KYUNGPOOK Math. J. 56(2016), 625-638
http://dx.doi.org/10.5666/KMJ.2016.56.2.625
pISSN 1225-6951 eISSN 0454-8124

Semi-Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds

Shiv Sharma Shukla and Akhilesh Yadav*
Department of Mathematics, University of Allahabad, Allahabad-211002, India
e-mail : ssshukla_au@rediffmail.com *and* akhilesh_mathau@rediffmail.com

Abstract. In this paper, we introduce the notion of semi-slant lightlike submanifolds of indefinite Sasakian manifolds giving characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2 and $RadTM$ on semi-slant lightlike submanifolds of an indefinite Sasakian manifold have been obtained. We also obtain necessary and sufficient conditions for foliations determined by above distributions to be totally geodesic.

1. Introduction

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu ([11]). A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be lightlike submanifold if the induced metric g on M is degenerate, i.e. there exists a non-zero $X \in \Gamma(TM)$ such that $g(X,Y) = 0$, $\forall Y \in \Gamma(TM)$. Lightlike geometry has its applications in general relativity, particularly in black hole theory, which gave impetus to study lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures. Various classes of lightlike submanifolds of indefinite Sasakian manifolds are defined according to the behaviour of distributions on these submanifolds with respect to the action of (1,1) tensor field ϕ in Sasakian structure of the ambient manifolds. Such submanifolds have been studied by Duggal and Sahin in ([12], [13]). In [3], Shahin studied screen-slant lightlike submanifolds. Further Sahin and Yildirim studied slant lightlike submanifolds of indefinite Sasakian manifolds in [4].
In [1], A. Lotta introduced the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold. The geometry of slant and semi-slant

* Corresponding Author.
Received May 1, 2015; revised November 3, 2015; accepted February 5, 2016.
2010 Mathematics Subject Classification: 53C15; 53C40; 53C50.
Key words and phrases: Semi-Riemannian manifold, degenerate metric, radical distribution, screen distribution, screen transversal vector bundle, lightlike transversal vector bundle, Gauss and Weingarten formulae.
This work was supported by Council of Scientific and Industrial Research (C.S.I.R.), India.

submanifolds of Sasakian manifolds was studied by Cabrerizo, J. L., Carriazo, A., Fernandez, L. M. and Fernandez, M., in ([9], [10]). On the other hand the theory of slant, contact Cauchy-Riemann lightlike submanifolds of indefinite Sasakian manifolds have been studied in ([4], [13]). Thus motivated sufficiently, we introduce the notion of semi-slant lightlike submanifolds of indefinite Sasakian manifolds. This new class of lightlike submanifolds of an indefinite Sasakian manifold includes slant, contact Cauchy-Riemann lightlike submanifolds as its sub-cases. The paper is arranged as follows. There are some basic results in section 2 . In section 3, we study semi-slant lightlike submanifolds of an indefinite Sasakian manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on semi-slant lightlike submanifolds of indefinite Sasakian manifolds.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [11] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM). \tag{2.1}$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{2.4}$$

Following are four cases of a lightlike submanifold $\left(M, g, S(TM), S(TM^{\perp})\right)$:

Case.1 r-lightlike if $r < min\,(m, n)$,

Case.2 co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,

Case.3 isotropic if $r = m < n$, $S\,(TM) = \{0\}$,

Case.4 totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y, A_V X$ belong to $\Gamma(TM)$ and $h(X,Y), \nabla^t_X V$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^\perp)$ called the lightlike connection and screen transversal connection on M respectively.
Now by using (2.5), (2.7)-(2.9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.10}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{2.11}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi. \tag{2.13}$$

By using above equations, we obtain

$$\overline{g}(h^l(X,\overline{P}Y),\xi) = g(A^*_\xi X, \overline{P}Y), \tag{2.14}$$

$$\overline{g}(h^*(X,\overline{P}Y),N) = g(A_N X, \overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \qquad A^*_\xi \xi = 0. \tag{2.16}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.7), we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y). \tag{2.17}$$

A semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called an ϵ-almost contact metric manifold [12] if there exists a (1, 1) tensor field ϕ, a vector field V called characteristic vector field and a 1-form η, satisfying

$$\phi^2 X = -X + \eta(X)V, \qquad \eta(V) = \epsilon, \qquad \eta \circ \phi = 0, \qquad \phi V = 0, \tag{2.18}$$

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.19}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\epsilon = 1$ or -1. It follows that

$$\overline{g}(V, V) = \epsilon, \tag{2.20}$$

$$\overline{g}(X, V) = \eta(X), \tag{2.21}$$

$$\overline{g}(X, \phi Y) = -\overline{g}(\phi X, Y), \qquad \forall X, Y \in \Gamma(T\overline{M}). \tag{2.22}$$

Then $(\phi, V, \eta, \overline{g})$ is called an ϵ-almost contact metric structure on $\overline{M}$.
An ϵ-almost contact metric structure $(\phi, V, \eta, \overline{g})$ is called an indefinite Sasakian structure iff

$$(\overline{\nabla}_X \phi)Y = \overline{g}(X, Y)V - \epsilon\eta(Y)X, \tag{2.23}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.
A semi-Riemannian manifold endowed with an indefinite Sasakian structure is called an indefinite Sasakian manifold. From (2.23), for any $X \in \Gamma(T\overline{M})$, we get

$$\overline{\nabla}_X V = -\phi X. \tag{2.24}$$

Let $(\overline{M}, \overline{g}, \phi, V, \eta)$ be an ϵ-almost contact metric manifold. If $\epsilon = 1$, then $\overline{M}$ is said to be a spacelike ϵ-almost contact metric manifold and if $\epsilon = -1$, then $\overline{M}$ is called a timelike ϵ-almost contact metric manifold. In this paper, we consider indefinite Sasakian manifolds with spacelike characteristic vector field V.

3. Semi-Slant Lightlike Submanifolds

In this section, we introduce the notion of semi-slant lightlike submanifolds of indefinite Sasakian manifolds. At first, we state the following Lemmas for later use:

Lemma 3.1 *Let M be a r-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M. Suppose that $\phi RadTM$ is* *a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi RadTM \cap \phi ltr(TM) = \{0\}$.*

Lemma 3.2 *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M. Suppose $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then any complementary distribution to $\phi RadTM \oplus \phi ltr(TM)$ in $S(TM)$ is Riemannian.*

The proofs of Lemma 3.1 and Lemma 3.2 follow as in Lemma 3.1 and Lemma 3.2 respectively of [4], so we omit them.

Definition 3.1 Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then we say that M is a semi-slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$,

(ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$,

(iii) the distribution D_1 is an invariant distribution, i.e. $\phi D_1 = D_1$,

(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between ϕX and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle θ is called the slant angle of distribution D_2. A semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$.
From the above definition, we have the following decomposition

$$TM = RadTM \oplus_{orth} (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}. \tag{3.1}$$

In particular, we have

(i) if $D_1 = 0$, then M is a slant lightlike submanifold,

(ii) if $D_1 \neq 0$ and $\theta = \pi/2$, then M is a contact CR-lightlike submanifold.

Thus the above new class of lightlike submanifolds of an indefinite Sasakian manifold includes slant, contact Cauchy-Riemann lightlike submanifolds as its subcases which have been studied in ([4], [13]).
Let $(R^{2m+1}_{2q}, \overline{g}, \phi, \eta, V)$ denote the manifold R^{2m+1}_{2q} with its usual Sasakian structure given by

$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), \qquad V = 2\partial z,$

$\overline{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$

$\phi(\sum_{i=1}^{m}(X_i \partial x_i + Y_i \partial y_i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,$

where (x^i, y^i, z) are the cartesian coordinates on R^{2m+1}_{2q}. Now we construct some examples of semi-slant lightlike submanifolds of an indefinite Sasakian manifold.

Example 1. Let $(R_2^{13}, \overline{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.
Suppose M is a submanifold of R_2^{13} given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = -y^4 = u_4$, $x^4 = y^3 = u_5$, $x^5 = u_6 \sin u_7$, $y^5 = u_6 \cos u_7$, $x^6 = \sin u_6$, $y^6 = \cos u_6$, $z = u_8$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where
$Z_1 = 2(-\partial x_1 + \partial y_2 - y^1 \partial z)$, $Z_2 = 2(\partial x_2 + y^2 \partial z)$, $Z_3 = 2\partial y_1$,
$Z_4 = 2(\partial x_3 - \partial y_4 + y^3 \partial z)$, $Z_5 = 2(\partial x_4 + \partial y_3 + y^4 \partial z)$,
$Z_6 = 2(\sin u_7 \partial x_5 + \cos u_7 \partial y_5 + \cos u_6 \partial x_6 - \sin u_6 \partial y_6 + \sin u_7 y^5 \partial z + \cos u_6 y^6 \partial z)$,
$Z_7 = 2(u_6 \cos u_7 \partial x_5 - u_6 \sin u_7 \partial y_5 + u_6 \cos u_7 y^5 \partial z)$, $Z_8 = V = 2\partial z$.

Hence $RadTM = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, V\}$.

Now $ltr(TM)$ is spanned by $N = \partial x_1 + \partial y_2 + y^1 \partial z$ and $S(TM^\perp)$ is spanned by
$W_1 = 2(\partial x_3 + \partial y_4 + y^3 \partial z)$, $W_2 = 2(\partial x_4 - \partial y_3 + y^4 \partial z)$,
$W_3 = 2(\sin u_7 \partial x_5 + \cos u_7 \partial y_5 - \cos u_6 \partial x_6 + \sin u_6 \partial y_6 + \sin u_7 y^5 \partial z - \cos u_6 y^6 \partial z)$,
$W_4 = 2(u_6 \sin u_6 \partial x_6 + u_6 \cos u_6 \partial y_6 + u_6 \sin u_6 y^6 \partial z)$.

It follows that $\phi Z_1 = Z_2 + Z_3$ and $\phi N = 1/2(Z_2 - Z_3)$, which implies that $\phi RadTM$ and $\phi ltr(TM)$ are distributions on M. On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\phi Z_4 = -Z_5$, $\phi Z_5 = Z_4$, which implies that D_1 is invariant with respect to ϕ and $D_2 = span\{Z_6, Z_7\}$ is a slant distribution with slant angle $\frac{\pi}{4}$. Hence M is a semi-slant 2-lightlike submanifold of R_2^{13}.

Example 2. Let $(R_2^{13}, \overline{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.
Suppose M is a submanifold of R_2^{13} given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = y^4 = u_4$, $x^4 = -y^3 = u_5$, $x^5 = u_6 \cos\theta$, $y^5 = u_7 \cos\theta$, $x^6 = u_7 \sin\theta$, $y^6 = u_6 \sin\theta$, $z = u_8$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where
$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z)$, $Z_2 = 2(\partial x_2 + y^2 \partial z)$, $Z_3 = 2\partial y_1$,
$Z_4 = 2(\partial x_3 + \partial y_4 + y^3 \partial z)$, $Z_5 = 2(\partial x_4 - \partial y_3 + y^4 \partial z)$,
$Z_6 = 2(\cos\theta \partial x_5 + \sin\theta \partial y_6 + y^5 \cos\theta \partial z)$,
$Z_7 = 2(\sin\theta \partial x_6 + \cos\theta \partial y_5 + y^6 \sin\theta \partial z)$, $Z_8 = V = 2\partial z$.

Hence $RadTM = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2 - y^1 \partial z$ and $S(TM^\perp)$ is spanned by $W_1 = 2(\partial x_3 - \partial y_4 + y^3 \partial z)$, $W_2 = 2(\partial x_4 + \partial y_3 + y^4 \partial z)$,
$W_3 = 2(\sin\theta \partial x_5 - \cos\theta \partial y_6 + y^5 \sin\theta \partial z)$,
$W_4 = 2(\cos\theta \partial x_6 - \sin\theta \partial y_5 + y^6 \cos\theta \partial z)$.

It follows that $\phi Z_1 = Z_2 - Z_3$ and $\phi N = 1/2(Z_2 + Z_3)$, which implies that $\phi RadTM$ and $\phi ltr(TM)$ are distributions on M. On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\phi Z_4 = Z_5$, $\phi Z_5 = -Z_4$, which implies that D_1 is invariant with respect to ϕ and $D_2 = span\{Z_6, Z_7\}$ is a slant distribution with slant angle 2θ. Hence M is a semi-slant 2-lightlike submanifold of R_2^{13}.

Now, for any vector field X tangent to M, we put $\phi X = PX + FX$, where PX and FX are tangential and transversal parts of ϕX respectively. We denote the projections on $RadTM$, $\phi RadTM$, $\phi ltr(TM)$, D_1 and D_2 in TM by P_1, P_2, P_3, P_4, and P_5 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ by Q_1 and Q_2 respectively. Then, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X + \eta(X)V. \tag{3.2}$$

Now applying ϕ to (3.2), we have

$$\phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + \phi P_5X, \tag{3.3}$$

which gives

$$\phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + fP_5X + FP_5X, \tag{3.4}$$

where fP_5X (resp. FP_5X) denotes the tangential (resp. transversal) component of ϕP_5X. Thus we get $\phi P_1X \in \Gamma(\phi RadTM)$, $\phi P_2X \in \Gamma(RadTM)$, $\phi P_3X \in \Gamma(ltr(TM))$, $\phi P_4X \in \Gamma(D_1)$, $fP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(S(TM^{\perp}))$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.5}$$

Applying ϕ to (3.5), we obtain

$$\phi W = \phi Q_1W + \phi Q_2W, \tag{3.6}$$

which gives

$$\phi W = \phi Q_1W + BQ_2W + CQ_2W, \tag{3.7}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of ϕQ_2W. Thus we get $\phi Q_1W \in \Gamma(\phi ltr(TM))$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^{\perp}))$.

Now, by using (2.23), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $RadTM$, $\phi RadTM$, $\phi ltr(TM)$, D_1, D_2, $ltr(TM)$, $S(TM^{\perp})$ and $\{V\}$, we obtain

$$\begin{aligned} &P_1(\nabla_X\phi P_1Y) + P_1(\nabla_X\phi P_2Y) + P_1(\nabla_X\phi P_4Y) + P_1(\nabla_X fP_5Y) \\ &= P_1(A_{FP_5Y}X) + P_1(A_{\phi P_3Y}X) + \phi P_2\nabla_XY - \eta(Y)P_1X, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &P_2(\nabla_X\phi P_1Y) + P_2(\nabla_X\phi P_2Y) + P_2(\nabla_X\phi P_4Y) + P_2(\nabla_X fP_5Y) \\ &= P_2(A_{FP_5Y}X) + P_2(A_{\phi P_3Y}X) + \phi P_1\nabla_XY - \eta(Y)P_2X, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &P_3(\nabla_X\phi P_1Y) + P_3(\nabla_X\phi P_2Y) + P_3(\nabla_X\phi P_4Y) + P_3(\nabla_X fP_5Y) \\ &= P_3(A_{FP_5Y}X) + P_3(A_{\phi P_3Y}X) + \phi h^l(X,Y) - \eta(Y)P_3X, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &P_4(\nabla_X\phi P_1Y) + P_4(\nabla_X\phi P_2Y) + P_4(\nabla_X\phi P_4Y) + P_4(\nabla_X fP_5Y) \\ &= P_4(A_{FP_5Y}X) + P_4(A_{\phi P_3Y}X) + \phi P_4\nabla_XY - \eta(Y)P_4X, \end{aligned} \tag{3.11}$$

$$P_5(\nabla_X \phi P_1 Y) + P_5(\nabla_X \phi P_2 Y) + P_5(\nabla_X \phi P_4 Y) + P_5(\nabla_X f P_5 Y) = P_5(A_{FP_5Y}X) + P_5(A_{\phi P_3 Y}X) + fP_5 \nabla_X Y + Bh^s(X,Y) - \eta(Y)P_5 X, \tag{3.12}$$

$$h^l(X,\phi P_1 Y) + h^l(X,\phi P_2 Y) + h^l(X,\phi P_4 Y) + h^l(X, fP_5 Y) = \phi P_3 \nabla_X Y - \nabla^l_X \phi P_3 Y - D^l(X, FP_5 Y), \tag{3.13}$$

$$h^s(X,\phi P_1 Y) + h^s(X,\phi P_2 Y) + h^s(X,\phi P_4 Y) + h^s(X, fP_5 Y) = Ch^s(X,Y) - \nabla^s_X FP_5 Y - D^s(X,\phi P_3 Y) + FP_5 \nabla_X Y, \tag{3.14}$$

$$\eta(\nabla_X \phi P_1 Y) + \eta(\nabla_X \phi P_2 Y) + \eta(\nabla_X \phi P_4 Y) + \eta(\nabla_X f P_5 Y) = \eta(A_{\phi P_3 Y}X) + \eta(A_{FP_5 Y}X) + \overline{g}(X,Y)V. \tag{3.15}$$

Theorem 3.3 *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M. Then M is a semi-slant lightlike submanifold if and only if*

(i) *$\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$,*

(ii) *the distribution D_1 is an invariant distribution, i.e. $\phi D_1 = D_1$,*

(iii) *there exists a constant $\lambda \in [0,1)$ such that $P^2 X = -\lambda X$.*

Moreover, there also exists a constant $\mu \in (0,1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then distribution D_1 is invariant with respect to ϕ and $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$.

Now for any $X \in \Gamma(D_2)$ we have $|PX| = |\phi X|\cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\phi X|}. \tag{3.16}$$

In view of (3.16), we get $\cos^2\theta = \frac{|PX|^2}{|\phi X|^2} = \frac{g(PX,PX)}{g(\phi X,\phi X)} = \frac{g(X,P^2X)}{g(X,\phi^2 X)}$, which gives

$$g(X,P^2X) = \cos^2\theta\, g(X,\phi^2 X). \tag{3.17}$$

Since M is semi-slant lightlike submanifold, $\cos^2\theta = \lambda(constant) \in [0,1)$ and therefore from (3.17), we get $g(X,P^2X) = \lambda g(X,\phi^2X) = g(X,\lambda\phi^2 X)$, which implies

$$g(X,(P^2 - \lambda\phi^2)X) = 0. \tag{3.18}$$

Since $(P^2 - \lambda\phi^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2\times D_2}$ is non-degenerate(positive definite), from (3.18), we have $(P^2 - \lambda\phi^2)X = 0$, which implies

$$P^2X = \lambda\phi^2 X = -\lambda X. \tag{3.19}$$

Now, for any vector field $X \in \Gamma(D_2)$, we have

$$\phi X = PX + FX, \tag{3.20}$$

where PX and FX are tangential and transversal parts of ϕX respectively. Applying ϕ to (3.20) and taking tangential component, we get

$$-X = P^2X + BFX. \tag{3.21}$$

From (3.19) and (3.21), we get

$$BFX = -\mu X, \tag{3.22}$$

where $1 - \lambda = \mu(constant) \in (0, 1]$.

This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (3.21), for any $X \in \Gamma(D_2)$, we get

$$-X = P^2X - \mu X, \tag{3.23}$$

which implies

$$P^2X = -\lambda X, \tag{3.24}$$

where $1 - \mu = \lambda(constant) \in [0, 1)$.

Now $\cos\theta = \frac{g(\phi X, PX)}{|\phi X||PX|} = -\frac{g(X, \phi PX)}{|\phi X||PX|} = -\frac{g(X, P^2X)}{|\phi X||PX|} = -\lambda\frac{g(X, \phi^2 X)}{|\phi X||PX|} = \lambda\frac{g(\phi X, \phi X)}{|\phi X||PX|}$.
From above equation, we get

$$\cos\theta = \lambda\frac{|\phi X|}{|PX|}. \tag{3.25}$$

Therefore (3.16) and (3.25) give $\cos^2\theta = \lambda(constant)$.
Hence M is a semi-slant lightlike submanifold. □

Corollary 3.1 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with slant angle θ, then for any $X, Y \in \Gamma(D_2)$, we have*

(i) $g(PX, PY) = \cos^2\theta\,(g(X, Y) - \eta(X)\eta(Y))$,

(ii) $g(FX, FY) = \sin^2\theta\,(g(X, Y) - \eta(X)\eta(Y))$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [3].

Lemma 3.4 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then for any $X, Y \in \Gamma(TM - \{V\})$, we have*

(i) $g(\nabla_X Y, V) = \overline{g}(Y, \phi X)$,

(ii) $g([X, Y], V) = 2\overline{g}(X, \phi Y)$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Since $\overline{\nabla}$ is a metric connection, from (2.7) and (2.24), for any $X, Y \in \Gamma(TM - \{V\})$, we have

$$g(\nabla_X Y, V) = \overline{g}(Y, \phi X). \tag{3.26}$$

From (2.22) and (3.26), for any $X, Y \in \Gamma(TM - \{V\})$, we have

$$g([X, Y], V) = 2\overline{g}(X, \phi Y). \tag{3.27}$$

□

Theorem 3.5 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $RadTM$ is integrable if and only if*

(i) $P_1(\nabla_X \phi P_1 Y) = P_1(\nabla_Y \phi P_1 X)$, $P_4(\nabla_X \phi P_1 Y) = P_4(\nabla_Y \phi P_1 X)$ *and* $P_5(\nabla_X \phi P_1 Y) = P_5(\nabla_Y \phi P_1 X)$,

(ii) $h^l(Y, \phi P_1 X) = h^l(X, \phi P_1 Y)$ *and* $h^s(Y, \phi P_1 X) = h^s(X, \phi P_1 Y)$, *for all* $X, Y \in \Gamma(RadTM)$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(RadTM)$. From (3.8), we have $P_1(\nabla_X \phi P_1 Y) = \phi P_2 \nabla_X Y$, which gives $P_1(\nabla_X \phi P_1 Y) - P_1(\nabla_Y \phi P_1 X) = \phi P_2[X, Y]$. From (3.11), we get $P_4(\nabla_X \phi P_1 Y) = \phi P_4 \nabla_X Y$, which gives $P_4(\nabla_X \phi P_1 Y) - P_4(\nabla_Y \phi P_1 X) = \phi P_4[X, Y]$. From (3.12), we have $P_5(\nabla_X \phi P_1 Y) = f P_5 \nabla_X Y + B h^s(X, Y)$, which gives $P_5(\nabla_X \phi P_1 Y) - P_5(\nabla_Y \phi P_1 X) = f P_5[X, Y]$. In view of (3.13), we obtain $h^l(X, \phi P_1 Y) = \phi P_3 \nabla_X Y$, which implies $h^l(X, \phi P_1 Y) - h^l(Y, \phi P_1 X) = \phi P_3[X, Y]$. Also from (3.14), we get $h^s(X, \phi P_1 Y) = C h^s(X, Y) + F P_5 \nabla_X Y$, which gives $h^s(X, \phi P_1 Y) - h^s(Y, \phi P_1 X) = F P_5[X, Y]$. This proves the theorem. □

Theorem 3.6 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_1 \oplus \{V\}$ is integrable if and only if*

(i) $P_1(\nabla_X \phi P_4 Y) = P_1(\nabla_Y \phi P_4 X)$, $P_2(\nabla_X \phi P_4 Y) = P_2(\nabla_Y \phi P_4 X)$ *and* $P_5(\nabla_X \phi P_4 Y) = P_5(\nabla_Y \phi P_4 X)$,

(ii) $h^l(Y, \phi P_4 X) = h^l(X, \phi P_4 Y)$ *and* $h^s(Y, \phi P_4 X) = h^s(X, \phi P_4 Y)$, *for all* $X, Y \in \Gamma(D_1 \oplus \{V\})$.

Proof. Let M be semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1 \oplus \{V\})$. From (3.8), we have $P_1(\nabla_X \phi P_4 Y) = \phi P_2 \nabla_X Y$, which gives $P_1(\nabla_X \phi P_4 Y) - P_1(\nabla_Y \phi P_4 X) = \phi P_2[X, Y]$. From (3.9), we get $P_2(\nabla_X \phi P_4 Y) = \phi P_1 \nabla_X Y$, which gives $P_2(\nabla_X \phi P_4 Y) - P_2(\nabla_Y \phi P_4 X) = \phi P_1[X, Y]$. From (3.12), we have $P_5(\nabla_X \phi P_4 Y) = f P_5 \nabla_X Y + B h^s(X, Y)$, which gives $P_5(\nabla_X \phi P_4 Y) - P_5(\nabla_Y \phi P_4 X) = f P_5[X, Y]$. In view of (3.13), we obtain

$h^l(X, \phi P_4 Y) = \phi P_3 \nabla_X Y$, which implies $h^l(X, \phi P_4 Y) - h^l(Y, \phi P_4 X) = \phi P_3[X, Y]$. Also from (3.14), we get $h^s(X, \phi P_4 Y) = Ch^s(X, Y) + FP_5 \nabla_X Y$, which gives $h^s(X, \phi P_4 Y) - h^s(Y, \phi P_4 X) = FP_5[X, Y]$. This concludes the theorem. □

Theorem 3.7 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ is integrable if and only if*

(i) $P_1(\nabla_X fP_5 Y - \nabla_Y fP_5 X) = P_1(A_{FP_5 Y} X - A_{FP_5 X} Y)$,

(ii) $P_2(\nabla_X fP_5 Y - \nabla_Y fP_5 X) = P_2(A_{FP_5 Y} X - A_{FP_5 X} Y)$,

(iii) $P_4(\nabla_X fP_5 Y - \nabla_Y fP_5 X) = P_4(A_{FP_5 Y} X - A_{FP_5 X} Y)$,

(iv) $h^l(X, fP_5 Y) - h^l(Y, fP_5 X) = D^l(Y, FP_5 X) - D^l(X, FP_5 Y)$, *for all $X, Y \in \Gamma(D_2 \oplus \{V\})$.*

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2 \oplus \{V\})$. From (3.8), we have $P_1(\nabla_X fP_5 Y) - P_1(A_{FP_5 Y} X) = \phi P_2 \nabla_X Y$, which gives $P_1(\nabla_X fP_5 Y - \nabla_Y fP_5 X) - P_1(A_{FP_5 Y} X - A_{FP_5 X} Y) = \phi P_2[X, Y]$. From (3.9), we get $P_2(\nabla_X fP_5 Y) - P_2(A_{FP_5 Y} X) = \phi P_1 \nabla_X Y$, which gives $P_2(\nabla_X fP_5 Y - \nabla_Y fP_5 X) - P_2(A_{FP_5 Y} X - A_{FP_5 X} Y) = \phi P_1[X, Y]$. In view of (3.11), we obtain $P_4(\nabla_X fP_5 Y) - P_4(A_{FP_5 Y} X) = \phi P_4 \nabla_X Y$, which implies
$P_4(\nabla_X fP_5 Y - \nabla_Y fP_5 X) - P_4(A_{FP_5 Y} X - A_{FP_5 X} Y) = \phi P_4[X, Y]$. Also from (3.13), we get $h^l(X, fP_5 Y) + D^l(X, FP_5 Y) = \phi P_3 \nabla_X Y$, which gives $h^l(X, fP_5 Y) - h^l(Y, fP_5 X) + D^l(X, FP_5 Y) - D^l(Y, FP_5 X) = \phi P_3[X, Y]$. Thus, we obtain the required results. □

Theorem 3.8 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field V tangent to M. Then induced connection ∇ is not a metric connection.*

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Suppose that the induced connection is a metric connection. Then $\nabla_X \phi P_2 Y \in \Gamma(RadTM)$ and $h^l(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$. Thus from (2.23), for any $Z \in \Gamma(\phi RadTM)$ and $W \in \Gamma(\phi ltr(TM))$, we have

$$\overline{\nabla}_W \phi Z - \phi \overline{\nabla}_W Z = \overline{g}(Z, W)V. \tag{3.28}$$

From (2.7), (3.28) and taking tangential components, we obtain

$$\begin{aligned} \nabla_W \phi Z - \phi P_1 \nabla_W Z - \phi P_2 \nabla_W Z - \phi P_4 \nabla_W Z \\ = fP_5 \nabla_W Z + Bh^s(Z, W) + \overline{g}(Z, W)V. \end{aligned} \tag{3.29}$$

Since $TM = RadTM \oplus_{orth} (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$, from (3.29), we get

$$\nabla_W \phi Z - \phi P_2 \nabla_W Z = 0, \ \phi P_1 \nabla_W Z = 0, \ \phi P_4 \nabla_W Z = 0, \tag{3.30}$$

$$(3.31)\qquad fP_5\nabla_W Z - Bh^s(Z,W) = 0, \qquad \overline{g}(Z,W)V = 0.$$

Now taking $W = \phi N$ and $Z = \phi\xi$ in (3.31), we get $\overline{g}(N,\xi)V = 0$. Thus $V = 0$, which is a contradiction. Hence M does not have a metric connection. □

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a semi-slant lightlike submanifold of an indefinite Sasakian manifold to be totally geodesic.

Definition 4.1. A semi-slant lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X,Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is mixed geodesic semi-slant lightlike submanifold if $h^l(X,Y) = 0$ and $h^s(X,Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $RadTM$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X\phi P_2 Z + \nabla_X\phi P_4 Z + \nabla_X fP_5 Z, \phi Y) = \overline{g}(A_{\phi P_3 Z}X + A_{FP_5 Z}X, \phi Y)$, for all $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.*

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. To prove $RadTM$ defines a totally geodesic foliation it is sufficient to show that $\nabla_X Y \in \Gamma(RadTM)$, for all $X, Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19), (2.23) and (3.4), for any $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X(\phi P_2 Z + \phi P_3 Z + \phi P_4 Z + fP_5 Z + FP_5 Z), \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{\phi P_3 Z}X + A_{FP_5 Z}X - \nabla_X\phi P_2 Z - \nabla_X\phi P_4 Z - \nabla_X fP_5 Z, \phi Y)$. Thus, the theorem is completed. □

Theorem 4.2 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_1 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

(i) $\overline{g}(A_{FZ}X, \phi Y) = \overline{g}(\nabla_X fZ, \phi Y)$,

(ii) *$A_{\phi W}X$ and $\nabla_X\phi N$ have no component in $D_1 \oplus \{V\}$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$, $Z \in \Gamma(D_2)$, $W \in \Gamma(\phi ltr(TM))$ and $N \in \Gamma(ltr(TM))$.*

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. The distribution $D_1 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1 \oplus \{V\})$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$. Since $\overline{\nabla}$ is metric connection, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X\phi Z, \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{FZ}X - \nabla_X fZ, \phi Y)$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X\phi N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \nabla_X\phi N)$. Now, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$

and $W \in \Gamma(\phi ltr(TM))$, we have $\overline{g}(\nabla_X Y, W) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi W)$, which gives $\overline{g}(\nabla_X Y, W) = \overline{g}(\phi Y, A_{\phi W} X)$. Thus, we obtain the required results. □

Theorem 4.3 *Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X \phi Z, fY) = -\overline{g}(h^s(X, \phi Z), FY)$,

(ii) $\overline{g}(fY, \nabla_X \phi N) = -\overline{g}(FY, h^s(X, \phi N))$,

(iii) $\overline{g}(fY, A_{\phi W} X) = \overline{g}(FY, D^s(X, \phi W))$,
for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(\phi ltr(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. The distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2 \oplus \{V\})$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$. Since $\overline{\nabla}$ is metric connection, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Z, \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \phi Z, fY) - \overline{g}(h^s(X, \phi Z), FY)$. From (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we get $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which gives $\overline{g}(\nabla_X Y, N) = -\overline{g}(fY, \nabla_X \phi N) - \overline{g}(FY, h^s(X, \phi N))$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $W \in \Gamma(\phi ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, W) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi W)$, which implies $\overline{g}(\nabla_X Y, W) = \overline{g}(fY, A_{\phi W} X) - \overline{g}(FY, D^s(X, \phi W))$. This concludes the theorem. □

Theorem 4.4 *Let M be a mixed geodesic semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

(i) *$\nabla_X \phi Z$ has no component in $D_2 \oplus \{V\}$,*

(ii) $\overline{g}(fY, \nabla_X \phi N) = -\overline{g}(FY, h^s(X, \phi N))$,

(iii) $\overline{g}(fY, A_{\phi W} X) = \overline{g}(FY, D^s(X, \phi W))$,
for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(\phi ltr(TM))$.

Proof. Let M be a mixed geodesic semi-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. The distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2 \oplus \{V\})$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$. Since $\overline{\nabla}$ is metric connection, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Z, \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \phi Z, fY) - \overline{g}(h^s(X, \phi Z), FY)$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(fY, \nabla_X \phi N) - \overline{g}(FY, h^s(X, \phi N))$. Now, from (2.7), (2.19) and

(2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $W \in \Gamma(\phi ltr(TM))$, we have $\overline{g}(\nabla_X Y, W) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi W)$, which gives $\overline{g}(\nabla_X Y, W) = \overline{g}(fY, A_{\phi W} X) - \overline{g}(FY, D^s(X, \phi W))$. This concludes the theorem. □

References

[1] C. L. Bejan and K. L. Duggal, *Global Lightlike Manifolds and Harmonicity*, Kodai Math. J., **28**(2005), 131-145.

[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203, Birkhauser Boston, Inc., Boston, MA, 2002.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Slant Submanifolds in Sasakian manifolds*, Glasgow Math J., **42**(2000), 125-138.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Semi-slant Submanifolds of a Sasakian manifold*, Geometriae Dedicata, **78**(1999), 183-199.

[5] B. Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit, Leuven, 1990.

[6] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Vol. 364 of Mathematics and its applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[7] K. L. Duggal and B. Sahin, *Differential Geomety of Lightlike Submanifolds*, Birkhauser Verlag AG, Basel, Boston, Berlin, 2010.

[8] K. L. Duggal and B. Sahin, *Lightlike Submanifolds of Indefinite Sasakian Manifolds*, Int. J. of Math. Math. Sci., Article ID 57585, 2007.

[9] D. L. Johnson and L. B. Whitt, *Totally Geodesic Foliations*, J. Differential Geometry, **15**(1980), 225-235.

[10] A. Lotta, *Slant Submanifolds in Contact geometry*, Bull. Math. Soc. Roumanie, **39**(1996), 183-198.

[11] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press New York, 1983.

[12] B. Sahin, *Screen Slant Lightlike Submanifolds*, Int. Electronic J. of Geometry, **2**(2009), 41-54.

[13] B. Sahin and C. Yildirim, *Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds*, Faculty of Sci. and Math., Univ. of Nis, Serbia, **2**(2012), 277-287.

[14] S. S. Shukla and A. Yadav, *Lightlike submanifolds of indefinite para-Sasakian manifolds*, Matematicki Vesnik, **66(4)**(2014), 371-386.

[15] S. S. Shukla and A. Yadav, *Radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds*, Demonstratio Math., **47(4)**(2014), 994-1011.