KYUNGPOOK Math. J. 56(2016), 577-582
http://dx.doi.org/10.5666/KMJ.2016.56.2.577
pISSN 1225-6951 eISSN 0454-8124

Equivalence of $\mathbb{Z}_4$-actions on Handlebodies of Genus g

Jesse Prince-Lubawy
Department of Mathematics, University of North Alabama, Florence, Alabama
e-mail : jprincelubawy@una.edu

Abstract. In this paper we consider all orientation-preserving $\mathbb{Z}_4$-actions on 3-dimensional handlebodies V_g of genus $g > 0$. We study the graph of groups $(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$, which determines a handlebody orbifold $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v})) \simeq V_g/\mathbb{Z}_4$. This algebraic characterization is used to enumerate the total number of $\mathbb{Z}_4$ group actions on such handlebodies, up to equivalence.

1. Introduction

A $\mathbf{G}$-action on a handlebody V_g, of genus $g > 0$, is a group monomorphism $\phi : \mathbf{G} \longrightarrow \mathrm{Homeo}^+(V_g)$, where $\mathrm{Homeo}^+(V_g)$ denotes the group of orientation-preserving homeomorphisms of V_g. Two actions ϕ_1 and ϕ_2 on V_g are said to be equivalent if and only if there exists an orientation-preserving homeomorphism h of V_g such that $\phi_2(x) = h \circ \phi_1(x) \circ h^{-1}$ for all $x \in \mathbf{G}$. From [4], the action of any finite group $\mathbf{G}$ on V_g corresponds to a collection of graphs of groups. We may assume these particular graphs of groups are in canonical form and satisfy a set of normalized conditions, which can be found in [2].

Let $\mathrm{v} = (r, s, t, m, n)$ be an ordered 5-tuple of nonnegative integers. The graph of groups $(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ in canonical form, shown in Figure 1, determines a handlebody orbifold $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$. The orbifold $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ is constructed in a similar manner as described in [2]. Note that the quotient of any $\mathbb{Z}_4$-action on V_g is an orbifold of this type, up to homeomorphism.

An explicit combinatorial enumeration of orientation-preserving $\mathbb{Z}_p$-actions on V_g, up to equivalence, is given in [2]. In this work we will be interested in examining the orientation-preserving geometric group actions on V_g for the group $\mathbb{Z}_4$. The case for $\mathbb{Z}_{p^2}$, when p is an odd prime is considered in [5] and gives a different result. As we will see, there is exactly one equivalence class of Z_4-actions on the handlebody of genus 2. This result coincides with [3]. In this paper we will prove the following

Received February 25, 2015 ; accepted November 23, 2015.
2010 Mathematics Subject Classification: 57M60.
Key words and phrases: handlebodies, orbifolds, graph of groups, orientation-preserving, cyclic actions.

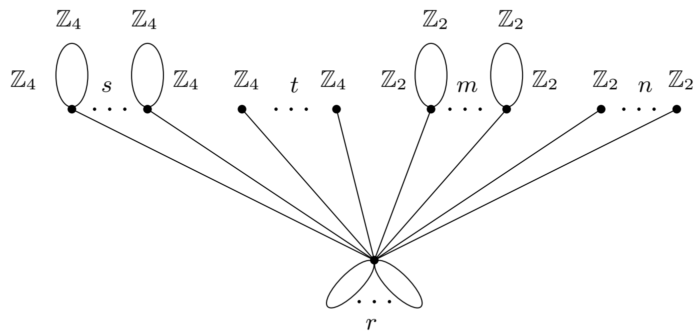

Figure 1: $(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$

main theorem:

Theorem 1.1 *If $\mathbb{Z}_4$ acts on V_g, where $g > 0$, then $V_g/\mathbb{Z}_4$ is homeomorphic to $V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))$ for some 5-tuple $\mathbf{v} = (r, s, t, m, n)$ of nonnegative integers with $r + s + t + m + n > 0$ and $g + 3 = 4(r + s + m) + 3t + 2n$. The number of equivalence classes of $\mathbb{Z}_4$-actions on V_g with this quotient type is m if $r + s + t = 0$, and $m + 1$ if $r + s + t > 0$.*

To illustrate the theorem, let $g = 3$. Then the genus equation becomes $6 = 4(r + s + m) + 3t + 2n$ so that $r + s + m$ must equal 0 or 1, and (r, s, t, m, n) is one of $(0, 0, 2, 0, 0)$, $(1, 0, 0, 0, 1)$, $(0, 1, 0, 0, 1)$, or $(0, 0, 0, 1, 1)$. Applying Theorem 1.1 to these four possibilities shows that there are a total of $1 + 1 + 1 + 1 = 4$ equivalence classes of orientation-preserving $\mathbb{Z}_4$-actions on V_3. Some results that follow directly from Theorem 1.1:

Corollary 1.2 *Every $\mathbb{Z}_4$-action on a handlebody of even genus must have an interval of fixed points and at least two fixed points on the boundary of the handlebody.*

Corollary 1.3 *Every $\mathbb{Z}_4$-action that is free on the boundary of the handlebody will have $t = n = 0$ and $g \equiv 1 (mod\ 4)$.*

2. The Main Theorem

The orbifold fundamental group of $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ is an extension of $\pi_1(V_g)$ by the group $\mathbf{G}$. We may view the fundamental group as a free product $G_1 * G_2 * G_3 * \cdots * G_{r+s+t+m+n}$, where G_i is isomorphic to either $\mathbb{Z}$, $\mathbb{Z}_4 \times \mathbb{Z}$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}$, or $\mathbb{Z}_2$. We establish notation similar to [2] and denote the generators of the orbifold fundamental group by $\{a_i : 1 \leq i \leq r\} \cup \{b_j, c_j : 1 \leq j \leq s\} \cup \{d_k : 1 \leq k \leq t\} \cup \{e_l, f_l : 1 \leq l \leq m\} \cup \{g_q : 1 \leq q \leq n\}$ such that $b_j^4 = d_k^4 = 1$, $[b_j, c_j] = 1$, $e_l^2 = g_q^2 = 1$, and $[e_l, f_l] = 1$.

Consider the set of pairs $((\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})), \lambda)$, where λ is a finite injective epimorphism from $\pi_1^{orb}(V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$ onto $\mathbb{Z}_4$. We say λ is finite injective since the kernel of λ is a free group of rank g. We consider only finite injective epimorphisms such that ker(λ)=im(ν_*) for some orbifold covering $\nu : V \longrightarrow V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v}))$. Since V is a handlebody with torsion free fundamental group, V is homeomorphic to a handlebody V_g of genus $g = 1 - 4\chi(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v}))$. Define an equivalence relation on this set of pairs by setting $((\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})),\lambda) \equiv ((\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})),\lambda')$ if and only if there exists an orbifold homeomorphism $h : V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})) \longrightarrow V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v}))$ such that $\lambda' = \lambda \circ h_*$. We define the set $\Delta(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$ to be the set of equivalence classes $[((\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})),\lambda)]$ under this relation.

Denote the set of equivalence classes $\mathscr{E}(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$ to be the set $\{[\phi] \mid \phi : \mathbb{Z}_4 \longrightarrow Homeo^+(V_g) \text{ and } V_g/\phi \simeq V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v}))\}$. Note that given any $\mathbb{Z}_4$-action $\phi : \mathbb{Z}_4 \longrightarrow Homeo^+(V_g)$, it must be the case that for some $V(\Gamma(\text{v}),\mathbf{G}(\mathbf{v}))$, $[\phi] \in \mathscr{E}(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$. The following proposition has a similar proof technique as found in [2].

Proposition 2.1 *Let* $\mathbf{v} = (r,s,t,m,n)$*. The set* $\mathscr{E}(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}))$ *is in one-to-one correspondence with the set* $\Delta(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$ *for every* $g > 0$*.*

To prove the main theorem, we count the number of elements in the delta set and use the one-to-one correspondence given in Proposition 2.1 to give the total count for the set $\mathscr{E}(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$. We resort to the following lemma to help count the number of elements in the delta set. The proof is an adaptation from [2].

Lemma 2.2 *If* α *is an automorphism of* $\pi_1^{orb}(V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$*, then* α *is realizable* [$\alpha = h_*$ *for some orientation-preserving homeomorphism* $h : V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})) \longrightarrow V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v}))$] *if and only if*

$$\begin{aligned}
\alpha(b_j) &= x_j b_{\sigma(j)}^{\varepsilon_j} x_j^{-1},\\
\alpha(c_j) &= x_j b_{\sigma(j)}^{v_j} c_{\sigma(j)}^{\varepsilon_j} x_j^{-1},\\
\alpha(d_k) &= y_k d_{\tau(k)}^{\delta_k} y_k^{-1},\\
\alpha(e_l) &= u_l e_{\gamma(l)}^{\varepsilon'_l} u_l^{-1},\\
\alpha(f_l) &= u_l e_{\gamma(l)}^{w_l} f_{\gamma(l)}^{\varepsilon'_l} u_l^{-1}, and\\
\alpha(g_q) &= z_q g_{\xi(q)}^{\delta'_q} z_q^{-1},
\end{aligned}$$

for some $x_j, y_k, u_l, z_q \in \pi_1^{orb}(V(\Gamma(\mathbf{v}),\mathbf{G}(\mathbf{v})))$*;* $\sigma \in \sum_s$*,* $\tau \in \sum_t$*,* $\gamma \in \sum_m$*,* $\xi \in \sum_n$*;* $\varepsilon_j, \delta_k, \varepsilon'_l, \delta'_q \in \{+1,-1\}$*; and* $0 \le v_j < 4$*,* $0 \le w_l < 2$*.*
Note that Σ_l *is the permutation group on* l *letters.*

Note that from [1], a generating set for the automorphisms of the handlebody orbifold fundamental group $\pi_1^{orb}(V(\Gamma(\text{v}),\mathbf{G}(\mathbf{v})))$ is the set of mappings $\{\rho_{ji}(x), \lambda_{ji}(x), \mu_{ji}(x), \omega_{ij}, \sigma_i, \phi_i\}$ whose definitions may be found in [1]. The first five maps are realizable. The realizable ϕ_i's are of the form found in Lemma 2.2 and will be used in the remaining arguments of this paper.

Lemma 2.3 *Let* $\mathbf{v} = (r, s, t, m, n)$ *with* $m > 0$ *and let*

$$\lambda_1, \lambda_2 : \pi_1^{orb}(V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))) \longrightarrow \mathbb{Z}_4$$

be two finite injective epimorphisms such that there exists a j *with* $\lambda_1(f_j)$ *being a generator of* $\mathbb{Z}_4$ *and* $\lambda_2(f_i)$ *is not a generator of* $\mathbb{Z}_4$ *for all* i. *Then* λ_1 *and* λ_2 *are not equivalent.*

Proof. Let $\lambda_1, \lambda_2 : \pi_1^{orb}(V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))) \longrightarrow \mathbb{Z}_4$ be two finite injective epimorphisms such that λ_1 sends f_j to a generator of $\mathbb{Z}_4$ for some j and λ_2 does not send f_i to a generator of $\mathbb{Z}_4$ for all i. We may assume that $\lambda_2(f_i) = 0$ for all i by composing λ_2 with the realizable automorphism $\prod \phi_i$, where ϕ_i sends the generator f_i to the element $e_i^{w_i} f_i$ and leaves all other generators fixed. Note that $w_i = 0$ if $\lambda_2(f_i) = 0$ and $w_i = 1$ if $\lambda_2(f_i) = 2$. To show that λ_1 and λ_2 are not equivalent we will consider the element f_j such that $\lambda_1(f_j)$ generates $\mathbb{Z}_4$. For contradiction, assume that λ_1 is equivalent to λ_2. Then by Lemma , there exists a realizable automorphism α such that $\alpha(f_j) = u e_m^w f_m^{\pm 1} u^{-1}$, where $u \in \pi_1^{orb}(V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v})))$ and $0 \le w < 2$. Hence $\lambda_1(f_j) = w\lambda_2(e_m)$, where $\lambda_2(e_m)$ is a multiple of 2, and hence $\lambda_1(f_j)$ is a multiple of 2. This is impossible since $\lambda_1(f_j)$ is a generator of $\mathbb{Z}_4$. Therefore λ_1 and λ_2 cannot be equivalent, proving the lemma. □

Lemma 2.4 *Let* $\mathbf{v} = (r, s, t, m, n)$ *and let* $\lambda : \pi_1^{orb}(V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))) \longrightarrow \mathbb{Z}_4$ *be a finite injective epimorphism. There exists a finite injective epimorphism* $\tilde{\lambda} : \pi_1^{orb}(V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))) \longrightarrow \mathbb{Z}_4$ *equivalent to* λ *such that the following hold:*

(1) $\tilde{\lambda}(a_1) = \cdots = \tilde{\lambda}(a_r) = 1$.

(2) $\tilde{\lambda}(b_1) = \cdots = \tilde{\lambda}(b_s) = 1$.

(3) $\tilde{\lambda}(c_i) = 0$ for all $1 \le i \le s$.

(4) $\tilde{\lambda}(d_1) = \cdots = \tilde{\lambda}(d_t) = 1$.

(5) $\tilde{\lambda}(e_1) = \cdots = \tilde{\lambda}(e_m) = 2$.

(6) $\tilde{\lambda}(f_i) = 1$ for all $i \le k$ some $0 \le k \le m$.

(7) $\tilde{\lambda}(f_i) = 0$ for all $k < i \le m$.

(8) $\tilde{\lambda}(g_1) = \cdots = \tilde{\lambda}(d_n) = 2$.

Proof. Let $\lambda : \pi_1^{orb}(V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))) \longrightarrow \mathbb{Z}_4$ be a finite injective epimorphism.

Properties (5) and (8) must occur since λ is finite injective.

Property (4) follows by composing λ with the realizable automorphism $\prod \phi_i$, where ϕ_i sends the generator d_i to the element $d_i^{\varepsilon_i}$ and leaves all other generators fixed. Note that $\varepsilon_i = 1$ if $\lambda(d_i) = 1$ and $\varepsilon_i = -1$ if $\lambda(d_i) = 3$.

Property (2) follows by a similar technique. Assuming property (2) holds, property (3) follows by composing λ with the realizable automorphism $\prod \phi_i$, where ϕ_i sends the generator c_i to the element $b_i^{-\lambda(c_i)} c_i$ and leaves all other generators fixed.

To show properties (6) and (7) hold, we may compose λ with the realizable automorphism $\prod \phi_i$, where ϕ_i sends the generator f_i to the element $e_i^{z_i} f_i$ and leaves all other generators fixed. Note that $z_i = 1$ if $\lambda(f_i) = 2$, $z_i = 2$ if $\lambda(f_i) = 0$ or $\lambda(f_i) = 1$, and $z_i = -1$ if $\lambda(f_i) = 3$. Furthermore, composing λ with the realizable automorphisms ω_{ij} we may interchange f_i as needed so that the first k generators map to 1 and the last $m-k$ generators map to 0.

Finally, to prove property (1) we may assume that there exists an element $x \in G_j$ (where G_j is either $\mathbb{Z}$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}$, or $\mathbb{Z}_2 \times \mathbb{Z}$) such that $\lambda(x) = 1$. Note that we may compose λ with a realizable automorphism that sends x to x^{-1} if needed. Now compose λ with the realizable automorphism $\prod \rho_{ji}(x^{-\lambda(a_i)+1})$. It may be shown that $(\lambda \circ \alpha)(a_i) = 1$ for all i. □

Proposition 2.5 *Let* $\mathbf{v} = (r, s, t, m, n)$ *with* $m > 0$ *and let*

$$\lambda, \lambda' : \pi_1^{orb}(V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))) \longrightarrow \mathbb{Z}_4$$

be two finite injective epimorphisms that satisfy the conclusion of Lemma 2.4, where $\lambda(f_i) = 1$ *for all* $1 \leq i \leq k$ *and* $\lambda'(f_i) = 1$ *for all* $1 \leq i \leq k'$. *Then* λ *is equivalent to* λ' *if and only if* $k = k'$.

Proof. For a contradiction, assume that λ is equivalent to λ' and $k \neq k'$. Without loss of generality we may assume that $k > k'$. Hence, λ maps at least one more generator f_i to 1 as does λ'. This would mean that there must exist a realizable automorphism α such that $(\lambda \circ \alpha)(f_{k'+1}) = 0$. By Lemma 2.2, this is impossible. Thus, $k = k'$. For the reverse implication suppose that $k = k'$. Then $\lambda = \lambda'$, proving the proposition. □

We will now prove the main theorem.

Proof. Define $\Delta_0(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ to be the set of equivalence classes $[(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})), \lambda]$ such that $\lambda(f_i) = 0$ for all $1 \leq i \leq m$. We now define $\Delta_1(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ to be the set of equivalence classes $[(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})), \lambda]$ such that $\lambda(f_i) = 1$ for at least one i such that $1 \leq i \leq m$. By Lemma 2.3, the delta set $\Delta(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ may be viewed as the disjoint union $\Delta_0(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))) \dot{\bigcup} \Delta_1(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$. Hence, the order of the delta set is the sum of the orders of the two sets $\Delta_0(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ and $\Delta_1(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$. Applying Lemma 2.4 and Proposition 2.5, we see that $|\Delta_1(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))| = m$ and $|\Delta_0(\mathbb{Z}_4, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))| = 1$. Hence by Proposition 2.1, the theorem follows. □

References

[1] D. I. Fuchs-Rabinovitch, *On the automorphism group of free products*, I, Mat. Sb., **8**(1940), 265-276.

[2] J. Kalliongis and A. Miller, *Equivalence and strong equivalence of actions on handlebodies*, Trans. Amer. Math. Soc., **308**(1988), 721–745.

[3] J. Kim, *Structures of geometric quotient orbifolds of three-dimensional g-manifolds of genus 2*, J. Korean Math. Soc., **46(4)**(2009), 859–893.

[4] D. McCullough, A. Miller, and B. Zimmerman, *Group actions on handlebodies*, Proc. London Math. Soc., **59(3)**(1989), 373–416.

[5] J. Prince-Lubawy, *Equivalence of cyclic p^2-actions on handlebodies*, In preparation.