

## Areas associated with a Strictly Locally Convex Curve

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ABSTRACT. Archimedes showed that for a point  $P$  on a parabola  $X$  and a chord  $AB$  of  $X$  parallel to the tangent of  $X$  at  $P$ , the area  $S$  of the region bounded by the parabola  $X$  and chord  $AB$  is four thirds of the area  $T$  of triangle  $\triangle ABP$ . It is well known that the area  $U$  formed by three tangents to a parabola is half of the area  $T$  of the triangle formed by joining their points of contact. Recently, the first and third authors of the present paper and others proved that among strictly locally convex curves in the plane  $\mathbb{R}^2$ , these two properties are characteristic ones of parabolas.

In this article, in order to generalize the above mentioned property  $S = \frac{4}{3}T$  for parabolas

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As we study strictly locally convex curves in the plane  $\mathbb{R}^2$  satisfying  $S = \lambda T + \nu U$ , where  $\lambda$  and  $\nu$  are some functions on the curves. As a result, we present two conditions which are necessary and sufficient for a strictly locally convex curve in the plane to be an open arc of a parabola.

## 1. Introduction

Usually, a regular plane curve  $X : I \rightarrow \mathbb{R}^2$  defined on an open interval is called *convex* if, for all  $t \in I$ , the trace  $X(I)$  lies entirely on one side of the closed half-plane determined by the tangent line at  $X(t)$  ([6]). A regular plane curve  $X : I \rightarrow \mathbb{R}^2$  is called *locally convex* if, for each  $t \in I$  there exists an open subinterval  $J \subset I$  containing  $t$  such that the curve  $X|_J$  restricted to  $J$  is a convex curve.

Hereafter, we will say that a locally convex curve  $X$  in the plane  $\mathbb{R}^2$  is *strictly locally convex* if the curve is smooth (that is, of class  $C^{(3)}$ ) and is of positive curvature  $\kappa$  with respect to the unit normal  $N$  pointing to the convex side. Hence, in this case we have  $\kappa(s) = \langle X''(s), N(X(s)) \rangle > 0$ , where  $X(s)$  is an arc-length parametrization of  $X$ .

When  $f : I \rightarrow \mathbb{R}$  is a smooth function defined on an open interval, we will also say that  $f$  is *strictly convex* if the graph of  $f$  has positive curvature  $\kappa$  with respect to the upward unit normal  $N$ . This condition is equivalent to the positivity of  $f''(x)$  on  $I$ .

Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  with the unit normal  $N$  pointing to the convex side. For a fixed point  $P = A \in X$  and a sufficiently small number  $h > 0$ , we consider the line  $m$  passing through  $P + hN(P)$  which is parallel to the tangent  $\ell$  to  $X$  at  $P$  and the points  $A_1$  and  $A_2$  where the line  $m$  intersects the curve  $X$ .

Let us denote by  $\ell_1, \ell_2$  the tangent lines of  $X$  at the points  $A_1, A_2$  and by  $B, B_1, B_2$  the points of intersection  $\ell_1 \cap \ell_2, \ell \cap \ell_1, \ell \cap \ell_2$ , respectively. We let  $L_P(h), \ell_P(h)$  and  $H_P(h)$  denote the lengths  $|A_1A_2|$  and  $|B_1B_2|$  of the corresponding segments and the height of the triangle  $\triangle BA_1A_2$  from the vertex  $B$  to the edge  $A_1A_2$ , respectively.

We also consider  $T_P(h), U_P(h), V_P(h)$  and  $W_P(h)$  defined by the area  $|\triangle AA_1A_2|, |\triangle BB_1B_2|, |\triangle BA_1A_2|$  of corresponding triangles and the area  $|\square A_1A_2B_2B_1|$  of trapezoid  $\square A_1A_2B_2B_1$ , respectively. Then, obviously we have

$$T_P(h) = \frac{1}{2}hL_P(h)$$

and

$$U_P(h) = \frac{1}{2}\{H_P(h) - h\}\ell_P(h).$$

If we put  $S_P(h)$  the area of the region bounded by the curve  $X$  and chord  $A_1A_2$ , then we have ([18])

$$S'_P(h) = L_P(h).$$

It is well known that parabolas satisfy the following properties

**Proposition 1.1.** *Suppose that  $X$  denotes an open arc of a parabola. For an arbitrary point  $P \in X$  and a sufficiently small number  $h > 0$ , it satisfies*

$$(1.1) \quad S_P(h) = \frac{4}{3}T_P(h),$$

$$(1.2) \quad U_P(h) = \frac{1}{2}T_P(h)$$

and

$$(1.3) \quad S_P(h) = \frac{8}{3}U_P(h).$$

In fact, Archimedes showed that parabolas satisfy (1.1) ([25]).

Recently, in [18] the first and third authors of the present paper and others proved that (1.1) is a characteristic property of parabolas and established some characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([25]). For the higher dimensional analogues of some results in [18], see [16] and [17]. Some characterizations of hyperspheres, ellipsoids, elliptic hyperboloids, hypercylinders and  $W$ -curves in the  $(n+1)$ -dimensional Euclidean space  $\mathbb{E}^{n+1}$  were given in [1, 4, 7, 8, 13, 15, 22]. In [19], some characteristic properties for hyperbolic spaces embedded in the Minkowski space were established.

In [12], it was shown that (1.2) is also a characteristic property of parabolas, which gives an affirmative answer to Question 3 in [21]. Thus, we have

**Proposition 1.2.** *Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then  $X$  is an open arc of a parabola if and only if it satisfies one of the following conditions.*

1) *For all  $P \in X$  and sufficiently small  $h > 0$ ,*

$$(1.4) \quad S_P(h) = \lambda(P)T_P(h),$$

*where  $\lambda(P)$  is a function of  $P$ .*

2) *For all  $P \in X$  and sufficiently small  $h > 0$ ,*

$$(1.5) \quad U_P(h) = \eta(P)T_P(h),$$

*where  $\eta(P)$  is a function of  $P$ .* □

*Proof.* For a proof of 1), see Theorem 3 of [18].

For 2), we refer to Theorem 1.3 of [12].

In Proposition 1.2, obviously we have  $\lambda(P) = 4/3$  and  $\eta(P) = 1/2$ .

In this article, in order to generalize the property (1.4) for parabolas we study strictly locally convex  $C^{(3)}$  curves in the plane  $\mathbb{R}^2$ .

As a result, first of all in Section 3 we establish a characterization theorem for parabolas, which shows that (1.3) is a characteristic property of parabolas.

**Theorem 1.3.** *Let  $X$  denote a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then the following are equivalent.*

1) *There exists a function  $\nu(P)$  of  $P \in X$  such that for all  $P \in X$  and sufficiently small  $h > 0$  the curve  $X$  satisfies*

$$(1.6) \quad S_P(h) = \nu(P)U_P(h).$$

2)  *$X$  is an open arc of a parabola.*

In Theorem 1.3, obviously we have  $\nu(P) = 8/3$ .

Combining (1.4) and (1.6), it follows from (1.1) and (1.3) that for an arbitrary point  $P$  and a sufficiently small number  $h > 0$ , parabolas satisfy

$$(1.7) \quad S_P(h) = \lambda(P)T_P(h) + \nu(P)U_P(h),$$

whenever  $\lambda(P)$  and  $\nu(P)$  are functions of  $P$  with

$$(1.8) \quad \lambda(P) + \frac{1}{2}\nu(P) = \frac{4}{3}.$$

We now naturally raise a question as follows:

**Question 1.4.** *Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  satisfying (1.7) for some functions  $\lambda(P)$  and  $\nu(P)$  of  $P$  with (1.8). Then, is it an open arc of a parabola?*

In Section 4, we give a partial affirmative answer to Question 1.4 as follows:

**Theorem 1.5.** *Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then the following are equivalent.*

1) *There exist functions  $\lambda(P)$  and  $\nu(P)$  of  $P \in X$  with*

$$(1.9) \quad \nu = \nu(P) \in \mathbb{R} - (0, \frac{8}{27}]$$

*such that for all  $P \in X$  and sufficiently small  $h > 0$  the curve  $X$  satisfies*

$$S_P(h) = \lambda(P)T_P(h) + \nu(P)U_P(h).$$

2)  *$X$  is an open arc of a parabola.*

In Theorem 1.5,  $\lambda(P)$  and  $\nu(P)$  necessarily satisfy (1.8).

In [5], it was shown that parabolas satisfy  $T_P(h) = 2U_P(h)$  for all points  $P$  and  $h > 0$ . This property of parabolas was proved to be a characteristic one of

parabolas ([12, 21, 23]). For some characterizations of parabolas or conic sections by properties of tangent lines, see [9] and [20]. In [14], using curvature function  $\kappa$  and support function  $h$  of a plane curve, the first and third authors of the present paper gave a characterization of ellipses and hyperbolas centered at the origin.

Among the graphs of functions, in [2, 3] Á. Bényi et al. gave some characterizations of parabolas. B. Richmond and T. Richmond established a dozen characterizations of parabolas using elementary techniques ([24]). In their papers, a parabola means the graph of a quadratic polynomial in one variable.

Throughout this article, all curves are of class  $C^{(3)}$  and connected, unless otherwise mentioned.

**2. Preliminaries**

In order to prove Theorems 1.3 and 1.5 in Section 1, we need the following lemma.

**Lemma 2.1.** *Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  with the unit normal  $N$  pointing to the convex side. Then we have*

$$(2.1) \quad \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}},$$

$$(2.2) \quad \lim_{h \rightarrow 0} \frac{1}{h\sqrt{h}} S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}},$$

$$(2.3) \quad \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} \ell_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}},$$

$$(2.4) \quad \lim_{h \rightarrow 0} \frac{1}{h\sqrt{h}} T_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}$$

and

$$(2.5) \quad \lim_{h \rightarrow 0} \frac{1}{h\sqrt{h}} U_P(h) = \frac{\sqrt{2}}{2\sqrt{\kappa(P)}} \quad ,$$

where  $\kappa(P)$  is the curvature of  $X$  at  $P$  with respect to the unit normal  $N$ .

*Proof.* It follows from [18] that (2.1) and (2.2) hold. For proofs of (2.3) and (2.5), see [12]. For a proof of (2.4), we refer to [21]. □

Next, we need the following lemma.

**Lemma 2.2.** *Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  with the unit normal  $N$  pointing to the convex side. Then we have*

$$(2.6) \quad hL'_P(h) = L_P(h) - \ell_P(h),$$

where  $L'_P(h)$  means the derivative of  $L_P(h)$  with respect to  $h$ .

*Proof.* For a proof, see Lemma 2.3 in [12]. □

Let us recall that  $L_P(h)$ ,  $\ell_P(h)$  and  $H_P(h)$  denote the lengths  $|A_1A_2|$  and  $|B_1B_2|$  of the corresponding segments and the height of the triangle  $\triangle BA_1A_2$  from the vertex  $B$  to the edge  $A_1A_2$ , respectively. Then, we get

$$(2.7) \quad L_P(h) : \ell_P(h) = H_P(h) : H_P(h) - h$$

,which yields

$$(2.8) \quad H_P(h) = \frac{hL_P(h)}{L_P(h) - \ell_P(h)}.$$

Together with Lemma 2.2, it follows from (2.8) that the following holds:

$$(2.9) \quad H_P(h) = \frac{L_P(h)}{L'_P(h)}.$$

At last, we need a lemma which is useful in the proof of Theorems 1.3 and 1.5 in Section 1 ([11]). For reader's convenience, we give a brief proof here.

**Lemma 2.3.** *Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then, the height function  $H_P(h)$  satisfies*

$$(2.10) \quad \lim_{h \rightarrow 0} \frac{H_P(h)}{h} = 2.$$

*Proof.* It follows from (2.8) that

$$(2.11) \quad \begin{aligned} \lim_{h \rightarrow 0} \frac{H_P(h)}{h} &= \lim_{h \rightarrow 0} \frac{L_P(h)}{L_P(h) - \ell_P(h)} \\ &= \lim_{h \rightarrow 0} \left\{ 1 - \frac{\ell_P(h)}{L_P(h)} \right\}^{-1}. \end{aligned}$$

Thus, together with (2.11), (2.1) and (2.3) complete the proof of Lemma 2.3. □

### 3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

It is trivial to show that any open arcs of parabolas satisfy 1) in Theorem 1.3 with  $\nu(P) = 8/3$ .

Conversely, suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  which satisfies for all  $P \in X$  and sufficiently small  $h > 0$

$$S_P(h) = \nu(P)U_P(h),$$

where  $\nu = \nu(P)$  is a function of  $P \in X$ . Then, it follows from Lemma 2.1 that  $\nu = 8/3$ .

We fix an arbitrary point  $P \in X$ .

Since  $2U_P(h) = (H_P(h) - h)\ell_P(h)$ , (2.8) shows that

$$(3.1) \quad \begin{aligned} 2U_P(h) &= \frac{L_P(h)}{H_P(h)}\{H_P(h) - h\}^2 \\ &= L'_P(h)\{H_P(h) - h\}^2, \end{aligned}$$

where the second equality follows from (2.9). Hence we see that (1.6) becomes

$$(3.2) \quad S_P(h) = \frac{4}{3}L'_P(h)\{H_P(h) - h\}^2.$$

By differentiating (3.2) with respect to  $h$ , we get

$$(3.3) \quad L_P(h) = \frac{4}{3}\{L''_P(h)(H_P(h) - h)^2 + 2L'_P(h)(H_P(h) - h)(H'_P(h) - 1)\}.$$

On the other hand, by differentiating (2.9) with respect to  $h$ , we obtain

$$(3.4) \quad L''_P(h) = \frac{L'_P(h)^2}{L_P(h)}(1 - H'_P(h)).$$

Hence, using (2.9) and (3.4), (3.3) may be rewritten as

$$(3.5) \quad L_P(h) = \frac{4}{3} \frac{(H_P(h)^2 - h^2)(H'_P(h) - 1)}{H_P(h)^2} L_P(h).$$

Dividing the both sides of (3.3) by  $L_P(h)$ , we have

$$(3.6) \quad (H_P(h)^2 - h^2)(H'_P(h) - 1) = \frac{3}{4}H_P(h)^2.$$

Let us denote by  $y = f(x)$  the height function  $H_P(x)$  for sufficiently small  $x > 0$ . Then, it follows from (3.6) that the function  $y = f(x)$  satisfies

$$(3.7) \quad \frac{dy}{dx} = 1 + \frac{3y^2}{4(y^2 - x^2)},$$

which is a homogeneous differential equation.

If we put  $v = y/x$ , from (3.7) we get

$$(3.8) \quad x \frac{dv}{dx} = -\frac{g(v)}{4(v^2 - 1)},$$

where we denote

$$(3.9) \quad g(v) = 4v^3 - 7v^2 - 4v + 4.$$

It follows from Lemma 2.3 that

$$(3.10) \quad \lim_{x \rightarrow 0} v(x) = 2.$$

Note that

$$(3.11) \quad g(v) = (v-2)(4v^2 + v - 2) = 4(v-2)(v-\alpha)(v-\beta),$$

where

$$(3.12) \quad \alpha = \frac{-1 - \sqrt{33}}{8}, \quad \beta = \frac{-1 + \sqrt{33}}{8}.$$

We consider two cases as follows.

**Case 1.** Suppose that  $dv/dx = 0$  on an interval  $(0, \epsilon)$  for some  $\epsilon > 0$ .

Then  $v$  is constant and hence it follows from (3.10) that  $v = 2$ . This shows that

$$(3.13) \quad H_P(h) = 2h,$$

and hence using (2.9), (3.13) becomes  $2hL'_P(h) = L_P(h)$ . By integrating this equation we get

$$(3.14) \quad L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}.$$

Integrating (3.14) implies that

$$(3.15) \quad S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}h\sqrt{h}.$$

Together with (3.14), (3.15) shows that at the point  $P$  for sufficiently small  $h > 0$  the curve  $X$  satisfies

$$S_P(h) = \frac{4}{3}T_P(h).$$

**Case 2.** Suppose that  $dv/dx \neq 0$ .

In this case, the differential equation (3.8) becomes

$$(3.16) \quad \frac{v^2 - 1}{(v-2)(v-\alpha)(v-\beta)}dv + \frac{1}{x}dx = 0,$$



which may be rewritten as

$$(3.17) \quad \left\{ \frac{\gamma}{v-\alpha} + \frac{\delta}{v-\beta} + \frac{\eta}{v-2} \right\} dv + \frac{1}{x} dx = 0,$$

where we put

$$\gamma = \frac{11 - 3\sqrt{33}}{88}, \quad \delta = \frac{11 + 3\sqrt{33}}{88}, \quad \eta = \frac{3}{4}.$$

By integrating (3.17), we get

$$(3.18) \quad x|v - 2|^\eta |v - \alpha|^\gamma |v - \beta|^\delta = C,$$

where  $C$  is a constant of integration. By letting  $x \rightarrow 0$ , Lemma 2.3 yields that the left hand side of (3.18) tends to zero, which implies that the constant  $C$  must be zero. Hence  $v$  must be 2. This contradiction shows that this case cannot occur.

Combining Cases 1 and 2, we see that at the point  $P \in X$  for sufficiently small  $h > 0$  the curve  $X$  satisfies

$$S_P(h) = \frac{4}{3} T_P(h).$$

Since  $P \in X$  was arbitrary, Proposition 1.2 completes the proof of Theorem 1.3.

#### 4. Proof of Theorem 1.5

In this section, we prove Theorem 1.5.

It follows from Proposition 1.1 that any open arcs of parabolas satisfy 1) in Theorem 1.5.

Conversely, suppose that  $X$  is a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  which satisfies for all  $P \in X$  and sufficiently small  $h > 0$

$$S_P(h) = \lambda(P)T_P(h) + \nu(P)U_P(h),$$

where  $\lambda = \lambda(P)$  and  $\nu = \nu(P)$  are some functions of  $P \in X$  with

$$\nu = \nu(P) \in \mathbb{R} - \left(0, \frac{8}{27}\right]$$

Then, it follows from Lemma 2.1 that for all  $P \in X$  the curve  $X$  satisfies

$$\lambda(P) + \frac{1}{2}\nu(P) = \frac{4}{3}.$$

Now, we fix an arbitrary point  $P \in X$ .

From now on, we may assume that  $\nu = \nu(P) \neq 0$  because otherwise, at the point  $P$  the curve  $X$  satisfies (1.1) for all sufficiently small  $h > 0$ . Hence we assume that  $\nu < 0$  or  $\nu > 8/27$ .

Since  $2T_P(h) = hL_P(h)$  and  $U_P(h)$  is given by (3.1), from (1.7) we get

$$(4.1) \quad 2S_P(h) = \lambda(P)hL_P(h) + \nu(P)L'_P(h)\{H_P(h) - h\}^2.$$

By differentiating (4.1) with respect to  $h$ , we obtain

$$(4.2) \quad \begin{aligned} 2L_P(h) &= \lambda\{L_P(h) + hL'_P(h)\} \\ &+ \nu(H_P(h) - h)\{L''_P(h)(H_P(h) - h) + 2L'_P(h)(H'_P(h) - 1)\}. \end{aligned}$$

Together with (2.9), (3.4) shows that (4.2) may be rewritten as

$$(4.3) \quad \begin{aligned} 2L_P(h) &= \lambda \frac{L_P(h)}{H_P(h)} (H_P(h) + h) \\ &+ \nu \frac{L_P(h)}{H_P(h)^2} (H_P(h)^2 - h^2) (H'_P(h) - 1). \end{aligned}$$

Multiplying the both sides of (4.3) by  $H_P(h)^2/L_P(h)$ , we have

$$(4.4) \quad \lambda H_P(h)(H_P(h) + h) + \nu(H_P(h)^2 - h^2)(H'_P(h) - 1) = 2H_P(h)^2.$$

Or equivalently, we get

$$(4.5) \quad (2 - \lambda + \nu)H_P(h)^2 = \lambda h H_P(h) + \nu h^2 + \nu(H_P(h)^2 - h^2)H'_P(h).$$

Let us denote by  $y = f(x)$  the height function  $H_P(x)$  for sufficiently small  $x > 0$ . Then, it follows from (4.5) that the height function  $y = f(x)$  satisfies

$$(4.6) \quad \nu \frac{dy}{dx} = \frac{ay^2 - \lambda xy - \nu x^2}{y^2 - x^2},$$

where we put  $a = 2 - \lambda + \nu = 3\nu/2 + 2/3$ . By letting  $v = y/x$ , from (4.6) we obtain

$$(4.7) \quad \nu x \frac{dv}{dx} = -\frac{g(v)}{v^2 - 1},$$

where we denote

$$(4.8) \quad g(v) = \nu v^3 - av^2 + (\lambda - \nu)v + \nu.$$

We decompose  $g(v)$  as follows:

$$(4.9) \quad \begin{aligned} g(v) &= (v - 2)\left\{\nu v^2 + \left(\frac{\nu}{2} - \frac{2}{3}\right)v - \frac{\nu}{2}\right\} \\ &= \nu(v - 2)(v - \alpha)(v - \beta), \end{aligned}$$

where we put

$$(4.10) \quad \alpha, \beta = \frac{1}{2}\left\{-\left(\frac{1}{2} - \frac{2}{3\nu}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{2}{3\nu}\right)^2 + 2}\right\}$$

with  $\alpha < 0$  and  $\alpha < \beta$ . Since  $\nu \neq 8/27$ , the polynomial  $g(v)$  has distinct three roots  $2, \alpha$  and  $\beta$ .

We consider two cases as follows.

**Case 1.** Suppose that  $dv/dx = 0$  on an interval  $(0, \epsilon)$  for some  $\epsilon > 0$ .

Then  $v$  is constant and hence it follows from Lemma 2.3 that  $v = 2$ . As in the proof of Case 1 in Section 3, we may prove that at the point  $P$  the curve  $X$  satisfies (1.1) for sufficiently small  $h > 0$ .

**Case 2.** Suppose that  $dv/dx \neq 0$ .

Note that the polynomial  $g(v)$  has distinct three roots  $2, \alpha$  and  $\beta$ . The differential equation (4.7) becomes

$$(4.11) \quad \frac{1}{x}dx + \frac{v^2 - 1}{(v - 2)(v - \alpha)(v - \beta)}dv = 0.$$

Or equivalently, we get

$$(4.12) \quad \frac{1}{x}dx + \left\{ \frac{\gamma}{(v - \alpha)} + \frac{\delta}{(v - \beta)} + \frac{\eta}{v - 2} \right\}dv = 0,$$

where we put

$$(4.13) \quad \gamma = \frac{\alpha^2 - 1}{(\alpha - \beta)(\alpha - 2)}, \quad \delta = \frac{\beta^2 - 1}{(\beta - \alpha)(\beta - 2)}$$

and

$$(4.14) \quad \eta = \frac{3}{(\alpha - 2)(\beta - 2)}.$$

Integrating (4.12) yields

$$(4.15) \quad x|v - \alpha|^\gamma |v - \beta|^\delta |v - 2|^\eta = C,$$

where  $C$  is a constant of integration. It follows from  $\nu < 0$  or  $\nu > 8/27$  that  $\eta > 0$ . Hence, by letting  $x \rightarrow 0$ , the left hand side of (4.15) goes to zero, which implies that  $C$  is zero and hence  $v = 2$ . This contradiction shows that this case cannot occur.

Combining Cases 1 and 2 shows that at the point  $P \in X$  for sufficiently small  $h > 0$  the curve  $X$  satisfies

$$S_P(h) = \frac{4}{3}T_P(h).$$

Since  $P \in X$  was arbitrary, Proposition 1.2 completes the proof of Theorem 1.5.

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