligned}&[u_1,v_1]d(x)g([u_1,v_1])+d([u_1,v_1])g(x[u_1,v_1])\\ &=g([u_1,v_1])[u_1,v_1]d(x)+g([u_1,v_1])d([u_1,v_1])g(x).\end{aligned}$$

In view of (3.1), (3.9) we find that

$$d\big([u_1,v_1]\big)\mathcal{N}\Big(g\big(x[u_1,v_1]\big)-g\big([u_1,v_1]\big)g(x)\Big)=\{0\} \quad \text{for all } \ x\in\mathcal{N}. \tag{3.10}$$

Since $\mathcal{N}$ is 3-prime, (3.10) gives

$$d\big([u_1,v_1]\big)=0 \ \text{ or } \ g\big(x[u_1,v_1]\big)=g\big([u_1,v_1]\big)g(x) \quad \text{for all } \ x\in\mathcal{N}. \tag{3.11}$$

If $g\big(x[u_1,v_1]\big)=g\big([u_1,v_1]\big)g(x)$ for all $x\in\mathcal{N}$, replacing x by $x[u_1,v_1]$ in (3.9) and invoking our hypothesis, we arrive at

$$g([u_1,v_1])xd([u_1,v_1])=xd([u_1,v_1])g([u_1,v_1]) \quad \text{for all } \ x\in\mathcal{N}.$$

From this relation, we get

$$d([u_1,v_1])\mathcal{N}\big[g([u_1,v_1]),x\big]=\{0\} \quad \text{for all } \ x\in\mathcal{N}.$$

By the 3-primeness of $\mathcal{N}$, the above expression implies that

$$d([u_1, v_1]) = 0 \text{ or } \; xg([u_1, v_1]) = g([u_1, v_1])x \quad \text{for all } \; x \in \mathcal{N}.$$

If $xg([u_1, v_1]) = g([u_1, v_1])x$ for all $x \in \mathcal{N}$, then replacing x by xt in (3.9), we find that

$$g([u_1, v_1])xd(t) = xd(t)g([u_1, v_1]) \quad \text{for all } \; x, t \in \mathcal{N}.$$

This implies that

$$g([u_1, v_1])\mathcal{N}[d(t), x] = \{0\} \quad \text{for all } \; x, t \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, $g([u_1, v_1]) = 0$ or $d(\mathcal{N}) \subseteq Z(\mathcal{N})$. By Lemma 2.3, we conclude that $g([u_1, v_1]) = 0$ or $\mathcal{N}$ is a commutative ring.

If $g([u_1, v_1]) = 0$, by (3.1), we have $d([u_1, v_1])[x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$ and Lemma 2.1 assures that $d([u_1, v_1]) = 0$ or $[x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$. By application of Lemma 2.5, we find that $d([u_1, v_1]) = 0$ or $\mathcal{N}$ is a commutative ring. Therefore, in all cases, (3.8) becomes

$$d^2([u, v]) = 0 \quad \text{for all } \; u, v \in \mathcal{N} \text{ or } \mathcal{N} \text{ is a commutative ring.} \tag{3.12}$$

Now, let $d^2([u, v]) = 0$ for all $u, v \in \mathcal{N}$. Further replacing v by uv and invoking the fact that $d(g(u)) = g(d(u))$ for all $u \in \mathcal{N}$, we get

$$\begin{aligned} 0 &= d^2([u, uv]) \\ &= d^2(u[u, v]) \\ &= d^2(u)[u, v] + 2d(g(u))d([u, v]) + ud^2([u, v]) \\ &= d^2(u)[u, v] + 2d(g(u))d([u, v]) \quad \text{for all } \; u, v \in \mathcal{N}. \end{aligned}$$

Taking $[r, s]$ instead of u in the latter expression and using the 2-torsion freeness of $\mathcal{N}$, we obtain

$$d(g([r, s]))\mathcal{N}d([[r, s]), v]) = \{0\} \quad \text{for all } \; r, s, v \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$, we find that

$$d(g([r, s])) = 0 \; \text{ or } \; d([[r, s]), v]) = 0 \quad \text{for all } \; r, s, v \in \mathcal{N}. \tag{3.13}$$

Suppose there exist two elements r_0, s_0 of $\mathcal{N}$ such that $d([[r_0, s_0], v]) = 0$ for all $v \in \mathcal{N}$. Using the same techniques as used after equation (3.7), we can easily arrive at $d([r_0, s_0]) = 0$ or $[r_0, s_0] \in Z(\mathcal{N})$. If there are two elements r_1, s_1 such that $d(g([r_1, s_1])) = 0$, by definition of d and Lemma 2.6, we get

$$d(x)g\big(d([r_1, s_1])\big) + xd^2([r_1, s_1]) = d(x)d([r_1, s_1]) + g(x)d^2([r_1, s_1]) \text{ for all } x \in \mathcal{N}.$$

This yields that,

$$d(x)\mathcal{N}d([r_1, s_1]) = \{0\} \quad \text{for all } \; x \in \mathcal{N}.$$

Since $\mathcal{N}$ is a 3-prime and $d \neq 0$, $d([r_1, s_1]) = 0$. Hence in all cases, (3.13) becomes

$$d\big([r, s]\big) = 0 \; \text{ or } \; [r, s] \in Z(\mathcal{N}) \quad \text{for all } \; r, s \in \mathcal{N}. \tag{3.14}$$

Suppose that there exist two elements r_2, s_2 of $\mathcal{N}$ such that $[r_2, s_2] \in Z(\mathcal{N})$. Then

$$\begin{aligned}0 &= d^2\Big([[r_2, s_2]u, v]\Big)\\ &= d^2([u, v][r_2, s_2])\\ &= d^2([u, v])[r_2, s_2] + 2d(g([u, v]))d([r_2, s_2]) + [u, v]d^2([r_2, s_2])\\ &= 2d(g([u, v]))d([r_2, s_2]) \quad \text{for all } \; u, v \in \mathcal{N}.\end{aligned}$$

By 2-torsion freeness of $\mathcal{N}$, the last expression implies that

$$d(g([u, v]))\mathcal{N}d([r_2, s_2]) = \{0\} \quad \text{for all } \; u, v \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, $d(g([u, v])) = 0$ for all $u, v \in \mathcal{N}$ or $d([r_2, s_2]) = 0$. If $d(g([u, v])) = 0$ for all $u, v \in \mathcal{N}$, by definition of d and Lemma 2.6, we get

$$d(x)g(d([u, v]))+xd^2([u, v]) = d(x)d([u, v])+d(x)d^2([u, v]) \quad \text{for all } \; u, v, x \in \mathcal{N}.$$

This yields that

$$d(x)\mathcal{N}d([u, v]) = \{0\} \quad \text{for all } \; u, v, x \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$ and $d \neq 0$, the above relation gives $d([u, v]) = 0$ for all $u, v \in \mathcal{N}$. In all cases, by (3.14) we have the remaining possibility that $d([r, s]) = 0$ for all $r, s \in \mathcal{N}$. By Lemma 2.4, we conclude that $\mathcal{N}$ is a commutative ring. □

The following corollaries earlier obtained in [2, Theorem 4.1] and [1, Theorem 4.1] respectively are direct consequences of Theorem 3.1.

Corollary 3.1 ([2, Theorem 4.1])**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 3.2 ([1, Theorem 4.1])**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring which admits a nonzero derivation d. Then the following assertions are equivalent*

(i) $d([x, y]) \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$.
(ii) $\mathcal{N}$ *is a commutative ring.*

It would be further interesting to know that whether Theorem 3.1 can be proved if we replace commutator by an anti-commutator.

Theorem 3.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. There exists no nonzero semiderivation d associated with an onto map g such that $d(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.*

Proof. By our hypotheses, we have

$$d(x \circ y) \in Z(\mathcal{N}) \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.15}$$

Replacing y by xy in (3.15), we find that

$$xd(x \circ y) + d(x)g(x \circ y) \in Z(\mathcal{N}) \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.16}$$

Application of Lemma 2.2, together with (3.16) yields that

$$\begin{aligned} &xd(x \circ y)g(z) + d(x)g((x \circ y)z) \\ &= g(z)xd(x \circ y) + g(z)d(x)g(x \circ y) \quad \text{for all } x, y, z \in \mathcal{N}. \end{aligned} \tag{3.17}$$

Replacing x by $d(u \circ v)x$ in (3.15) and using Lemma 2.8, we find that

$$d^2(u \circ v)\mathcal{N}\Big(g\big((x \circ y)z\big) - g(z)g\big(x \circ y\big)\Big) = \{0\} \quad \text{for all } u, v, x, y, z \in \mathcal{N}. \tag{3.18}$$

By 3-primeness of $\mathcal{N}$, (3.18) implies that

$$\text{either } d^2(u \circ v) = 0 \text{ or } g\big((x \circ y)z\big) = g(z)g\big(x \circ y\big) \text{ for all } u, v, x, y, z \in \mathcal{N}. \tag{3.19}$$

Assume that

$$g\big((x \circ y)z\big) = g(z)g\big(x \circ y\big) \quad \text{for all } x, y, z \in \mathcal{N}.$$

Putting $r \circ s$ instead of x in (3.17) and invoking the last equation, we find that

$$(r \circ s)d\Big((r \circ s) \circ y\Big)g(z) = g(z)(r \circ s)d\Big((r \circ s) \circ y\Big) \quad \text{for all } r, s, y, z \in \mathcal{N},$$

which implies that

$$d\Big((r \circ s) \circ y\Big)\mathcal{N}\Big[g(z), r \circ s\Big] = \{0\} \quad \text{for all } r, s, y, z \in \mathcal{N}. \tag{3.20}$$

By 3-primeness of $\mathcal{N}$, (3.20) becomes

$$d\Big((r \circ s) \circ y\Big) = 0 \text{ or } g(z)(r \circ s) = (r \circ s)g(z) \quad \text{for all } r, s, y, z \in \mathcal{N}.$$

If there exist two elements r_0, s_0 of $\mathcal{N}$ such that $g(z)(r_0 \circ s_0) = (r_0 \circ s_0)g(z)$ for all $z \in \mathcal{N}$, then since g is onto, we find that

$$z(r_0 \circ s_0) = (r_0 \circ s_0)z \quad \text{for all } z \in \mathcal{N}. \tag{3.21}$$

Replacing x and y by r_0 and s_0 respectively in (3.17), we get

$$r_0 d(r_0 \circ s_0)g(z) = g(z)r_0 d(r_0 \circ s_0) \quad \text{for all } z \in \mathcal{N}.$$

By (3.15), the last equation becomes

$$d(r_0 \circ s_0)\mathcal{N}[r_0, g(z)] = \{0\} \quad \text{for all } z \in \mathcal{N}.$$

Since g is onto and $\mathcal{N}$ is 3-prime, we arrive at

$$d(r_0 \circ s_0) = 0 \text{ or } r_0 \in Z(\mathcal{N}).$$

If $r_0 \in Z(\mathcal{N})$, then (3.21) becomes $z(2r_0 s_0) = (2r_0 s_0)z$ for all $z \in \mathcal{N}$ which implies that $r_0\mathcal{N}[z, 2s_0] = \{0\}$ for all $z \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, we find that $r_0 = 0$ or $2s_0 \in Z(\mathcal{N})$.

If $r_0 = 0$, then $d(r_0 \circ s_0) = 0$. Otherwise $2s_0 \in Z(\mathcal{N})$. Using (3.15), we have $d((2s_0)(2s_0)) = d(2s_0 \circ s_0) \in Z(\mathcal{N})$. This yields that

$$(2s_0)d(2s_0) + d(2s_0)g(2s_0) \in Z(\mathcal{N})$$

that is,

$$\Big\{(2s_0)d(2s_0)+d(2s_0)g(2s_0)\Big\}g(z)$$
$$= g(z)\Big\{(2s_0)d(2s_0)+d(2s_0)g(2s_0)\Big\} \quad \text{for all } z\in\mathcal{N}.$$

By a simple calculation and applications of Lemmas 2.2 & 2.6, we find that

$$d(2s_0)\mathcal{N}\big(g((2s_0)z)-g(z)g(2s_0)\big)=\{0\} \quad \text{for all } z\in\mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, we find that

$$d(2s_0)=0 \text{ or } g((2s_0)z)=g(z)g(2s_0) \quad \text{for all } z\in\mathcal{N}.$$

By 2-torsion freeness of $\mathcal{N}$, we obtain

$$d(s_0)=0 \text{ or } g((2s_0)z)=g(z)g(2s_0) \quad \text{for all } z\in\mathcal{N}.$$

If $g((2s_0)z)=g(z)g(2s_0)$ for all $z\in\mathcal{N}$, then by (3.15) we have $d((2x^2)(2s_0)=d((2s_0)\circ x^2)\in Z(\mathcal{N})$ for all $x\in\mathcal{N}$. Another time by a simple calculation and using Lemmas 2.2 & 2.6, we conclude that

$$d(2s_0)\mathcal{N}[2x^2,g(z)]=\{0\} \quad \text{for all } x,z\in\mathcal{N}.$$

Since g is onto, by 3-primeness of $\mathcal{N}$, we obtain $d(s_0)=0$ or $2x^2\in Z(\mathcal{N})$ for all $x\in\mathcal{N}$. If $2x^2\in Z(\mathcal{N})$ for all $x\in\mathcal{N}$, then $2x^4=x^2(2x^2)\in Z(\mathcal{N})$ for all $x\in\mathcal{N}$, and hence by Lemma 2.1, $2x^2=0$ or $x^2\in Z(\mathcal{N})$. But $2x^2=0$ gives $x^2=0\in Z(\mathcal{N})$, which implies that $x^2\in Z(\mathcal{N})$ for all $x\in\mathcal{N}$. In this case (3.15) implies

$$d((2y^2)x^2)=d(x^2\circ y^2)\in Z(\mathcal{N}) \quad \text{for all } x,y\in\mathcal{N},$$

and hence by definition of d, the latter expression becomes

$$2y^2d(x^2)+d(2y^2)g(x^2)\in Z(\mathcal{N}) \quad \text{for all } x,y\in\mathcal{N}.$$

In view of Lemma 2.2 this yields that

$$2y^2d(x^2)g(z)+d(2y^2)g(x^2z)$$
$$= g(z)2y^2d(x^2)g(z)+g(z)d(2y^2)g(x^2) \quad \text{for all } x,y,z\in\mathcal{N}.$$

Using Lemma 2.8 and the fact that $2y^2\in Z(\mathcal{N})$ for all $y\in\mathcal{N}$, we arrive at

$$d(2y^2)\mathcal{N}\big(g(x^2z)-g(z)g(x^2)\big)=\{0\} \quad \text{for all } x,y,z\in\mathcal{N}.$$

By 3-primeness of $\mathcal{N}$, we obtain

$$d(2y^2)=0 \text{ or } g(x^2z)=g(z)g(x^2) \quad \text{for all } x,y,z\in\mathcal{N}. \tag{3.22}$$

Suppose that $g(x^2z)=g(z)g(x^2)$ for all $x,z\in\mathcal{N}$. By (3.15), we have

$$d((u\circ v)x^2)=d(uvx^2+vux^2)=d(u\circ vx^2)\in Z(\mathcal{N}) \quad \text{for all } x,u,v\in\mathcal{N}.$$

This yields that $(u \circ v)d(x^2) + d(u \circ v)g(x^2) \in Z(\mathcal{N})$ for all $x, y, z \in \mathcal{N}$, and by Lemma 2.2 the latter expression reduced to

$$\begin{aligned} &(u \circ v)d(x^2)g(z) + d(u \circ v)g(x^2 z) \\ &= g(z)(u \circ v)d(x^2) + g(z)d(u \circ v)g(x^2) \quad \text{for all } x, u, v, z \in \mathcal{N}. \end{aligned}$$

This implies that

$$d(x^2)\mathcal{N}[u \circ v, g(z)] = \{0\} \quad \text{for all } x, u, v, z \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime and g is onto, we conclude that $d(x^2) = 0$ or $u \circ v \in Z(\mathcal{N})$ for all $x, u, v \in \mathcal{N}$, which shows that equation (3.22) becomes

$$d(x^2) = 0 \text{ or } u \circ v \in Z(\mathcal{N}) \quad \text{for all } x, u, v \in \mathcal{N}.$$

If $u \circ v \in Z(\mathcal{N})$ for all $u, v \in \mathcal{N}$, by Lemma 2.9, we conclude that $\mathcal{N}$ is a commutative ring. Otherwise, by (3.15) we have $x^2 d(y + y) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$. By Lemma 2.1, we obtain

$$x^2 = 0 \text{ or } d(y + y) \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}.$$

If $x^2 = 0$ for all $x \in \mathcal{N}$, then $x(x + y)^2 = 0$ for all $x, y \in \mathcal{N}$. Hence by the simple calculation, we obtain that $xyx = 0$ for all $x, y \in \mathcal{N}$ and by 3-primeness of $\mathcal{N}$, we conclude that $x = 0$ for all $x \in \mathcal{N}$; a contradiction.

If $d(y + y) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$, replacing y by ys_0 we get $d(y(s_0 + s_0)) = yd(2s_0) + d(y)g(2s_0) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$, and by Lemma 2.2 the last upshot implies that

$$yd(2s_0)g(z) + d(y)g((2s_0)z) = g(z)yd(2s_0) + g(z)d(y)g(2s_0) \quad \text{for all } y, z \in \mathcal{N}.$$

Using the fact that $g((2s_0)z) = g(z)g(2s_0)$ for all $z \in \mathcal{N}$, the last expression becomes

$$d(2s_0)\mathcal{N}[y + y, g(z)] = \{0\} \quad \text{for all } y \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime and g is onto, we find that $d(2s_0) = 0$ or $y + y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. If $d(2s_0) = 0$ by 2-torsion freeness we get $d(s_0) = 0$. If $y + y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$, then taking y^2 instead of y, we get $y(y + y) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$ and by Lemma 2.1, we arrive at $y + y = 0$ or $y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. Since $\mathcal{N}$ is 2-torsion free, in both the cases we arrive at $y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. This implies that $\mathcal{N} \subseteq Z(\mathcal{N})$. Hence by Lemma 2.7, we conclude that $\mathcal{N}$ is a commutative ring.

Suppose there exist two elements r_1, s_1 of $\mathcal{N}$ such that

$$d\big((r_1 \circ s_1) \circ y\big) = 0 \quad \text{for all } y \in \mathcal{N}.$$

Substituting $(r_1 \circ s_1)y$ for y, we obtain

$$d(r_1 \circ s_1)\Big((r_1 \circ s_1) \circ y\Big) + g(r_1 \circ s_1)d\Big((r_1 \circ s_1) \circ y\Big) = 0 \quad \text{for all } y \in \mathcal{N}.$$

In view of (3.15), the above yields that

$$d(r_1 \circ s_1)\mathcal{N}\Big((r_1 \circ s_1) \circ y\Big) = \{0\} \quad \text{for all } y \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, the last relation implies that

$$d\big(r_1 \circ s_1\big) = 0 \ \text{ or } \ (r_1 \circ s_1) \circ y = 0 \quad \text{for all } \ y \in \mathcal{N}. \tag{3.23}$$

Suppose that

$$(r_1 \circ s_1) \circ y = 0 \quad \text{for all } \ y \in \mathcal{N}. \tag{3.24}$$

Thus in view of (3.24) we have $d(y(r_1 \circ s_1)) = -d((r_1 \circ s_1)y)$. By definition of d and Lemma 2.6, we have

$$\begin{aligned} d(y)(r_1 \circ s_1) + g(y)d(r_1 \circ s_1) &= -(d(r_1 \circ s_1)g(y) + (r_1 \circ s_1)d(y)) \\ &= -(r_1 \circ s_1)d(y) - d(r_1 \circ s_1)g(y) \quad \text{for all } \ y \in \mathcal{N}, \end{aligned}$$

which implies that

$$(r_1 \circ s_1) \circ d(y) + 2g(y)d(r_1 \circ s_1) = 0 \quad \text{for all } \ y \in \mathcal{N}.$$

This yields that

$$2g(y)d(r_1 \circ s_1) = 0 \quad \text{for all } \ y \in \mathcal{N}.$$

Since $\mathcal{N}$ is 2-torsion free and g is onto, we find that

$$y\mathcal{N}d(r_1 \circ s_1) = \{0\} \quad \text{for all } \ y \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, we conclude that $d(r_1 \circ s_1) = 0$. Thus in all cases, we find that $d(u_1 \circ v_1) = 0$. Returning to (3.19), we obtain $d^2(u \circ v) = 0$ for all $u, v \in \mathcal{N}$. Replacing v by uv and invoking the fact that $d(g(u)) = g(d(u))$ for all $u \in \mathcal{N}$, we get

$$\begin{aligned} 0 &= d^2(u \circ uv) \\ &= d^2(u(u \circ v)) \\ &= d^2(u)(u \circ v) + 2d(g(u))d(u \circ v) + ud^2(u \circ v) \\ &= d^2(u)(u \circ v) + 2d(g(u))d(u \circ v) \quad \text{for all } \ u, v \in \mathcal{N}. \end{aligned}$$

Taking $r \circ s$ instead of u in the last expression and using 2-torsion freeness of $\mathcal{N}$, we obtain

$$d(g(r \circ s))\mathcal{N}d((r \circ s) \circ v) = \{0\} \quad \text{for all } \ r, s, v \in \mathcal{N}.$$

Again, 3-primeness of $\mathcal{N}$ gives

$$d(g(r \circ s)) = 0 \ \text{ or } \ d((r \circ s) \circ v) = 0 \quad \text{for all } \ r, s, v \in \mathcal{N}. \tag{3.25}$$

If there are two elements r_0, s_0 of $\mathcal{N}$ such that $d((r_0 \circ s_0) \circ v) = 0$, using the same techniques as used after equation (3.22), we can easily obtain that $d(r_0 \circ s_0) = 0$.

Now suppose there exist two elements r_1, s_1 of $\mathcal{N}$ such that $d(g(r_1 \circ s_1)) = 0$. By definition of d and Lemma 2.6, we get

$$d(x)g\big(d(r_1 \circ s_1)\big) + xd^2(r_1 \circ s_1) = d(x)d(r_1 \circ s_1) + g(x)d^2(r_1 \circ s_1) \quad \text{for all } \ x \in \mathcal{N}.$$

This leads to

$$d(x)\mathcal{N}d(r_1 \circ s_1) = \{0\} \quad \text{for all } \ x \in \mathcal{N}.$$

Since $\mathcal{N}$ is a 3-prime and $d \neq 0$, $d(r_1 \circ s_1) = 0$. Hence, in all cases, we arrive at $d(r \circ s) = 0$ for all $r, s \in \mathcal{N}$. Replacing s by rs and using the definition of d, we get $d(r)(r \circ s) = 0$ for all $r, s \in \mathcal{N}$, it follows that $d(r)rs = -d(r)sr$ for all $r, s \in \mathcal{N}$. Putting st instead of s we arrive at

$$d(-r)\mathcal{N}(-tr + rt) = \{0\} \quad \text{for all } r, t \in \mathcal{N}.$$

This yields that $d(r) = 0$ or $r \in Z(\mathcal{N})$ for all $r \in \mathcal{N}$.

If there is an element $r_0 \in \mathcal{N}$ such that $r_0 \in Z(\mathcal{N})$, then by 2-torsion freeness of $\mathcal{N}$, we have $d(sr_0) = 0$ for all $s \in \mathcal{N}$ and by definition of d, we find that $sd(r_0) + d(s)g(r_0) = 0$ for all $s \in \mathcal{N}$. Now replacing s by sr_0, we arrive at $r_0 sd(r_0) = 0$ for all $s \in \mathcal{N}$ and by 3-primeness of $\mathcal{N}$, we conclude that $d(r_0) = 0$. Finally, $d(r) = 0$ for all $r \in \mathcal{N}$; a contradiction. This completes the proof of the theorem. □

The following corollaries are the immediate consequences of the above theorem.

Corollary 3.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. There exists no nonzero semiderivation d of $\mathcal{N}$ such that $d(x \circ y) = 0$ for all $x, y \in \mathcal{N}$.*

Corollary 3.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. There exists no nonzero derivation d of $\mathcal{N}$ such that $d(x \circ y) = 0$ for all $x, y \in \mathcal{N}$.*

Corollary 3.5. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. There exists no nonzero derivation d of $\mathcal{N}$ such that $d(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.*

Corollary 3.6. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-rings which admits a nonzero semiderivation d. Then $\mathcal{N}$ is commutative if and only if $d(xy) \in Z(\mathcal{N})$.*

Theorem 3.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring which admits a semiderivation d associated with a map g. Then the following assertions are equivalent*:

(i) *$d([x, y]) + x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.*
(ii) *$d([x, y]) - x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.*
(iii) *$\mathcal{N}$ is a commutative ring.*

Proof. It is easy to verify that (iii)⇒(i) and (iii)⇒(ii).

(i)⇒(iii). Assume that

$$d([x, y]) + x \circ y \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.26}$$

For $x = y$, (3.26) becomes $2x^2 \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$ which implies that $x^2 \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. In this case, replacing x by x^2 in (3.26), we get

$$x^2(y + y) \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.27}$$

By Lemma 2.1, (3.27) gives

$$x^2 = 0 \text{ or } y + y \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}.$$

If $x^2 = 0$ for all $x \in \mathcal{N}$, then $x(x+y)^2 = 0$ for all $x, y \in \mathcal{N}$. By the simple calculation, we obtain $xyx = 0$ for all $x, y \in \mathcal{N}$ and by 3-primeness of $\mathcal{N}$, we conclude that $x = 0$ for all $x \in \mathcal{N}$; a contradiction.

If $y + y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$, then taking y^2 instead of y, we get $y(y+y) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$ and by Lemma 2.1, we arrive at $y + y = 0$ or $y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. Since $\mathcal{N}$ is 2-torsion free, in both cases give $y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$ which implies that $\mathcal{N} \subseteq Z(\mathcal{N})$. By Lemma 2.7, we conclude that $\mathcal{N}$ is a commutative ring.

(ii)⇒(iii). Using the same techniques as we have used in the proof of (i)⇒(iii), we find that $\mathcal{N}$ is a commutative ring. □

Remark. The results in this paper remain true for right near-rings with the obvious variations.

The following example shows that the hypothesis "2-torsion free" is an essential condition in Theorems 3.1 & 3.2.

Example 3.1. Let $\mathcal{N} = M_2(\mathbb{Z}_2)$ and d be the inner derivation induced by the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{N}$ is a non-commutative prime ring and $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d+a \\ 0 & c \end{pmatrix}$. It is easy to verify that $d([A, B]) \in Z(\mathcal{N})$ and $d(A \circ B) \in Z(\mathcal{N})$ for all $A, B \in \mathcal{N}$. But $\mathcal{N}$ is not 2-torsion free.

Example 3.2. Let $\mathcal{N} = M_2(\mathbb{Z}_3)$ and d be the inner derivation induced by the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{N}$ is a non-commutative 2-torsion free prime ring and $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$. Take $x = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $d([x, y]) = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \notin Z(\mathcal{N})$ and $d(x \circ y) = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \notin Z(\mathcal{N})$ which show that the condition "$d([x, y]) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$" in Theorem 3.1, the condition $d(x \circ y) \in Z(\mathcal{N})$" in Theorem 3.2 are not superfluous.

The following example demonstrates that the 3-primeness of $\mathcal{N}$ in the above theorems can not be omitted.

Example 3.3. Let S be a 2-torsion free zero-symmetric left near ring and let

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z \in S \right\}.$$

Define $d, g : N \to N$ by

$$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}.$$

Then it can be seen easily that $\mathcal{N}$ is a zero-symmetric left near-ring which is not 3-prime and the maps d is a semiderivation on $\mathcal{N}$ associated with an onto map g satisfying all the requirements of Theorems 3.1 & 3.2. However, $\mathcal{N}$ is not a commutative ring.

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