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Some Inclusion Properties of New Subclass of Starlike and Convex Functions associated with Hohlov Operator

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Abstract. For a sufficiently adequate special case of the Dziok-Srivastava linear operator defined by means of the Hadamard product (or convolution) with Srivastava-Wright convolution operator, the authors investigate several mapping properties involving various subclasses of analytic and univalent functions, $G(\lambda, \alpha)$ and $M(\lambda, \alpha)$. Furthermore we discuss some inclusion relations for these subclasses to be in the classes of k-uniformly convex and k-starlike functions.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathbb{U}. \tag{1.1}$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ the subclass

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of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n > 0. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \le \alpha < 1)$, if and only if $\Re\left(zf'(z)/f(z)\right) > \alpha \quad (z \in \mathbb{U})$. This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \le \alpha < 1)$ if and only if $\Re\left(1 + zf''(z)/f'(z)\right) > \alpha \quad (z \in \mathbb{U})$. This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

Furthermore, we denote by $k - \mathcal{UCV}$ and $k - \mathcal{ST}$, $(0 \le k < \infty)$, two interesting subclasses of $\mathcal{S}$ consisting respectively of functions which are k-uniformly convex and k-starlike in $\mathbb{U}$. Namely, we have for $0 \le k < \infty$

$$k - \mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathbb{U}) \right\}$$

and

$$k - \mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathbb{U}) \right\}.$$

The class $1 - \mathcal{UCV}$ was defined and discussed by Goodman [7]. Further the classes $k - \mathcal{UCV}$ and $k - \mathcal{ST}$ were introduced and its geometric definitions, connections with the conic domains were investigated in [11, 12].

The Gaussian hypergeometric function $F(a, b; c, z)$ given by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathbb{U}) \tag{1.3}$$

is the solution of the homogenous hypergeometric differential equation

$$z(1 - z)w''(z) + [c - (a + b + 1)z]w'(z) - abw(z) = 0$$

and has rich applications in various fields such as conformal mappings, quasi conformal theory, continued fractions, and so on. Here, a, b, c are complex numbers such that $c \ne 0, -1, -2, -3, \dots$, $(a)_0 = 1$ for $a \ne 0$, and for each positive integer n, $(a)_n = a(a + 1)(a + 2) \dots (a + n - 1)$ is the Pochhammer symbol. In the case of $c = -k$, $k = 0, 1, 2, \dots$, the function $F(a, b; c; z)$ is defined if $a = -j$ or $b = -j$ where $j \le k$. We refer to [2, 14] and references therein for some important results.

Also for functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.4}$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear operator involving the generalized hypergeometric function, was introduced and studied systematically by Dziok and Srivastava [6, 5] and (subsequently) by many other authors. Here, in our present investigation, we recall a familiar convolution operator $I_{a,b,c}$ due to Hohlov [8], which indeed is a very specialized case of the widely- (and extensively-) investigated Dziok-Srivastava operator. For $f \in \mathcal{A}$, we recall the operator $I_{a,b,c}(f)$ of which maps $\mathcal{A}$ into itself defined by means of Hadamard product as

$$I_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z). \tag{1.5}$$

Therefore, for a function f defined by (1.1), we have

$$I_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n. \tag{1.6}$$

The Hohlov operator $I_{a,b,c}$ (which has been emphasized upon in this paper) is a very specialized case of the Dziok-Srivastava linear operator [5, 6] which, in turn, is contained in the Srivastava-Wright convolution operator [15] (see also [9]). It is the Srivastava-Wright convolution operator [15] (see also [9]) that is defined by using the Fox-Wright generalized hypergeometric function.

Using the integral representation,

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{\mathrm{d}t}{(1-tz)^a}, \Re(c) > \Re(b) > 0,$$

we can write

$$I_{a,b,c}(f)(z) = \left(\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} \mathrm{d}t \right) * \frac{z}{(1-tz)^a}.$$

When $f(z)$ equals the convex function $z/(1-z)$, then the operator $I_{a,b,c}(f)$ in this case becomes $zF(a,b;c;z)$. If $a = 1,\ b = 1+\delta,\ c = 2+\delta$ with $\Re(\delta) > -1$, then the convolution operator $I_{a,b,c}(f)$ turns into Bernardi operator

$$B_f(z) = I_{a,b,c}(f)(z) = \frac{1+\delta}{z^\delta} \int_0^1 t^{\delta-1} f(t)\mathrm{d}t.$$

Indeed, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively.

Let us denote (see [11], [12])

$$P_1(k) = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)} & \text{for} \quad 0 \le k < 1, \\ 8/\pi^2 & \text{for} \quad k = 1, \\ \frac{\pi^2}{4\sqrt{t}(1+t)(k^2-1)K^2(t)} & \text{for} \quad k > 1, \end{cases} \tag{1.7}$$

where $t \in (0,1)$ is determined by $k = \cosh(\pi K'(t)/[4K(t)])$, K is the Legendre's complete Elliptic integral of the first kind

$$K(t) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $K'(t) = K(\sqrt{1-t^2})$ is the complementary integral of $K(t)$. Let Ω_k be a domain such that $1 \in \Omega_k$ and

$$\partial\Omega_k = \left\{ w = u + iv : \quad u^2 = k^2(u-1)^2 + k^2v^2 \right\}, \quad 0 \le k < \infty.$$

The domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic when $k = 1$, and a right half-plane when $k = 0$. If p is an analytic function with $p(0) = 1$ which maps the unit disc $\mathbb{U}$ conformally onto the region Ω_k, then $P_1(k) = p'(0)$. $P_1(k)$ is strictly decreasing function of the variable k and it values are included in the interval $(0, 2]$.

Let $f \in \mathcal{A}$ be of the form (1.1). If $f \in k - \mathcal{UCV}$, then the following coefficient inequalities hold true (cf. [11]):

$$|a_n| \le \frac{(P_1(k))_{n-1}}{n!}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{1.8}$$

Similarly, if f of the form (1.1) belongs to the class $k - \mathcal{ST}$, then (cf., [12])

$$|a_n| \le \frac{(P_1(k))_{n-1}}{(n-1)!}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{1.9}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$, $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \le B < A \le 1)$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathcal{R}^\tau(A,B)$ was introduced earlier by Dixit and Pal [4]. Two of the many interesting subclasses of the class $\mathcal{R}^\tau(A,B)$ are worthy of mention here. First of all, by setting

$$\tau = e^{i\eta}\ \cos\eta\ (-\pi/2 < \eta < \pi/2),\ A = 1 - 2\beta\ (0 \le \beta < 1) \text{ and } B = -1,$$

the class $\mathcal{R}^\tau(A,B)$ reduces essentially to the class $\mathcal{R}_\eta(\beta)$ introduced and studied by Ponnusamy and Rønning [14], where

$$\mathcal{R}_\eta(\beta) = \left\{ f \in \mathcal{A} :\ \Re(e^{i\eta}(f'(z) - \beta)) > 0 \quad z \in \mathbb{U} \right\}.$$

Secondly, if we put

$$\tau = 1,\ A = \beta \text{ and } \ B = -\beta\ (0 < \beta \le 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta, \quad z \in \mathbb{U}$$

which was studied by (among others) Padmanabhan [13] and Caplinger and Causey [3].

Motivated by the earlier work of Srivastava et al. [16], in this paper we introduce two new subclasses of $\mathcal{S}$ namely $G(\lambda, \alpha)$ and $M(\lambda, \alpha)$ to obtain coefficient bounds and to discuss some inclusion properties involving Hohlov operator.

For some $\alpha (0 \le \alpha < 1)$ and $\lambda (0 \le \lambda < 1)$, we let $G(\lambda, \alpha)$ and $M(\lambda, \alpha)$ be two new subclass of $\mathcal{S}$ consisting of functions of the form (1.1) satisfying the analytic criteria

$$G(\lambda, \alpha) := \left\{ f \in \mathcal{S} : \Re\left(\frac{(1-\lambda)f(z)}{zf'(z)} + \lambda\right) > \alpha, \ z \in \mathbb{U} \right\}, \tag{1.10}$$

$$M(\lambda, \alpha) := \left\{ f \in \mathcal{S} : \Re\left(\frac{f'(z)+\lambda z f''(z)}{f'(z)+zf''(z)}\right) > \alpha, \ z \in \mathbb{U} \right\}. \tag{1.11}$$

Also denote $G^*(\lambda, \alpha) = G(\lambda, \alpha) \cap \mathcal{T}$ and $M^*(\lambda, \alpha) = M(\lambda, \alpha) \cap \mathcal{T}$, the subclasses of $\mathcal{T}$ defined in (1.2).

2. Coefficient Bounds

In his section, we obtain the necessary and sufficient conditions for functions $f \in G(\lambda, \alpha)$ and $f \in M(\lambda, \alpha)$.

Lemma 2.1. *A function $f \in \mathcal{A}$ of the form* (1.1) *belongs to the class $G(\lambda, \alpha)$ if $f(z)/(zf'(z)) \in \mathcal{H}$ and if*

$$\sum_{n=2}^{\infty} (1 + n\lambda - \lambda - \alpha n)|a_n| \le 1 - \alpha. \tag{2.3}$$

Proof. It is suffices to show that the values for $\frac{(1-\lambda)f(z)}{zf'(z)} + \lambda$ lie in a circle centered at $\omega = 1$ whose radius is $1 - \alpha$. We have

$$\begin{aligned}\left|\frac{(1-\lambda)f(z)}{zf'(z)} + (\lambda - 1)\right| &= \left|\frac{\sum_{n=2}^{\infty}(1-\lambda+n\lambda-n)a_n z^n}{z+\sum_{n=2}^{\infty} n a_n z^n}\right| \\ &\le \frac{\sum_{n=2}^{\infty}(1-\lambda+n\lambda-n)|a_n||z^{n-1}|}{1-\sum_{n=2}^{\infty} n|a_n||z^{n-1}|} \\ &\le \frac{\sum_{n=2}^{\infty}(1-\lambda+n\lambda-n)|a_n|}{1-\sum_{n=2}^{\infty} n|a_n|}. \end{aligned} \tag{2.4}$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda-n)|a_n| \le (1-\alpha)(1-\sum_{n=2}^{\infty} n|a_n|), \tag{2.5}$$

which is equivalent to

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda-\alpha n)|a_n| \le 1-\alpha. \tag{2.6}$$

But (2.6) is true by hypothesis. Hence

$$\left|\frac{(1-\lambda)f(z)}{zf'(z)}+(\lambda-1)\right| \le 1-\alpha \tag{2.7}$$

and the theorem is proved. □

Corollary 2.2. *A function $f \in \mathcal{A}$ of the form* (1.1) *belongs to the class $M(\lambda,\alpha)$ if $f(z)/(zf'(z)) \in \mathcal{H}$ and if*

$$\sum_{n=2}^{\infty} n(1+n\lambda-\lambda-\alpha n)|a_n| \le 1-\alpha.$$

Proof. It is well known that $f \in M(\lambda,\alpha)$ if and only if $zf^{'} \in G(\lambda,\alpha)$. Since $zf^{'} = z+\sum_{n=2}^{\infty} na_n z^n$ we may replace a_n with na_n in Lemma 2.1. For functions in $\mathcal{T}$ the converse of Lemma 2.1 is also true. □

Lemma 2.3. *A function $f \in \mathcal{T}$ belongs to the class $G^*(\lambda,\alpha)$ if and only if $f(z)/(zf'(z)) \in \mathcal{H}$ and if*

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda-\alpha n)|a_n| \le 1-\alpha.$$

Proof. In view of Lemma(2.1), it suffices to show the only if part. Assume that

$$\Re\left(\frac{(1-\lambda)f(z)}{zf'(z)}+\lambda\right) = \Re\left\{\frac{z-\sum_{n=2}^{\infty}((1-\lambda)+n\lambda)a_n z^{n-1}}{z-\sum_{n=2}^{\infty} na_n z^{n-1}}\right\} > \alpha, \quad (|z|<1). \tag{2.8}$$

Choose values of z on the real axis so that $\left(\frac{(1-\lambda)f(z)}{zf'(z)}+\lambda\right)$ is real. Upon clearing the denominator in (2.8) and letting $z \to 1$ through real values, we obtain

$$1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)|a_n| \ge \alpha(1-\sum_{n=2}^{\infty} n|a_n|). \tag{2.9}$$

Thus $\sum_{n=2}^{\infty}(1+n\lambda-\lambda-\alpha n)|a_n| \le 1-\alpha$, and the proof is complete. □

Corollary 2.4. *A function $f \in \mathcal{T}$ of the form* (1.1) *belongs to the class $M^*(\lambda, \alpha)$ if and only if $f(z)/(zf'(z)) \in \mathcal{H}$ and*

$$\sum_{n=2}^{\infty} n(1 + n\lambda - \lambda - \alpha n)|a_n| \leq 1 - \alpha.$$

3. Inclusion Properties

Making use of the following lemma, we will study the action of the hypergeometric function on the classes $k - \mathcal{UCV}$, $k - \mathcal{ST}$.

Lemma 3.5. [4] *If $f \in \mathcal{R}^\tau(A, B)$ is of form* (1.1)*, then*

$$|a_n| \leq (A - B)\frac{|\tau|}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{3.4}$$

The result is sharp.

Theorem 3.6. *Let $a, b \in \mathbb{C} \setminus \{0\}$, $|a| \neq 1$, $|b| \neq 1$. Also, let c be a real number such that $c > |a| + |b| + 1$. If $f \in \mathcal{R}^\tau(A, B)$, $I_{a,\,b,\,c}(f)/(zI'_{a,\,b,\,c}(f)) \in \mathcal{H}$ and if the inequality*

$$\frac{\Gamma(c)\Gamma(c - |a| - |b| - 1)}{\Gamma(c - |a|)\Gamma(c - |b|)}\left[(\lambda - \alpha)(c - |a| - |b| - 1) + \frac{(1 - \lambda)}{(|a| - 1)(|b| - 1)}\right]$$

$$\leq (1 - \alpha)\left(\frac{1}{(A - B)|\tau|} + 1\right) + (1 - \lambda)\frac{c - 1}{(|a| - 1)(|b| - 1)} \tag{3.5}$$

is satisfied, then $I_{a,\,b,\,c}(f) \in G(\lambda, \alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\mathcal{R}^\tau(A, B)$. By virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty}(1 + n\lambda - \lambda - n\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq 1 - \alpha.$$

Taking into account the inequality (3.4) and the relation $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda-n\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|$$
$$\leq (A-B)|\tau|(\lambda-\alpha)\sum_{n=2}^{\infty}\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|$$
$$+(A-B)|\tau|(1-\lambda)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}}$$
$$\leq (A-B)|\tau|$$
$$\left\{(\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}+\frac{(c-1)(1-\lambda)}{(|a|-1)(|b|-1)}\sum_{n=2}^{\infty}\frac{(|a|-1)_n(|b|-1)_n}{(c-1)_n(1)_n}\right\}$$
$$=(\lambda-\alpha)(A-B)|\tau|\left(F(|a|,|b|,c;1)-1\right)$$
$$+(A-B)|\tau|(1-\lambda)\frac{(c-1)}{(|a|-1)(|b|-1)}$$
$$\left(F(|a|-1,|b|-1,c-1;1)-\frac{(|a|-1)(|b|-1)}{c-1}-1\right),$$

where we use the relation

$$(a)_n = a(a+1)_{n-1}. \tag{3.6}$$

The proof now follows by an application of Gauss summation theorem and (3.5). □

Theorem 3.7. *Let $a,b\in\mathbb{C}\setminus\{0\}$, $|a|\neq 1$, $|b|\neq 1$. Also, let c be a real number such that $c>|a|+|b|+2$. If $f\in\mathcal{S}$, $I_{a,b,c}(f)/(zI'_{a,b,c}(f))\in\mathcal{H}$ and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$\left[1-\alpha+\frac{(\lambda-\alpha)(a)_2(b)_2}{(1-\alpha)(c-|a|-|b|-2)_2}+\frac{|ab|(2\lambda-3\alpha+1)}{c-|a|-|b|-1}(c-|a|-|b|-1)\right]$$
$$\leq 2(1-\alpha)$$

is satisfied, then $I_{a,b,c}(f)\in G(\lambda,\alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\mathcal{S}$. By virtue of Lemma 2.1, it suffices to show that

$$S(a,b,c,\lambda,\alpha):=\sum_{n=2}^{\infty}(1+n\lambda-\lambda-n\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\leq 1-\alpha.$$

Applying the well known estimate for the coefficients of the functions $f \in \mathcal{S}$, due to de Branges [1], we need to show that

$$\sum_{n=2}^{\infty} n(1+n\lambda-\lambda-n\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq 1-\alpha. \tag{3.7}$$

Taking into account the inequality $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$S(a,b,c,\lambda,\alpha) \leq \sum_{n=2}^{\infty}\left(n^2(\lambda-\alpha)+n(1-\lambda)\right)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

writing $n=(n-1)+1$, and $n^2=(n-1)(n-2)+3(n-1)+1$, we can rewrite the above term as

$$\begin{aligned} S(a,b,c,\lambda,\alpha) &\leq (\lambda-\alpha)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &+ (2\lambda-3\alpha+1)\sum_{n=2}^{\infty}(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &+ (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \end{aligned}$$

Repeatedly using the relation given in (3.6),

$$\begin{aligned} S(a,b,c,\lambda,\alpha) &\leq (\lambda-\alpha)\sum_{n=3}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} \\ &+ (2\lambda-3\alpha+1)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} \\ &+ (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \end{aligned}$$

The inequality (3.7) now follows by applying Gauss summation theorem and by the hypothesis. □

Theorem 3.8. *Let $a,b \in \mathbb{C}\setminus\{0\}$. Also, let c be a real number and $P_1 = P_1(k)$ be given by (1.7). If $f \in k-\mathcal{UCV}$, for some $k\,(0 \leq k < \infty)$, and the inequality*

$$(\lambda-\alpha)\,{}_3F_2(|a|,|b|,P_1;c,1;1)+(1-\lambda)\;{}_3F_2(|a|,|b|,P_1;c,2;1) \leq 2(1-\alpha) \tag{3.8}$$

is satisfied, then $I_{a,b,c}(f) \in G(\lambda,\alpha)$.

Proof. Let f be given by (1.1). By (2.3), to show $I_{a,b,c}(f) \in M(\lambda, \alpha)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda-n\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq 1-\alpha. \tag{3.9}$$

We will repeat the method of proving used in the proof of Theorem 1. Applying the estimates for the coefficients given by (1.8), and making use of the relations (3.6) and $|(a)_n| \leq (|a|)_n$, we get

$$\begin{aligned} & \sum_{n=2}^{\infty}(1+n\lambda-\lambda-n\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\ \leq & \sum_{n=2}^{\infty}[n(\lambda-\alpha)+(1-\lambda)]\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_n} \\ = & (\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} + (1-\lambda)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_n} \\ = & (\lambda-\alpha)\left[{}_3F_2(|a|,|b|,P_1;c,1;1)-1\right] + (1-\lambda)\ \left[{}_3F_2(|a|,|b|,P_1;c,2;1)-1\right] \\ \leq & 1-\alpha \end{aligned}$$

provided the condition (3.8) is satisfied. □

Theorem 3.9. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + 1$. If $f \in \mathcal{R}^{\tau}(A, B)$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(\lambda-\alpha)|ab|+(1-\alpha)(c-|a|-|b|-1)\right]$$

$$\leq (1-\alpha)\left(\frac{1}{(A-B)|\tau|}+1\right) \tag{3.10}$$

is satisfied, then $I_{a,b,c}(f) \in M(\lambda, \alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\mathcal{R}^{\tau}(A, B)$. By virtue of Lemma 3.6, it suffices to show that

$$\sum_{n=2}^{\infty} n(1+n\lambda-\lambda-\alpha n)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq 1-\alpha. \tag{3.11}$$

Taking into account the inequality (3.4) and the relation $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$\sum_{n=2}^{\infty} n(1+n\lambda-\lambda-\alpha n)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|$$
$$\leq (A-B)|\tau|(\lambda-\alpha)\sum_{n=2}^{\infty} n\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| + (A-B)|\tau|(1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_n}$$
$$\leq (A-B)|\tau|\left\{(\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right\}$$
$$= (A-B)|\tau|\left\{(\lambda-\alpha)\frac{|ab|}{c}F(1+|a|,1+|b|,1+c;1) + (1-\alpha)\left(F(|a|,|b|,c;1)-1\right)\right\}$$

$$= (A-B)|\tau|\left\{(\lambda-\alpha)\frac{|ab|}{c}\frac{\Gamma(c-a-b-1)\Gamma(c+1)}{\Gamma(c-a)\Gamma(c-a)}\right.$$
$$\left. +(1-\alpha)\left\{\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}-1\right\}\right\}$$
$$= (A-B)|\tau|\frac{\Gamma(c-a-b-1)\Gamma(c)}{\Gamma(c-a)\Gamma(c-a)}\left\{(\lambda-\alpha)ab+(1-\alpha)(c-a-b-1)\right\}$$
$$-(A-B)|\tau|(1-\alpha)$$
$$= (A-B)|\tau|\left\{(1-\alpha)\left\{\frac{1}{(A-B)|\tau|}+1\right\}\right\} - (A-B)|\tau|(1-\alpha)$$
$$\leq (1-\alpha)$$

provided the condition (3.10) is satisfied. □

Theorem 3.10. *Let $a,b\in\mathbb{C}\setminus\{0\}$. Also, let c be a real number and $P_1 = P_1(k)$ be given by* (1.7)*. If, for some k $(0\leq k<\infty)$, $f\in k-\mathcal{UCV}$, and the inequality*

$$\begin{aligned}
&(\lambda-\alpha)\frac{|ab|P_1}{c}\ {}_3F_2(1+|a|,1+|b|,1+P_1;1+c,2;1)\\
+\ &(1-\alpha){}_3F_2(|a|,|b|,P_1;c,1;1)\\
\leq\ &2(1-\alpha)
\end{aligned}\tag{3.12}$$

is satisfied, then $I_{a,b,c}(f)\in M(\lambda,\alpha)$.

Proof. Let f be given by (1.1). By (2.3), to show $I_{a,b,c}(f)\in M(\lambda,\alpha)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(1+n\lambda-\lambda-\alpha n)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\leq 1-\alpha. \tag{3.13}$$

We will repeat the method of proving used in the proof of the first Theorem. Applying the estimates for the coefficients given by (1.8), and making use of the relations (3.6) and $|(a)_n| \leq (|a|)_n$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[(1+n\lambda-\lambda-\alpha n)] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\
\leq\ & \sum_{n=2}^{\infty} n[n(\lambda-\alpha)+(1-\lambda)]\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
=\ & (\lambda-\alpha)\sum_{n=2}^{\infty}\frac{|ab|P_1}{c}\frac{(1+|a|)_{n-2}(1+|b|)_{n-2}(1+P_1)_{n-2}}{(1+c)_{n-2}(1)_{n-2}(2)_{n-2}} \\
& +(1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
=\ & (\lambda-\alpha)\frac{|ab|P_1}{c}[{}_3F_2(1+|a|,1+|b|,1+P_1;1+c,2;1)] \\
& +(1-\alpha)\ [{}_3F_2(|a|,|b|,P_1;c,1;1)-1] \\
\leq\ & 1-\alpha
\end{aligned}$$

provided the condition (3.12) is satisfied. □

Theorem 3.11. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number and $P_1 = P_1(k)$ be given by* (1.7). *If $f \in k - \mathcal{ST}$, for some $k\,(0 \leq k < \infty)$, and the inequality*

$$\begin{aligned}
& (\lambda-\alpha)\frac{|ab|P_1}{c}{}_3F_2(1+|a|,1+|b|,1+P_1;1+c,1;1) \\
& +(1+\lambda-2\alpha)\frac{|ab|P_1}{c}{}_3F_2(1+|a|,1+|b|,1+P_1;1+c,2;1) \\
& +(1-\alpha){}_3F_2(|a|,|b|,P_1;c,1;1) \\
\leq\ & 2(1-\alpha).
\end{aligned}$$

is satisfied, then $I_{a,b,c}(f) \in M(\lambda, \alpha)$.

Proof. Let f be given by (1.1). We will repeat the method of proving used in the proof of Theorem 3.7. Applying the estimates for the coefficients given by (1.9), and making use of the relations (3.6) and $|(a)_n| \leq (|a|)_n$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+n\lambda-\lambda-\alpha n) \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\
\leq\ & \sum_{n=2}^{\infty} n[n(\lambda-\alpha)+(1-\lambda)]\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
=\ & \sum_{n=2}^{\infty}(n-1)[(n-1)(\lambda-\alpha)+(1-\alpha)]\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
& +\sum_{n=2}^{\infty}[(n-1)(\lambda-\alpha)+(1-\alpha)]\ \frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}
\end{aligned}$$

$$\begin{aligned}
&= (\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-2}(1)_{n-2}} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-2}}\\
&\quad +(\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-2}} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}\\
&= (\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-2}(1)_{n-2}} + (1+\lambda-2\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-2}}\\
&\quad +(1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}\\
&= (\lambda-\alpha)\ \frac{|ab|P_1}{c}\ {}_3F_2(1+|a|,1+|b|,1+P_1;1+c,1;1)\\
&\quad +(1+\lambda-2\alpha)\frac{|ab|P_1}{c}\ {}_3F_2(1+|a|,1+|b|,1+P_1;1+c,2;1)\\
&\quad +(1-\alpha)\left[\ {}_3F_2(|a|,|b|,P_1;c,1;1)-1\right]\\
&\le 1-\alpha
\end{aligned}$$

provided the hypothesis is satisfied. □

We state the following theorems without proof.

Theorem 3.12. (i) *If* $a, b > -1,\ c > 0$ *and* $ab < 0$*, then* $zF(a, b, c)$ *is in* $G(\lambda, \alpha)$ *if and only if* $c > a + b + 1 - \frac{(\lambda-\alpha)}{(1-\alpha)}ab$.

(ii) *If* $a, b > 0,\ c > a+b+1$*, then* $F_1(a, b, c; z) = z[2-F(a, b, c; z)]$ *is in* $G(\lambda, \alpha)$ *iff*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(\lambda-\alpha)ab}{(1-\alpha)(c-a-b-1)}\right]\le 2.$$

Theorem 3.13. (i) *If* $a, b > -1,\ c > 0$ *and* $ab < 0$*, then* $zF(a, b, c)$ *is in* $M(\lambda, \alpha)$ *if and only if* $(\lambda-\alpha)(a)_2(b)_2 + (1-4\alpha+3\lambda)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \ge 0$.

(ii) *If* $a,\ b > 0,\ c > a+b+2$*, then* $F_1(a, b, c; z) = z[2-F(a, b, c; z)]$ *is in* $M(\lambda, \alpha)$ *iff*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(\lambda-\alpha)(a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2}+\frac{1-4\alpha+3\lambda}{1-\alpha}\frac{ab}{c-a-b-1}\right]\le 2.$$

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