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Jacobi Operators with Respect to the Reeb Vector Fields on Real Hypersurfaces in a Nonflat Complex Space Form

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Abstract. Let M be a real hypersurface of a complex space form with almost contact metric structure (ϕ, ξ, η, g). In this paper, we prove that if the structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the structure tensor ϕ, then M is a homogeneous real hypersurface of Type A provided that $\mathrm{Tr}R_\xi$ is constant.

1. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature $4c \neq 0$ is called a complex space form, which is denoted by $M_n(c)$. So naturally there exists a Kähler structure J and Kähler metric $\tilde{g}$ on $M_n(c)$. As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n(\mathbb{C})$, or complex hyperbolic space $H_n(\mathbb{C})$ as $c > 0$ or $c < 0$. Now let us consider a real hypersurface M in $M_n(c)$. Then we also denote by g the induced Riemannian metric of $M$and by N a local unit normal vector field of M in $M_n(c)$. Further, A denotes by the shape operator of M in $M_n(c)$. Then, an almost contact metric structure (ϕ, ξ, η, g) of M is naturally induced from the

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Kähler structure of $M_n(c)$ as follows:

$$\phi X = (JX)^T,\ \xi = -JN,\ \eta(X) = g(X, \xi), X \in TM,$$

where TM denotes the tangent bundle of M and $(\quad)^T$ the tangential component of a vector. The Reeb vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$. A real hypersurface is said to a *Hopf hypersurface* if the Reeb vector ξ of M is principal. Hopf hypersurfaces is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map (see Cecil and Ryan [2]). By making use of those results and the mentioned work of Takagi ([17], [18]), Kimura [11] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ whose all principal curvatures are constant. For the case $H_n\mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ $(0 \leq k \leq n-1)$ adding a horosphere in $H_n\mathbb{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [13](resp. Montiel and Romero [12]) showed that a real hypersurface in $P_n\mathbb{C}$ (resp. $H_n\mathbb{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the Reeb flow ξ is isometric or equivalently the structure operator ϕ commutes with the shape operator A.

The Reeb vector field ξ plays an important role in the theory of real hypersurfaces in a complex space form $M_n(c)$. Related to the Reeb vector field ξ the Jacobi operator R_ξ defined by $R_\xi = R(\cdot, \xi)\xi$ for the curvature tensor R on a real hypersurface M in $M_n(c)$ is said to be a *structure Jacobi operator* on M. The structure Jacobi operator has a fundamental role in contact geometry. In [3], Cho and first author started the study on real hypersurfaces in complex space form by using the operator R_ξ. In particular the structure Jacobi operator has been studied under the various commutative conditions ([4], [5], [7], [16]). For example, Pérez *et al.* [16] called that real hypersurfaces M has commuting structure Jacobi operator if $R_\xi R_X = R_X R_\xi$ for any vector field X on M, and proved that there exist no real hypersurfaces in $M_n(c)$ with commuting structure Jacobi operator. On the other hand Ortega *et al.* [14] have proved that there are no real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator R_ξ, that is, $\nabla_X R_\xi = 0$ for any vector field X on M. More generally, such a result has been extended by [15]. In this situation, if naturally leads us to be consider another condition weaker than parallelness. In the preceding work, we investigate the weaker condition ξ-parallelness, that is, $\nabla_\xi R_\xi = 0$ (cf. [4], [7], [8]). Moreover some works have studied several conditions on the structure Jacobi operator R_ξ ([3], [5], [7] and [8]). The following facts are used in this paper without proof.

Theorem 1.1. (Ki, Kim and Lim [5]) *Let M be a real hypersurface in a nonflat complex space form $M_n(c), c \neq 0$ which satisfies $R_\xi(A\phi - \phi A) = 0$. Then M is a Hopf hypersurface in $M_n(c)$. Further, M is locally congruent to one of the following hypersurfaces:*

(I) *In cases that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, \dots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(II) *In cases $M_n(c) = H_n\mathbb{C}$,*

(A_0) *a horosphere;*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;*

(A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, \dots, n-2\}$.*

Theorem 1.2. (Ki, Nagai and Takagi [9]) *Let M be a real hypersurface in a nonflat complex space form $M_n(c), c \neq 0$ If M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$. Then M is the same types as those in Theorem 1.1, where S denotes the Ricci tensor of M.*

In [6], the authors started the study on real hypersurfaces in a complex space form with $\phi\nabla_\xi\xi$-parallel structure Jacobi operator R_ξ, that is, $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ for the vector $\phi\nabla_\xi\xi$ orthogonal to ξ. In this paper we invetigate the structure Jacobi operator is $\phi\nabla_\xi\xi$-parallel under the condition that the structure Jacobi operator commute with the structure tensor ϕ. We prove that if the structure Jacobi operator R_ξ is $\phi\nabla_\xi\xi$-parallel and R_ξ commute with the structure tensor ϕ, then M is homogeneous real hypersurfaces of Type A provided that $\mathrm{Tr}R_\xi$ is constant.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c), c \neq 0$ with almost complex structure J, and N be a unit normal vector field on M. The Riemannian connection $\tilde\nabla$ in $M_n(c)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde\nabla_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde\nabla_X N = -AX$$

where g denotes the Riemannian metric of M induced from that of $M_n(c)$ and A denotes the shape operator of M in direction N. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

We call ξ the structure vector field (or the Reeb vector field) and its flow also denoted by the same latter ξ. The Reeb vector field ξ is said to be principal if $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$.

A real hypersurface M is said to be a Hopf hypersurface if the Reeb vector field ξ is principal. It is known that the aggregate (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi,\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1,\ \phi\xi = 0,\ \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M. From Kähler condition $\tilde{\nabla} J = 0$, and taking account of above equations, we see that

$$\nabla_X \xi = \phi AX, \tag{2.1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.2}$$

for any vector fields X and Y tangent to M.

Since we consider that the ambient space is of constant holomorphic sectional curvature $4c$, equations of Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X,Y)Z = \ & c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

In what follows, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$ and $h = \mathrm{Tr}A$, and for a function f we denote by ∇f the gradient vector field of f.

From the Gauss equation (2.3), the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2.5}$$

for any vector field X on M.

Now, we put

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where W is a unit vector field orthogonal to ξ. In the sequel, we put $U = \nabla_\xi \xi$, then by (2.1) we see that

$$U = \mu\phi W \tag{2.7}$$

and hence U is orthogonal to W. So we have $g(U,U) = \mu^2$. Using (2.7), it is clear that

$$\phi U = -A\xi + \alpha\xi, \tag{2.8}$$

which shows that $g(U,U) = \beta - \alpha^2$. Thus it is seen that

$$\mu^2 = \beta - \alpha^2. \tag{2.9}$$

Making use of (2.1), (2.7) and (2.8), it is verified that

$$\mu g(\nabla_X W, \xi) = g(AU, X), \tag{2.10}$$

$$g(\nabla_X \xi, U) = \mu g(AW, X) \tag{2.11}$$

because W is orthogonal to ξ.

Now, differentiating (2.8) covariantly and taking account of (2.1) and (2.2), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.12}$$

which together with (2.4) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{2.13}$$

Applying (2.12) by ϕ and making use of (2.11), we obtain

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu g(AW, X)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi, \tag{2.14}$$

which connected to (2.1), (2.9) and (2.13) gives

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.15}$$

Using (2.3), the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = R(X, \xi)\xi = c\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{2.16}$$

for any vector field X on M. Differentiating this covariantly along M, we find

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) =\ & g(\nabla_X(R_\xi Y) - R_\xi(\nabla_X Y), Z) \\ =\ & -c(\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)) \\ & +(X\alpha)g(AY, Z) + \alpha g((\nabla_X A)Y, Z) \\ & -\eta(AZ)\{g((\nabla_X A)\xi, Y) + g(A\phi AX, Y)\} \\ & -\eta(AY)\{g((\nabla_X A)\xi, Z) + g(A\phi AX, Z)\}. \end{aligned} \tag{2.17}$$

From (2.5) and (2.16), we have

$$\begin{aligned} (R_\xi S - SR_\xi)(X) =\ & -\eta(AX)A^3\xi + \eta(A^3X)A\xi - \eta(A^2X)(hA\xi - c\xi) \\ & +(h\eta(AX) - c\eta(X))A^2\xi - ch(\eta(AX)\xi - \eta(X)A\xi). \end{aligned} \tag{2.18}$$

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M; A\xi - \alpha\xi \neq 0\}.$$

At each point of Ω, the Reeb vector field ξ is not principal. That is, ξ is not an eigenvector of the shape operator A of M if $\Omega \neq \emptyset$.

In what follows we assume that Ω is not an empty set in order to prove our main theorem by reductio ad absurdum, unless otherwise stated, all discussion concerns the set Ω.

3. Real Hypersurfaces Satisfying $R_\xi \phi = \phi R_\xi$

Let M be a real hypersurface in $M_n(c), c \neq 0$. We suppose that $R_\xi \phi = \phi R_\xi$. Then by using (2.16) we have

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi. \tag{3.1}$$

Then, using (3.1), it is clear that $\alpha \neq 0$ on Ω. So a function λ given by $\beta = \alpha\lambda$ is defined. Because of (2.9), we have

$$\mu^2 = \alpha\lambda - \alpha^2. \tag{3.2}$$

Replacing X by U in (3.1) and taking account of (2.8), we find

$$\phi AU = \lambda A\xi - A^2\xi, \tag{3.3}$$

which implies

$$\phi A^2\xi = AU + \lambda U \tag{3.4}$$

because U is orthogonal to $A\xi$. From this and (2.6) we have

$$\mu\phi AW = AU + (\lambda - \alpha)U, \tag{3.5}$$

which together with (2.7) yields

$$g(AW, U) = 0. \tag{3.6}$$

Using (2.6) and (3.3), we can write (2.15) as

$$\nabla_\xi U = (3\lambda - 2\alpha)A\xi - 3\mu AW - \alpha\lambda\xi + \phi\nabla\alpha. \tag{3.7}$$

Since $\alpha \neq 0$ on Ω, (3.1) reformed as

$$(\phi A - A\phi)X = \eta(X)U + u(X)\xi + \tau(u(X)W + w(X)U), \tag{3.8}$$

where a 1-form u is defined by $u(X) = g(U, X)$ and w by $w(X) = g(W, X)$, where we put

$$\alpha\tau = \mu, \ \lambda - \alpha = \mu\tau. \tag{3.9}$$

Differentiating (3.8) covariantly and taking the inner product with any vector field Z, we find

$$
\begin{aligned}
&g(\phi(\nabla_Y A)X, Z) + g(\phi(\nabla_Y A)Z, X) \\
&= -\eta(AX)g(AY, Z) - g(AX, Y)\eta(AZ) \\
&\quad + g(A^2X, Y)\eta(Z) + \eta(X)g(A^2Y, Z) \\
&\quad + (\eta(X) + \tau w(X))g(\nabla_Y U, Z) \\
&\quad + g(\nabla_Y U, X)(\eta(Z) + \tau w(Z)) \\
&\quad + u(X)g(\nabla_Y \xi, Z) + g(\nabla_Y \xi, X)u(Z) \\
&\quad + (Y\tau)(u(X)w(Z) + u(Z)w(X)) \\
&\quad + \tau(u(X)g(\nabla_Y W, Z) + g(\nabla_Y W, X)u(Z))
\end{aligned}
\tag{3.10}
$$

because of (2.1) and (2.2). From this, taking the skew-symmetric part with respect to X and Y, and making use of the Codazzi equation (2.4), we find

(3.11)

$$
\begin{aligned}
&c(\eta(X)g(Y, Z) - \eta(Y)g(X, Z)) + g((\nabla_X A)\phi Y, Z) - g((\nabla_Y A)\phi X, Z) \\
&= -\eta(AX)g(AY, Z) + \eta(AY)g(AX, Z) + \eta(X)g(A^2Y, Z) - \eta(Y)g(A^2X, Z) \\
&\quad + (\eta(X) + \tau w(X))g(\nabla_Y U, Z) - (\eta(Y) + \tau w(Y))g(\nabla_X U, Z) \\
&\quad + (g(\nabla_Y U, X) - g(\nabla_X U, Y))(\eta(Z) + \tau w(Z)) \\
&\quad + u(X)g(\nabla_Y \xi, Z) - u(Y)g(\nabla_X \xi, Z) + (g(\nabla_Y \xi, X) - g(\nabla_X \xi, Y))u(Z) \\
&\quad + (Y\tau)(u(X)w(Z) + u(Z)w(X)) - (X\tau)(u(Y)w(Z) + u(Z)w(Y)) \\
&\quad + \tau\{u(X)g(\nabla_Y W, Z) - u(Y)g(\nabla_X W, Z)\} \\
&\quad + \tau\{(g(\nabla_Y W, X) - g(\nabla_X W, Y))u(Z)\}.
\end{aligned}
$$

Interchanging Y and Z in (3.10), we obtain

$$
\begin{aligned}
&g(\phi(\nabla_Z A)X, Y) + g(\phi(\nabla_Z A)Y, X) \\
&\quad = -\eta(AX)g(AY, Z) - g(AX, Z)\eta(AY) \\
&\quad\quad + g(A^2X, Z)\eta(Y) + \eta(X)g(A^2Y, Z) + (\eta(X) + \tau w(X))g(\nabla_Z U, Y) \\
&\quad\quad + g(\nabla_Z U, X)(\eta(Y) + \tau w(Y)) + u(X)g(\nabla_Z \xi, Y) + g(\nabla_Z \xi, X)u(Y) \\
&\quad\quad + (Z\tau)(u(X)w(Y) + u(Y)w(X)) + \tau(u(X)g(\nabla_Z W, Y) + g(\nabla_Z W, X)u(Y)),
\end{aligned}
$$

or, using (2.4)

$$
\begin{aligned}
&g(\phi(\nabla_X A)Z, Y) + g(\phi(\nabla_Y A)Z, X) + c(\eta(X)g(Z, Y) + \eta(Y)g(Z, X) - 2\eta(Z)g(X, Y)) \\
&= -\eta(AX)g(AY, Z) - g(AX, Z)\eta(AY) + g(A^2X, Z)\eta(Y) + \eta(X)g(A^2Y, Z) \\
&\quad + (\eta(X) + \tau w(X))g(\nabla_Z U, Y) + g(\nabla_Z U, X)(\eta(Y) + \tau w(Y)) \\
&\quad + u(X)g(\nabla_Z \xi, Y) + g(\nabla_Z \xi, X)u(Y) \\
&\quad + (Z\tau)(u(X)w(Y) + u(Y)w(X)) + \tau(u(X)g(\nabla_Z W, Y) + g(\nabla_Z W, X)u(Y)).
\end{aligned}
$$

Combining this to (3.11), we have

$$
\begin{aligned}
&2g((\nabla_Y A)\phi X, Z) + 2c(\eta(Z)g(X,Y) - \eta(X)g(Y,Z)) \\
&+2\eta(X)g(A^2Z,Y) - 2\eta(AX)g(AZ,Y) \\
&+(g(\nabla_Z U, X) - g(\nabla_X U, Z))(\eta(Y) + \tau w(Y)) \\
&+(g(\nabla_Y U, X) - g(\nabla_X U, Y))(\eta(Z) + \tau w(Z)) \\
&+(g(\nabla_Z U, Y) + g(\nabla_Y U, Z))(\eta(X) + \tau w(X)) \\
&+(g(\nabla_Z \xi, X) - g(\nabla_X \xi, Z))u(Y) + (g(\nabla_Y \xi, X) - g(\nabla_X \xi, Y))u(Z) \\
&+(g(\nabla_Z \xi, Y) + g(\nabla_Y \xi, Z))u(X) + (Y\tau)(u(X)w(Z) + u(Z)w(X)) \\
&+(Z\tau)(u(X)w(Y) + u(Y)w(X)) - (X\tau)(u(Y)w(Z) + u(Z)w(Y)) \\
&+\tau\{u(X)(g(\nabla_Z W, Y) + g(\nabla_Y W, Z)) \\
&+u(Z)(g(\nabla_X W, Y) - g(\nabla_Y W, X)) \\
&+u(Y)(g(\nabla_Z W, X) - g(\nabla_X W, Z))\} = 0.
\end{aligned} \tag{3.12}
$$

If we put $X = \xi$ in (3.12), then we have

$$
\begin{aligned}
&g(\nabla_Y U, Z) + g(\nabla_Z U, Y) + 2c(\eta(Z)\eta(Y) - g(Z,Y)) \\
&+2g(A^2Y, Z) - 2\alpha g(AY, Z) - du(\xi, Z)(\eta(Y) + \tau w(Y)) \\
&-du(\xi, Y)(\eta(Z) + \tau w(Z)) - 2u(Y)u(Z) \\
&-(\xi\tau)(u(Y)w(Z) + u(Z)w(Y)) \\
&-\tau\{u(Z)dw(\xi, Y) + u(Y)dw(\xi, Z)\} = 0,
\end{aligned} \tag{3.13}
$$

where d denotes the operator of the exterior derivative.

4. Real Hypersurfaces Satisfying $R_\xi\phi = \phi R_\xi$ and $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$

We will continue our discussions under the same hypothesis $R_\xi\phi = \phi R_\xi$ as in Section 3. Furthermore, suppose that $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ and then $\nabla_W R_\xi = 0$ since we assume that $\mu \neq 0$. Replacing X by W in (2.17), we find

$$
\begin{aligned}
&(W\alpha)g(AY, Z) - c(\eta(Z)g(\phi AW, Y) + \eta(Y)g(\phi AW, Z)) \\
&+\alpha g((\nabla_W A)Y, Z) - \eta(AZ)\{g((\nabla_W A)\xi, Y) + g(A\phi AW, Y)\} \\
&-\eta(AY)\{g((\nabla_W A)\xi, Z) + g(A\phi AW, Z)\} = 0
\end{aligned} \tag{4.1}
$$

by virtue of $\nabla_W R_\xi = 0$. Putting $Y = \xi$ in this and making use of (2.13) and (3.6), we obtain

$$
\alpha A\phi AW + c\phi AW = 0 \tag{4.2}
$$

because U and W are mutually orthogonal. From this and (2.16), it is seen that $R_\xi \phi AW = 0$ by virtue of (3.6), and hence $R_\xi AW = 0$ which together with (2.16) implies that

$$
\alpha A^2 W = -cAW + c\mu\xi + \mu(\alpha + g(AW, W))A\xi, \tag{4.3}
$$

which tells us that

$$\alpha g(A^2W,W) = (\mu^2 - c)g(AW,W) + \alpha\mu^2. \tag{4.4}$$

Since $\alpha \neq 0$, $\beta = \alpha\lambda$ and (3.2), we see that

$$g(A^2W,W) = \Big(\lambda - \alpha - \frac{c}{\alpha}\Big)\, g(AW,W) + \mu^2. \tag{4.5}$$

Combining (3.5) to (4.2), we get

$$\alpha A^2U = -(\mu^2 + c)AU - c(\lambda - \alpha)U. \tag{4.6}$$

If we apply μW to (3.3) and make use of (2.6), then we find

$$g(AU,U) = \mu^2(g(AW,W) + \alpha - \lambda). \tag{4.7}$$

Using (4.2), we see from (4.1)

$$\begin{aligned}\alpha(\nabla_W A)X =& -(W\alpha)AX + \eta(AX)(\nabla_W A)\xi + g((\nabla_W A)\xi, X)A\xi\\ &- \frac{c}{\alpha}\mu(w(X)\phi AW + g(\phi AW,X)W)\end{aligned}$$

for any vector field X, which together with (3.5) yields

$$\begin{aligned}\alpha(\nabla_W A)X =& -(W\alpha)AX + \eta(AX)(\nabla_W A)\xi + g((\nabla_W A)\xi, X)A\xi\\ &- \frac{c}{\alpha}\{w(X)AU + u(AX)W + (\lambda - \alpha)(w(X)U + u(X)W)\}.\end{aligned} \tag{4.8}$$

Now, if we put $X = W$ in (2.12), and make use of (3.5) and (4.2), then we find

$$(\nabla_W A)\xi = -\phi\nabla_W U + (W\alpha)\xi + \frac{1}{\mu}\Big(\alpha + \frac{c}{\alpha}\Big)\{AU + (\lambda - \alpha)U)\}. \tag{4.9}$$

Also, if we take the inner product (2.12) with $A\xi$ and take account of (2.6), (3.2) and (3.4), then we obtain

$$\alpha(X\alpha) + \mu(X\mu) = g(\alpha\xi + \mu W, (\nabla_X A)\xi) - g(A^2U + \lambda AU, X),$$

which together with (2.4), (2.13) and (4.6) yields

$$\mu(\nabla_W A)\xi = -\Big(\alpha + \frac{c}{\alpha}\Big)AU - \frac{c}{\alpha}(\lambda + \alpha)U + \mu\nabla\mu. \tag{4.10}$$

If we take the inner product (4.10) with ξ and make use of (2.13) and (3.6), then we find

$$W\alpha = \xi\mu. \tag{4.11}$$

Using (4.10), we can write (4.8) as

$$\begin{aligned}&\alpha(\nabla_W A)X+(W\alpha)AX\\&+\frac{1}{\mu}\eta(AX)\left\{\left(\alpha+\frac{c}{\alpha}\right)AU+\frac{c}{\alpha}(\lambda+\alpha)U-\mu\nabla\mu\right\}\\&+\frac{1}{\mu}\left\{\left(\alpha+\frac{c}{\alpha}\right)u(AX)+\frac{c}{\alpha}(\lambda+\alpha)u(X)-\mu(X\mu)\right\}A\xi\\&+\frac{c}{\alpha}\{w(X)AU+u(AX)W+(\lambda-\alpha)(w(X)U+u(X)W)\}=0.\end{aligned}\tag{4.12}$$

Putting $X=W$ in this, we get

$$\alpha(\nabla_W A)W+(W\alpha)AW-(W\mu)A\xi+\left(\alpha+\frac{2c}{\alpha}\right)AU+\frac{2c\lambda}{\alpha}U-\mu\nabla\mu=0.\tag{4.13}$$

Combining (4.9) to (4.10), we obtain

$$\mu\phi\nabla_W U-\mu(W\alpha)\xi+\mu\nabla\mu=2\left(\alpha+\frac{c}{\alpha}\right)AU+\left(\mu^2+\frac{2c}{\alpha}\lambda\right)U.$$

If we apply ϕ to this and make use of (2.8), (2.11) and (3.3), then we find

$$\begin{aligned}&-\mu\nabla_W U-\mu^2 g(AW,W)\xi+\mu\phi\nabla\mu\\&=2\left(\alpha+\frac{c}{\alpha}\right)(\lambda A\xi-A^2\xi)-\mu\left(\mu^2+\frac{2c}{\alpha}\lambda\right)W,\end{aligned}$$

which together with (2.6) yields

$$\begin{aligned}\mu\nabla_W U=\ &\mu\phi\nabla\mu+(2c-\mu^2)A\xi+2\mu\left(\alpha+\frac{c}{\alpha}\right)AW\\&-(\alpha\mu^2+2c\lambda+\mu^2 g(AW,W))\xi.\end{aligned}\tag{4.14}$$

Now, we can take a orthonormal frame field $\{e_0=\xi,\ e_1=W,\ e_2,\dots,e_n,\ e_{n+1}=\phi e_1=(1/\mu)U,\ e_{n+2}=\phi e_2,\dots,e_{2n}=\phi e_n\}$ of M. Differentiating (2.6) covariantly and making use of (2.1), we find

$$(\nabla_X A)\xi+A\phi AX=(X\alpha)\xi+\alpha\phi AX+(X\mu)W+\mu\nabla_X W,\tag{4.15}$$

which implies

$$\mu\mathrm{div}W=\mu\sum_{i=0}^{2n}g(\nabla_{e_i}W,e_i)=\xi h-\xi\alpha-W\mu.\tag{4.16}$$

Taking the inner product with Y to (4.15) and taking the skew-symmetric part, we have

$$\begin{aligned}&-2cg(\phi X,Y)+2g(A\phi AX,Y)\\&=(X\alpha)\eta(Y)-(Y\alpha)\eta(X)+\alpha g((\phi A+A\phi)X,Y)\\&\quad+(X\mu)w(Y)-(Y\mu)w(X)\\&\quad+\mu(g(\nabla_X W,Y)-g(\nabla_Y W,X)).\end{aligned}\tag{4.17}$$

Putting $X = \xi$ in this and using (2.10) and (4.11), we have

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (W\alpha)W. \tag{4.18}$$

Putting $X = \mu W$ in (4.15) and taking account of (4.10), we get

$$\begin{aligned}&-\left(\alpha + \frac{c}{\alpha}\right) AU - \frac{c}{\alpha}(\lambda+\alpha)U + \mu\nabla\mu + \mu A\phi AW \\ &= \mu(W\alpha)\xi + \mu(W\mu)W + \mu\alpha\phi AW + \mu^2\nabla_W W,\end{aligned}$$

or, using (3.5) and (4.2),

$$\mu^2\nabla_W W = -2\left(\alpha + \frac{c}{\alpha}\right) AU - \left(\mu^2 + \frac{2c}{\alpha}\lambda\right) U + \mu\nabla\mu - \mu(W\alpha)\xi - \mu(W\mu)W. \tag{4.19}$$

Now, putting $X = U$ in (4.17) and making use of (2.6) and (3.3), we have

$$\begin{aligned}&\mu(g(\nabla_U W, Y) - g(\nabla_Y W, U)) \\ &= (2c\mu - U\mu)w(Y) - (U\alpha)\eta(Y) \\ &\quad + \mu^2\eta(AY) + 2\lambda\mu w(AY) - 2\mu w(A^2Y),\end{aligned}$$

which together with (4.3) gives

$$\begin{aligned}\mu dw(U,Y) = {}& (2c\mu - U\mu)w(Y) - \{U\alpha + 2c(\lambda-\alpha)\}\eta(Y) \\ & -\{\mu^2 + 2(\lambda-\alpha)g(AW,W)\}\eta(AY) + 2\mu\left(\lambda + \frac{c}{\alpha}\right) w(AY).\end{aligned} \tag{4.20}$$

Because of (2.10) and (4.18), it is verified that

$$\mu dw(\xi, X) = 2u(AX) - \alpha u(X) - (\xi\alpha)\eta(X) - (W\alpha)w(X) + X\alpha. \tag{4.21}$$

Using (2.11) and (3.7), we obtain

$$du(\xi, X) = (3\lambda - 2\alpha)\eta(AX) - 2\mu w(AX) - \alpha\lambda\eta(X) + g(\phi\nabla\alpha, X). \tag{4.22}$$

Using above two equations, (3.13) is reduced to

$$\begin{aligned}&g(\nabla_X U, Y) + g(\nabla_Y U, X)) \\ &= 2c(g(X,Y) - \eta(X)\eta(Y)) - 2g(A^2X, Y) + 2\alpha g(AX, Y) \\ &\quad +(\xi\tau)(u(X)w(Y) + u(Y)w(X)) \\ &\quad +\frac{1}{\alpha}(2u(AX) + X\alpha - (\xi\alpha)\eta(X) - (W\alpha)w(X))u(Y) \\ &\quad +\frac{1}{\alpha}(2u(AY) + Y\alpha - (\xi\alpha)\eta(Y) - (W\alpha)w(Y))u(X) \\ &\quad +\{(3\lambda - 2\alpha)\eta(AX) - 2\mu w(AX) \\ &\qquad -\alpha\lambda\eta(X) + g(\phi\nabla\alpha, X)\}(\eta(Y) + \tau w(Y)) \\ &\quad +\{(3\lambda - 2\alpha)\eta(AY) - 2\mu w(AY) \\ &\qquad -\alpha\lambda\eta(Y) + g(\phi\nabla\alpha, Y)\}(\eta(X) + \tau w(X)),\end{aligned} \tag{4.23}$$

where we have used (4.21) and (4.22). Taking the trace of this and using (4.7), we find

$$\mathrm{div}U = 2c(n-1) + \alpha h - \mathrm{Tr}A^2 + \lambda(\lambda - \alpha). \tag{4.24}$$

Replacing X by U in (4.23) and using (4.6) and (4.7), we find

$$\begin{aligned} &g(\nabla_U U, Y) + g(\nabla_Y U, U)\\ &= (\lambda-\alpha)(Y\alpha) + 2\left(2\lambda - \alpha + \frac{c}{\alpha}\right)u(AY)\\ &\quad + \left\{\frac{U\alpha}{\alpha} + \frac{2c\lambda}{\alpha} + 2(\lambda-\alpha)(g(AW,W) + \alpha - \lambda)\right\}u(Y)\\ &\quad + \{\mu(W\alpha) - (\lambda-\alpha)\xi\alpha\}\eta(Y) + \mu^2(\xi\tau)w(Y). \end{aligned}$$

Since $g(\nabla_X U, U) = \mu(X\mu)$, it follows that

$$\begin{aligned} du(U,X) = &-2\mu(X\mu) + (\lambda-\alpha)(X\alpha) + 2\left(2\lambda - \alpha + \frac{c}{\alpha}\right)u(AX)\\ &+ \left\{\frac{U\alpha}{\alpha} + \frac{2c\lambda}{\alpha} + 2(\lambda-\alpha)(g(AW,W) + \alpha - \lambda)\right\}u(X)\\ &+\{\mu(W\alpha) - (\lambda-\alpha)\xi\alpha\}\eta(X) + \mu^2(\xi\tau)w(X), \end{aligned} \tag{4.25}$$

which implies that

$$du(U,W) = -2\mu(W\mu) + (\lambda-\alpha)W\alpha + \mu^2(\xi\tau). \tag{4.26}$$

5. The Exterior Derivative of 1-form u

We will continue our discussions under the hypotheses as those stated in Section 4.

Putting $Z = U$ in (3.12), we find

$$\begin{aligned} &-2\mu g((\nabla_Y A)X, W) + 2c\eta(X)u(Y) - du(U,X)(\eta(Y) + \tau w(Y))\\ &- du(U,Y)(\eta(X) + \tau w(X)) - d\eta(U,X)u(Y) - d\eta(U,Y)u(X)\\ &+ \mu^2(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)) + \mu^2((X\tau)w(Y) + (Y\tau)w(X))\\ &+ \tau\{\mu^2(g(\nabla_X W, Y) + g(\nabla_Y W, X)) - dw(U,Y)u(X) - dw(U,X)u(Y)\}\\ &- (U\tau)(u(X)w(Y) + u(Y)w(X)) = 0. \end{aligned}$$

Because of (2.1), (2.11) and (3.3), we see

$$d\eta(U,X) = (\lambda-\alpha)\eta(AX) - 2\mu w(AX).$$

Using this and (2.4), above equation reformed as

$$\begin{aligned}
&-2\mu g((\nabla_W A)Y,X)-2c(\eta(Y)u(X)+\eta(X)u(Y))-du(U,X)(\eta(Y)+\tau w(Y))\\
&-du(U,Y)(\eta(X)+\tau w(X))+\mu^2((X\tau)w(Y)+(Y\tau)w(X))\\
&-(U\tau)(u(X)w(Y)+u(Y)w(X))-\{(\lambda-\alpha)\eta(AX)-2\mu w(AX)\}u(Y)\\
&-\{(\lambda-\alpha)\eta(AY)-2\mu w(AY)\}u(X)+\mu^2(g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X))\\
&+\tau\{\mu^2(g(\nabla_X W,Y)+g(\nabla_Y W,X))-dw(U,Y)u(X)-dw(U,X)u(Y)\}=0.
\end{aligned}$$

Substituting (4.20) into this, we obtain

$$\begin{aligned}
&2\mu g((\nabla_W A)Y,X)\\
&=-2c(\eta(Y)u(X)+\eta(X)u(Y))-du(U,X)(\eta(Y)+\tau w(Y))\\
&\quad-du(U,Y)(\eta(X)+\tau w(X))+\mu^2((X\tau)w(Y)+(Y\tau)w(X))\\
&\quad-(U\tau)(u(X)w(Y)+u(Y)w(X))-\{(\lambda-\alpha)\eta(AX)-2\mu w(AX)\}u(Y)\\
&\quad-\{(\lambda-\alpha)\eta(AY)-2\mu w(AY)\}u(X)+\mu^2(g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X))\\
&\quad+\tau\mu^2(g(\nabla_X W,Y)+g(\nabla_Y W,X))\\
&\quad-\frac{1}{\alpha}u(X)\Big\{(2c\mu-U\mu)w(Y)-(U\alpha+2c(\lambda-\alpha))\eta(Y)\\
&\qquad-\{\mu^2+2(\lambda-\alpha)g(AW,W)\}\eta(AY)+2\mu\left(\lambda+\frac{c}{\alpha}\right)w(AY)\Big\}\\
&\quad-\frac{1}{\alpha}u(Y)\Big\{(2c\mu-U\mu)w(X)-\{U\alpha+2c(\lambda-\alpha)\}\eta(X)\\
&\qquad-\{\mu^2+2(\lambda-\alpha)g(AW,W)\}\eta(AX)+2\mu\left(\lambda+\frac{c}{\alpha}\right)w(AX)\Big\}.
\end{aligned}$$

Combining this to (4.12), we have

$$\begin{aligned}
&-2\mu(W\alpha)g(AY,X)\\
&+2\eta(AY)\left\{-\left(\alpha+\frac{c}{\alpha}\right)u(AX)-\frac{c}{\alpha}(\alpha+\lambda)u(X)+\mu X\mu\right\}\\
&+2\left\{-\left(\alpha+\frac{c}{\alpha}\right)u(AY)-\frac{c}{\alpha}(\alpha+\lambda)u(Y)+\mu(Y\mu)\right\}\eta(AX)\\
&-\frac{2c\mu}{\alpha}\{u(AX)w(Y)+u(AY)w(X)+(\lambda-\alpha)(w(X)u(Y)+w(Y)u(X))\}\\
&=-2\alpha c(\eta(Y)u(X)+\eta(X)u(Y))-\alpha du(U,X)(\eta(Y)+\tau w(Y))\\
&\quad-\alpha du(U,Y)(\eta(X)+\tau w(X))+\alpha\mu^2((X\tau)w(Y)+(Y\tau)w(X))\\
&\quad-\alpha(U\tau)(u(X)w(Y)+u(Y)w(X))-\mu^2(\eta(AX)u(Y)+\eta(AY)u(X))\\
&\quad+2\alpha\mu(w(AY)u(X)+w(AX)u(Y))+\alpha\mu^2(g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X))\\
&\quad+\mu^3(g(\nabla_X W,Y)+g(\nabla_Y W,X))\\
&\quad-u(X)\Big\{(2c\mu-U\mu)w(Y)-(U\alpha+2c(\lambda-\alpha))\eta(Y)\\
&\qquad-(\mu^2+2(\lambda-\alpha)g(AW,W))\eta(AY)+2\mu\left(\lambda+\frac{c}{\alpha}\right)w(AY)\Big\}\\
&\quad-u(Y)\Big\{(2c\mu-U\mu)w(X)-(U\alpha+2c(\lambda-\alpha))\eta(X)\\
&\qquad-(\mu^2+2(\lambda-\alpha)g(AW,W))\eta(AX)+2\mu\left(\lambda+\frac{c}{\alpha}\right)w(AX)\Big\}.
\end{aligned}\tag{5.1}$$

If we put $Y = W$ in (5.1) and take account of (2.1), (3.5) and (4.19), then we find

$$\begin{aligned}
&- 2\mu(W\alpha)w(AX) + \mu^2(X\mu)\\
&+ 2\mu(W\mu)\eta(AX) - \frac{2c\mu}{\alpha}\{u(AX) + (\lambda-\alpha)u(X)\}\\
&= -\mu du(U,X) - \alpha du(U,W)(\eta(X) + \tau w(X))\\
&\quad + \alpha\mu^2((X\tau) + (W\tau)w(X))\\
&\quad - \mu^2\{(W\alpha)\eta(X) + (W\mu)w(X)\}\\
&\quad + \left(U\mu - \alpha(U\tau) - \frac{2c}{\alpha}\mu g(AW,W)\right)u(X),
\end{aligned}$$

or, using (4.25) and (4.26)

$$\begin{aligned}
&2\mu(W\alpha)AW - 2c\mu U + \{\mu(\lambda-\alpha)\xi\alpha - 3\mu^2 W\alpha - \alpha\mu^2(\xi\tau)\}\xi\\
&- \{\mu^2(W\mu) + \tau\mu^2(W\alpha) + 2\mu^3(\xi\tau)\}W + \mu^2\nabla\mu - \mu(\lambda-\alpha)\nabla\alpha\\
&- 2\mu(2\lambda-\alpha)AU - \mu\left\{\frac{U\alpha}{\alpha} + 2(\lambda-\alpha)g(AW,W) - 2(\lambda-\alpha)^2\right\}U\\
&+ \alpha\mu^2((W\tau)W + \nabla\tau) + \left\{U\mu - \alpha(U\tau) - \frac{2c\mu}{\alpha}g(AW,W)\right\}U = 0.
\end{aligned}$$

By the way, since $\alpha\tau = \mu$, we find

$$\alpha\mu\nabla\tau = \mu\nabla\mu - (\lambda-\alpha)\nabla\alpha. \tag{5.2}$$

Using this, above equation is reduced to

$$\begin{aligned}
&\mu\nabla\mu - (\lambda-\alpha)\nabla\alpha\\
&= (2\lambda-\alpha)AU + \left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right)g(AW,W) - (\lambda-\alpha)^2 + c\right\}U\\
&\quad -(W\alpha)AW + \{2\mu(W\alpha) - (\lambda-\alpha)\xi\alpha\}\xi + (\lambda-\alpha)(2W\alpha - \tau(\xi\alpha))W.
\end{aligned} \tag{5.3}$$

If we take the inner product (5.3) with W, then we get

$$\mu(W\mu) = \{3(\lambda-\alpha) - g(AW,W)\}W\alpha - \tau(\lambda-\alpha)\xi\alpha. \tag{5.4}$$

Also, taking the inner product (5.3) with U and making use of (4.7), we obtain

$$\frac{U\mu}{\mu} - \frac{U\alpha}{\alpha} = \left(3\lambda - 2\alpha + \frac{c}{\alpha}\right)g(AW,W) + (\lambda-\alpha)(2\alpha - 3\lambda) + c. \tag{5.5}$$

On the other hand, replacing Y by W in (4.23) and using (4.3), we find

$$\begin{aligned}
&g(\nabla_X U, W) + g(\nabla_W U, X) - \frac{\mu}{\alpha}g(\phi\nabla\alpha, X) - (\xi\tau)u(X)\\
&- \{\mu(3\lambda-2\alpha) - 2\mu g(AW,W) + g(\phi\nabla\alpha, W)\}(\eta(X) + \tau w(X))\\
&+ 2\left(\lambda - 2\alpha - \frac{c}{\alpha}\right)g(AW,X) - 2cw(X)\\
&+ \frac{\mu}{\alpha}(4\alpha - 3\lambda + 2g(AW,W))\eta(AX) + \mu\left(\lambda + \frac{2c}{\alpha}\right)\eta(X) = 0,
\end{aligned}$$

or using (4.14),

$$
\begin{aligned}
&g(\nabla_X U, W) + g(\phi\nabla\mu, X) - \frac{\lambda-\alpha}{\mu} g(\phi\nabla\alpha, X) \\
&-(\xi\tau)u(X) + 2(\lambda-\alpha)w(AX) \\
&+\left\{\frac{U\alpha}{\alpha} + (\lambda-\alpha)(5\alpha - 6\lambda + 4g(AW,W))\right\} w(X) \\
&+\left\{\frac{U\alpha}{\mu} + \mu(4\alpha - 5\lambda + 3g(AW,W))\right\} \eta(X) = 0.
\end{aligned}
\tag{5.6}
$$

By the way, applying (5.3) by ϕ and making use of (2.6), (3.3) and (3.5), we have

$$
\begin{aligned}
&\mu\phi\nabla\mu - (\lambda-\alpha)\phi\nabla\alpha \\
&= -\frac{1}{\mu}(W\alpha)AU + \mu(\xi\tau)U + \mu^2(2\lambda-\alpha)\xi - \mu(2\lambda-\alpha)AW \\
&\quad - \mu\left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW,W) - (\lambda-\alpha)(3\lambda-2\alpha) + c\right\} W.
\end{aligned}
$$

Substituting this into (5.6), we find

$$
\begin{aligned}
&g(\nabla_X U, W) \\
&= \frac{W\alpha}{\mu^2} u(AX) + \alpha w(AX) \\
&\quad + \left\{3(\lambda-\alpha)^2 + \frac{c}{\alpha} g(AW,W) + c - \frac{U\alpha}{\alpha} - 3(\lambda-\alpha) g(AW,W)\right\} w(X) \\
&\quad + \left\{3\mu(\lambda - \alpha - g(AW,W)) - \frac{U\alpha}{\mu}\right\} \eta(X).
\end{aligned}
\tag{5.7}
$$

On the other hand, (4.12) turns out, using (2.4), to be

$$
\begin{aligned}
&\alpha(\nabla_X A)W \\
&= \frac{c\alpha}{\mu}(\eta(X)U + 2u(X)\xi) - (W\alpha)AX \\
&\quad + \frac{1}{\mu}\eta(AX)\left\{\mu\nabla\mu - \left(\alpha + \frac{c}{\alpha}\right) AU - \frac{c}{\alpha}(\lambda+\alpha)U\right\} \\
&\quad + \frac{1}{\mu}\left\{\mu(X\mu) - \left(\alpha + \frac{c}{\alpha}\right) u(AX) - \frac{c}{\alpha}(\lambda+\alpha)u(X)\right\} A\xi \\
&\quad - \frac{c}{\alpha}\left\{w(X)AU + u(AX)W + (\lambda-\alpha)(u(X)W + w(X)U)\right\}.
\end{aligned}
$$

If we apply by ϕ to this and make use of (3.3), then we find

$$
\begin{aligned}
&-\alpha\phi(\nabla_X A)W = (W\alpha)\phi AX + c\alpha\eta(X)W - (X\mu)U \\
&+\frac{1}{\mu}\eta(AX)\left\{\left(\alpha+\frac{c}{\alpha}\right)\{(\lambda-\alpha)A\xi-\mu AW\} - \frac{c}{\alpha}\mu(\lambda+\alpha)W - \mu\phi\nabla\mu\right\} \\
&+\frac{1}{\mu}\left\{\left(\alpha+\frac{c}{\alpha}\right)u(AX) + \frac{2c\lambda}{\alpha}u(X)\right\}U + \frac{c}{\alpha}w(X)(\mu^2\xi - \mu AW).
\end{aligned}
\tag{5.8}
$$

Now, if we put $Z = W$ in (3.12), then we find

$$
\begin{aligned}
&2g(\phi(\nabla_Y A)W, X) \\
&= 2\{(w(A^2Y) - cw(Y))\eta(X) - w(AY)\eta(AX)\} \\
&\quad + du(W,X)(\eta(Y)+\tau w(Y)) + \tau du(Y,X) + (W\tau)(w(Y)u(X) + w(X)u(Y)) \\
&\quad + (g(\nabla_W U, Y) + g(\nabla_Y U, W))(\eta(X) + \tau w(X)) \\
&\quad + \frac{2}{\mu}\{u(AX) + (\lambda-\alpha)u(X)\}u(Y) \\
&\quad + (Y\tau)u(X) - (X\tau)u(Y) + \tau(u(Y)g(\nabla_W W, X) \\
&\quad + u(X)g(\nabla_W W, Y)).
\end{aligned}
$$

Using (2.1), (2.10), (3.5) and (3.8), we can write the above equation as

$$
\begin{aligned}
&2\alpha g(\phi(\nabla_Y A)W, X) \\
&= \mu du(Y,X) - 2c\eta(X)w(AY) + 2\mu(c+\alpha^2+\alpha g(AW,W))\eta(X)\eta(Y) \\
&\quad + 2(\alpha\mu^2 + \mu^2 g(AW,W) - c\alpha)\eta(X)w(Y) \\
&\quad - 2\alpha\eta(AX)w(AY) + \alpha(W\tau)(w(X)u(Y) + w(Y)u(X)) \\
&\quad + \alpha g(\nabla_W U, X)(\eta(Y) + \tau w(Y)) \\
&\quad + \alpha g(\nabla_W U, Y)(\eta(X) + \tau w(X)) - g(\nabla_X U, W)\eta(AY) \\
&\quad + g(\nabla_Y U, W)\eta(AX) + \frac{2\alpha}{\mu}\{u(AX) + (\lambda-\alpha)u(X)\}u(Y) \\
&\quad + \alpha((Y\tau)u(X) - (X\tau)u(Y)) + \mu(u(X)g(\nabla_W W, Y) + u(Y)g(\nabla_W W, X)),
\end{aligned}
$$

or using (5.8),

$$
\begin{aligned}
&\mu du(X,Y) \\
&= (W\alpha)g((\phi A + A\phi)X, Y) + \frac{2c}{\alpha}\mu(w(X)w(AY) - w(Y)w(AX)) \\
&\quad + \eta(AX)g(\phi\nabla\mu, Y) - \eta(AY)g(\phi\nabla\mu, X) \\
&\quad + \frac{2c}{\mu\alpha}(u(X)u(AY) - u(Y)u(AX)) - (X\mu)u(Y) + (Y\mu)u(X) \\
&\quad + \alpha((X\tau)u(Y) - (Y\tau)u(X)) \\
&\quad + g(\nabla_Y U, W)\eta(AX) - g(\nabla_X U, W)\eta(AY) \\
&\quad + \{2c\alpha - 2c\lambda - \mu^2(\alpha + g(AW,W))\}(\eta(X)w(Y) - \eta(Y)w(X)),
\end{aligned}
$$

which together with (5.2) and (5.7) yields

$$
\begin{aligned}
&\mu du(X,Y)\\
&= (W\alpha)g((\phi A + A\phi)X,Y) + \frac{2c\mu}{\alpha}(w(X)w(AY) - w(Y)w(AX))\\
&\quad + \frac{W\alpha}{\mu^2}(\eta(AX)u(AY) - \eta(AY)u(AX))\\
&\quad + \eta(AX)g(\phi\nabla\mu, Y) - \eta(AY)g(\phi\nabla\mu, X)\\
&\quad + \alpha(\eta(AX)w(AY) - \eta(AY)w(AX))\\
&\quad + \frac{2c}{\mu\alpha}(u(X)u(AY) - u(Y)u(AX)) + \frac{\mu}{\alpha}((X\alpha)u(Y) - (Y\alpha)u(X))\\
&\quad + \{(\mu^2 + c)g(AW,W) + \alpha\mu^2 - c\alpha + 2c\lambda\}(\eta(X)w(Y) - \eta(Y)w(X)).
\end{aligned} \tag{5.9}
$$

Putting $X = \phi e_i$ and $Y = e_i$ in this and summing up for $i = 1, 2, \cdots, n$, we obtain

$$\mu\sum_{i=0}^{2n} du(\phi e_i, e_i) = (h - \alpha - g(AW,W))W\alpha - \mu(W\mu),$$

where we have used (2.6)–(2.8), (3.5) and (4.7). Taking the trace of (2.12), we obtain

$$\sum_{i=0}^{2n} g(\phi\nabla_{e_i}U, e_i) = \xi\alpha - \xi h.$$

Thus, it follows that

$$\mu(\xi h - \xi\alpha) = \mu(W\mu) + (g(AW,W) + \alpha - h)W\alpha, \tag{5.10}$$

which together with (4.16) gives

$$\mu^2(\mathrm{div}W) = (g(AW,W) + \alpha - h)W\alpha. \tag{5.11}$$

We notice here that

Remark 5.1. *If $AU = \sigma U$ for some function σ on Ω, then $AW \in \mathrm{span}\{\xi, W\}$ on Ω, where* span$\{\xi, W\}$ *is a linear subspace spanned by ξ and W.*

In fact, because of the hypothesis $AU = \sigma U$, (3.5) reformed as

$$\mu\phi AW = (\sigma + \lambda - \alpha)U,$$

which implies that $AW = \mu\xi + (\sigma + \lambda - \alpha)W \in \mathrm{span}\{\xi, W\}$.

Now, we prepare the following lemma for later use.

Lemma 5.2. *Let M be a real hypersurface of $M_n(c), c \neq 0$ which satisfies $R_\xi\phi = \phi R_\xi$ and $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$. If $AW \in \mathrm{span}\{\xi, W\}$, then $\Omega = \emptyset$.*

Proof. Since (3.5) and $AW = \mu\xi + g(AW, W)W$, we have

$$AU = (g(AW, W) + \alpha - \lambda)U. \tag{5.12}$$

From (4.2) we also have

$$g(AW, W)(\alpha AU + cU) = 0.$$

Now, suppose that $g(AW, W) \neq 0$ on Ω. Then we have $\alpha AU + cU = 0$ on this subset, which together with (5.12) gives

$$\mu^2 = \alpha g(AW, W) + c. \tag{5.13}$$

From this and (2.16) we have $R_\xi W = 0$ and consequently $R_\xi A\xi = 0$ on the subset because of (2.6) and (2.16). If we take (3.1) by R_ξ and using $R_\xi U = 0$ and $R_\xi A\xi = 0$, we obtain $R_\xi(A\phi - \phi A) = 0$, that is, $R_\xi(\mathcal{L}_\xi g) = 0$ on the subset, where $\mathcal{L}_\xi$ denotes the operator of the Lie derivative with respect to ξ. Owing to Theorem 5.1 of [5], it is verified that $A\xi = \alpha\xi$, a contradiction. Therefore we have the following

$$g(AW, W) = 0 \tag{5.14}$$

on Ω. So we have

$$AW = \mu\xi. \tag{5.15}$$

From (5.12) and (5.14), we get

$$AU = (\alpha - \lambda)U. \tag{5.16}$$

Differentiating (5.15) covariantly, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi.$$

Taking the inner product with W and making use of (2.11) and (5.16), we have

$$g((\nabla_X A)W, W) = 2(\lambda - \alpha)u(X)$$

Using (2.4) it reformed as

$$(\nabla_W A)W = 2(\lambda - \alpha)U. \tag{5.17}$$

On the other hand, (4.13) is reduced, using (5.16) and (5.17), to

$$(\mu^2 + 2c)U = -\mu(W\alpha)\xi + (W\mu)A\xi + \mu\nabla\mu. \tag{5.18}$$

Taking the inner product with W, we have

$$W\mu = 0. \tag{5.19}$$

Hence, it follows from (5.18) that

$$\mu\nabla\mu = \mu(W\alpha)\xi + (\mu^2 + 2c)U, \tag{5.20}$$

which shows that for any vector fields X

$$\mu(X\mu) = \mu(W\alpha)\eta(X) + (\mu^2 + 2c)u(X).$$

Differentiating this covariantly and using (2.1), we have

$$\begin{aligned}
&(Y\mu)(X\mu) + \mu(Y(X\mu)) \\
&= Y(\mu(W\alpha))\eta(X) + \mu(W\alpha)g(\phi AY, X) \\
&\quad + (2\mu(W\alpha)\eta(Y) + 2(\mu^2 + 2c)u(Y))u(X) + (\mu^2 + 2c)g(\nabla_Y U, X) \\
&\quad + \{\mu(W\alpha)\eta(\nabla_Y X) + (\mu^2 + 2c)u(\nabla_Y U)\}.
\end{aligned}$$

Taking the skew-symmtric part of this, we find

$$\begin{aligned}
&Y(\mu(W\alpha))\eta(X) - X(\mu(W\alpha))\eta(Y) \\
&+(\mu(W\alpha))g((\phi A + A\phi)Y, X) \\
&+2\mu(W\alpha)(\eta(Y)u(X) - \eta(X)u(Y)) \\
&+(\mu^2 + 2c)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0.
\end{aligned} \tag{5.21}$$

Replacing Y by ξ in this, and using (2.10) and (5.17), we have

$$\begin{aligned}
X(\mu(W\alpha)) - 2\mu(W\alpha)u(X) =&\xi(\mu(W\alpha))\eta(X) + (\mu(W\alpha))u(X) \\
&+ (\mu^2 + c)(g(\nabla_\xi U, X) - \mu^2\eta(X)).
\end{aligned}$$

Substituting this into (5.21), we obtain

$$\begin{aligned}
&\mu(W\alpha)(u(Y)\eta(X) - u(X)\eta(Y)) \\
&+ (\mu^2 + 2c)(g(\nabla_\xi U, Y)\eta(X) - g(\nabla_\xi U, X)\eta(Y)) \\
&+ \mu(W\alpha)g((\phi A + A\phi)Y, X) \\
&+ (\mu^2 + 2c)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0.
\end{aligned}$$

Putting $Y = U$ in this, and using (2.8), (3.3), (4.11), (5.15) and (5.16), we obtain

$$\begin{aligned}
&(\mu^2 + 2c)(g(\nabla_U U, X) - \mu(X\mu)) \\
&+\mu(\mu^2 + 2c)(W\alpha)\eta(X) + \mu^2(\lambda - \alpha)(W\alpha)w(X) = 0.
\end{aligned} \tag{5.22}$$

On the other hand, putting $Y = U$ in (5.9) and making use of (5.14), (5.15) and (5.19), we have

$$g(\nabla_U U, X) - \mu(X\mu) = 2(\lambda - \alpha)(W\alpha)w(X) + \frac{U\alpha}{\alpha}u(X) - (\lambda - \alpha)X\alpha.$$

Combining this to (5.22), we have

$$(\mu^2+2c)\left\{2(\lambda-\alpha)(W\alpha)w(X)+\frac{U\alpha}{\alpha}u(X)-(\lambda-\alpha)X\alpha\right\}$$
$$+\mu(\mu^2+2c)(W\alpha)\eta(X)-\mu^2(\lambda-\alpha)(W\alpha)w(X)=0.$$

If we put $X=W$ in this, then we have

$$(\mu^2+2c)(\lambda-\alpha)W\alpha=(\mu^2+4c)(\lambda-\alpha)W\alpha,$$

which, together with $\lambda\neq\alpha$, shows that

$$W\alpha=0. \tag{5.23}$$

Thus, (5.20) becomes

$$\mu\nabla\mu=(\mu^2+2c)U, \tag{5.24}$$

which implies

$$\phi\nabla\mu=-(\mu^2+2c)W. \tag{5.25}$$

Using (5.23), we can write (5.21) as

$$(\mu^2+2c)(g(\nabla_Y U,X)-g(\nabla_X U,Y))=0.$$

Now, suppose that $\mu^2+2c\neq 0$. Then we have $g(\nabla_Y U,X)-g(\nabla_X U,Y)=0$. Using (5.14)–(5.16), (5.20), (5.23) and (5.25), we can write (5.9) as

$$(\mu^2+c)(w(X)\eta(Y)-w(Y)\eta(X))=0,$$

which implies $\mu^2+c=0$. So μ is constant. Thus, (5.23) becomes $\mu^2+2c=0$, a contradiction. Therefore, we see that $\mu^2+2c=0$.

Accordingly we see that μ is constant, which together with (5.4) yields $\xi\alpha=0$. Hence (5.3) is reduced to

$$\mu^2\nabla\alpha=\{\mu^2(3\lambda-2\alpha)-c\alpha\}U. \tag{5.26}$$

Taking the inner product this to X and differentiating covariantly, we find

$$\begin{aligned}\mu^2(Y(X\alpha))=&\{\mu^2(3Y\lambda-2Y\alpha)-c\alpha\}u(X)\\&+\{\mu^2(3\lambda-2\alpha)-c\alpha\}(g(\nabla_Y U,X)+g(U,\nabla_Y X)).\end{aligned}$$

The skew-symmetric part of this is given by

$$\begin{aligned}&3\mu^2((Y\lambda)u(X)-(X\lambda)u(Y))+(2\mu^2+c)((X\alpha)u(Y)-(Y\alpha)u(X))\\&+\{\mu^2(3\lambda-2\alpha)-c\alpha\}(g(\nabla_Y U,X)-g(\nabla_X U,Y))=0,\end{aligned}$$

which implies that $\nabla\lambda = \chi U$ for some function χ, where we have used (4.24) and (5.26). Thus it follows that

$$\{\mu^2(3\lambda-\alpha)-c\alpha\}(g(\nabla_Y U, X)-g(\nabla_X U, Y)) = 0.$$

If $g(\nabla_Y U, X)-g(\nabla_X U, Y)=0$, then similarly as above we have a contradiction. Thus we have $\mu^2(3\lambda-2\alpha)-c\alpha = 0$, which together with $\mu^2+2c=0$ gives $2\lambda-\alpha=0$. i.e. $2\mu^2+\alpha^2=0$, a contradiction. Therefore Lemma 5.2 is proved. □

Lemma 5.3.

$$\alpha^2\phi(\nabla\lambda-\nabla h) = -4\mu(\mu^2+c)(AW-\mu\xi)+\frac{\alpha}{\mu}(h-\lambda)(W\alpha)U+fW,$$

for some function f on Ω.

Proof. Putting $X=Y=e_i$ in (3.12), summing up for $i=0,1,\cdots,2n$ and using (2.1) and (2.4), we find

$$\begin{aligned}&\mathrm{Tr}(\nabla_{\phi Z}A)-2c(n-1)\eta(Z)+(\mathrm{Tr}A^2)\eta(Z)-h\eta(AZ)\\&+g(\nabla_\xi U, Z)-g(\nabla_Z U,\xi)+\tau(g(\nabla_W U, Z)+g(\nabla_U W, Z))\\&+(\mathrm{div}U)(\eta(Z)+\tau w(Z))+g((\phi A+A\phi)U, Z)\\&+(W\tau)u(Z)+(U\tau)w(Z)+\tau(\mathrm{div}W)u(Z)=0,\end{aligned}$$

or using (2.10), (3.3), (3.7) and (4.24)

$$\begin{aligned}&\phi\nabla\alpha-\phi\nabla h+\frac{\mu}{\alpha}(\nabla_W U+\nabla_U W)-4\mu AW+(W\tau+\tau(\mathrm{div}W))U\\&+(U\tau+\tau(\mathrm{div}U)+\mu(4\lambda-3\alpha-h))W\\&+(\lambda-\alpha)(\lambda+3\alpha)\xi=0.\end{aligned}\tag{5.27}$$

On the other hand, combining (4.20) to (5.8) and making use of (5.7), we find

$$\begin{aligned}\nabla_U W =&\frac{1}{\mu}\{\mu\phi\nabla\mu-(\lambda-\alpha)\phi\nabla\alpha\}\\&-(\xi\tau)U+2\left(2\lambda-\alpha+\frac{c}{\alpha}\right)AW\\&+\left\{(\lambda-\alpha)(2\alpha-3\lambda)+c-\left(\lambda+\frac{c}{\alpha}\right)g(AW,W)\right\}W\\&+\mu\left\{g(AW,W)+3\alpha-5\lambda-\frac{2c}{\alpha}\right\}\xi.\end{aligned}$$

Substituting this and (4.14) into (5.27), we find

$$\begin{aligned}
&\alpha\phi(\nabla\alpha-\nabla h)+2\mu\phi\nabla\mu-(\lambda-\alpha)\phi\nabla\alpha\\
&=4\mu\left(\alpha-\lambda-\frac{c}{\alpha}\right)AW+\left(\mu(\xi\tau)-\alpha(W\tau)-\mu(\mathrm{div}W)\right)U\\
&\quad-4(\lambda-\alpha)(\mu^2+c)\xi-\left(\alpha(U\tau)+\mu(\mathrm{div}W)\right)W\\
&\quad-\mu\Big\{3c+(\lambda-\alpha)(\alpha-3\lambda)\\
&\quad-\left(\lambda+\frac{c}{\alpha}\right)g(AW,W)+4\alpha\lambda-3\alpha^2-h\alpha\Big\}W.
\end{aligned}\tag{5.28}$$

From (4.11), (4.16) and (5.2) we have

$$\begin{aligned}
&\alpha\mu\left(\mu(\xi\tau)-\alpha(W\tau)-\mu(\mathrm{div}W)\right)\\
&=2\mu^2(W\alpha)-\mu(\lambda-2\alpha)\xi\alpha-\alpha\mu(\xi h).
\end{aligned}$$

By the way, using (5.4) and (5.10) we have

$$\mu(\lambda-2\alpha)\xi\alpha+\alpha\mu(\xi h)=\alpha(3\lambda-2\alpha-h)W\alpha.$$

Thus, we have

$$\alpha\mu\left(\mu(\xi\tau)-\alpha(W\tau)-\mu(\mathrm{div}W)\right)=\alpha(h-\lambda)W\alpha.\tag{5.29}$$

Differentiating (3.2) covariantly, we find

$$2\mu\nabla\mu=(\lambda-2\alpha)\nabla\alpha+\alpha\nabla\lambda.\tag{5.30}$$

Using this and (5.29), the equation (5.28) reformed as

$$\alpha^2\phi(\nabla\lambda-\nabla h)=-4\mu(\mu^2+c)(AW-\mu\xi)+\frac{\alpha}{\mu}(h-\lambda)(W\alpha)U+fW,$$

where we have put

$$\begin{aligned}
f=&\alpha\mu\Big\{h\alpha+4\alpha^2-8\alpha\lambda+3\lambda^2-3c\\
&+\left(\lambda+\frac{c}{\alpha}\right)g(AW,W)-\mathrm{div}U-\frac{\alpha}{\mu}(U\tau)\Big\}.
\end{aligned}$$

This completes the proof of Lemma 5.3. □

6. Lemmas

We will continue our discussions under the same hypotheses as those in Section 4. Further we assume that $\mathrm{Tr}R_\xi$ is constant, that is, $g(S\xi,\xi)$ is constant. Then, from (2.5) we see that $\beta-h\alpha$ is constant, i.e.

$$\alpha(h-\lambda)=C,\tag{6.1}$$

where C is some constant. Differentiating this covariantly, we have

$$(\lambda - h)\nabla\alpha + \alpha(\nabla\lambda - \nabla h) = 0. \tag{6.2}$$

So we have $\alpha\phi(\nabla\lambda - \nabla h) = (h - \lambda)\phi\nabla\alpha$. Thus, from Lemma 5.3 we find

$$\frac{\alpha(h-\lambda)}{\mu}\phi(\nabla\alpha - (W\alpha)W) = -4(\mu^2 + c)(AW - \mu\xi) + \frac{\alpha}{\mu}fW,$$

which tells us that

$$\frac{\alpha(h-\lambda)}{\mu^2}(U\alpha) = 4(\mu^2 + c)g(AW, W) - \frac{\alpha}{\mu}f.$$

Combining the last two equations, it follows that

$$\begin{aligned}\frac{\alpha(h-\lambda)}{\mu}\phi\left(\nabla\alpha - (W\alpha)W - \frac{U\alpha}{\mu^2}U\right)\\ = -4(\mu^2 + c)(AW - \mu\xi - g(AW, W)W).\end{aligned} \tag{6.3}$$

Applying this by ϕ and using (3.5), we find

$$\begin{aligned}\alpha(h-\lambda)\left(\nabla\alpha - (\xi\alpha)\xi - (W\alpha)W - \frac{U\alpha}{\mu^2}U\right)\\ = 4(\mu^2 + c)\{AU + (\lambda - \alpha)U - g(AW, W)U\}.\end{aligned} \tag{6.4}$$

Taking the inner product with AW to this, and using (4.6), (5.4) and $\alpha \neq 0$, we see

$$(h - \lambda)(g(AW, \nabla\alpha) - \mu(\xi\alpha) - g(AW, W)(W\alpha)) = 0. \tag{6.5}$$

First of all, we prove the following:

Lemma 6.1. *$h - \lambda \neq 0$ on Ω.*

Proof. If not, then we have from (6.4)

$$(\mu^2 + c)\{AU - (g(AW, W) + \alpha - \lambda)U\} = 0$$

on this subset. Because of Remark 5.1 and Lemma 5.2, it is verified that $\mu^2 + c = 0$ on the set and hence μ is constant. Accordingly we see that $W\alpha = 0$ because of (4.11) and hence $\xi\alpha = 0$ and $\xi\tau = 0$ by virtue of (5.2) and (5.4). Thus, (5.3) reformed as

$$(\lambda - \alpha)\nabla\alpha + (2\lambda - \alpha)AU + \{c - (\lambda - \alpha)^2\}U = 0,$$

which together with $\mu^2 + c = 0$ implies that

$$X\alpha = \lambda u(X) + \varepsilon g(AU, X) \tag{6.6}$$

for any vector field X, where we have put $c\varepsilon = \alpha^2 - 2c$. Differentiating (6.6) covariantly with respect to a vector field Y and taking skew-symmetric part, we get

$$\begin{aligned}&(Y\lambda)u(X) - (X\lambda)u(Y) + \lambda(g(\nabla_Y U, X) - g(\nabla_X U, Y))\\&+ (Y\varepsilon)u(AX) - (X\varepsilon)u(AY)\\&+ \varepsilon\{c\mu(\eta(Y)w(X) - \eta(X)w(Y)) + g(A\nabla_Y U, X) - g(A\nabla_X U, Y)\} = 0.\end{aligned}$$

where we have used the Codazzi equation (2.4). Since $\xi\alpha = 0$ and (6.1), by replacing X by ξ in this, we get

$$\begin{aligned}&-\lambda(g(\nabla_\xi U, Y) + g(\nabla_Y \xi, U))\\&+ \varepsilon(g(\nabla_Y U, \alpha\xi + \mu W) - c\mu w(Y) - g(\nabla_\xi U, AY)) = 0,\end{aligned}$$

where we have used (2.6), which together with (2.10) and (5.9) implies that

$$\varepsilon A\nabla_\xi U + \lambda\nabla_\xi U + \mu\lambda AW \in \mathrm{span}\{\xi, W\}. \tag{6.7}$$

On the other hand, we can write (3.7) as

$$\nabla_\xi U = -\mu(\varepsilon + 3)AW + (\lambda - \alpha)(\varepsilon + 2)A\xi,$$

where we have used (3.3) and (6.6), which together with (2.6), (4.3) and the fact that $\mu^2 + c = 0$ yields

$$\begin{aligned}A\nabla_\xi U = &-\mu(\lambda - \alpha)AW + \{c - (\lambda - \alpha)(\varepsilon + 3)g(AW, W)\}A\xi\\&- c(\lambda - \alpha)(\varepsilon + 3)\xi.\end{aligned}$$

Combining the last three equations, it is seen that

$$\{(2\lambda - \alpha)\varepsilon + 2\lambda\}AW \in \mathrm{span}\{\xi, W\},$$

which shows that $(2\lambda - \alpha)\varepsilon + 2\lambda = 0$ by Lemma 5.2. So we have $(2\lambda - \alpha)(\alpha^2 - 2c) + 2c\lambda = 0$, a contradiction because of $\mu^2 + c = 0$. This completes the proof. □

If we combine (6.2) to (5.30), then we have

$$2\mu\nabla\mu = (h - 2\alpha)\nabla\alpha + \alpha\nabla h. \tag{6.8}$$

If we apply this by ξ, then we find

$$2\mu(\xi\mu) = (h - 2\alpha)\xi\alpha + \alpha(\xi h). \tag{6.9}$$

From (4.11), (5.6) and (5.12) we get $(h - \lambda)(\mu(\xi\alpha) - \alpha(W\alpha)) = 0$ and hence

$$\mu(\xi\alpha) = \alpha(W\alpha) \tag{6.10}$$

by virtue of Lemma 6.1, which together with (6.9) yields

$$\mu(\xi h) = (2\lambda - h)W\alpha. \tag{6.11}$$

From (5.2) and (6.10) we have $\xi\tau = 0$. Thus, using (6.8) and (6.10) we verify from (5.3)

$$\begin{aligned}&\frac{1}{2}(h\nabla\alpha + \alpha\nabla h) - \lambda\nabla\alpha + (W\alpha)AW \\ &= (2\lambda-\alpha)AU + \left\{\left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W) - (\lambda-\alpha)^2 + c\right\} U \\ &\quad +(W\alpha)\{\mu\xi + (\lambda-\alpha)W\}.\end{aligned} \tag{6.12}$$

Because of Lemma 6.1, (6.5) implies that

$$g(AW, \nabla\alpha) = (\alpha + g(AW, W))W\alpha, \tag{6.13}$$

with the aid of (6.10). Applying (5.3) by AW and making use of (4.3), (6.10), (6.13) and $\xi\tau = 0$, we find

$$\begin{aligned}&\mu g(AW, \nabla\mu) - (\lambda-\alpha)(\alpha + g(AW, W))W\alpha + g(A^2W, W)W\alpha \\ &= \{\mu^2 + (\lambda-\alpha)g(AW, W)\}W\alpha,\end{aligned}$$

which together with (4.5) gives

$$\mu\alpha g(AW, \nabla\mu) = \{(\mu^2 + c)g(AW, W) + \alpha\mu^2\}W\alpha. \tag{6.14}$$

In the next place, we will prove that

Lemma 6.2. $\xi\alpha = W\alpha = W\mu = \xi h = \xi\lambda = W\lambda = 0$ and $\xi(g(AW, W)) = 0$ *on* Ω.

Proof. Differentiating (4.4) covariantly, we get

$$\begin{aligned}&g(A^2W, W)(X\alpha) + \alpha(X(g(A^2W, W))) \\ &= 2\mu g(AW, W)(X\mu) + (\mu^2 - c)(X(g(AW, W))) + \mu^2(X\alpha) + 2\mu\alpha(X\mu).\end{aligned} \tag{6.15}$$

Replacing X by ξ in this, and using (4.11) and (6.10), we find

$$\begin{aligned}\alpha(\xi(g(A^2W, W))) =& \frac{\alpha}{\mu}(\mu^2 - g(A^2W, W))W\alpha + 2\mu(\alpha + g(AW, W))W\alpha \\ &+ (\mu^2 - c)(\xi(g(AW, W))).\end{aligned} \tag{6.16}$$

By the way, using (4.10), (4.18), (6.10) and (6.13), we verify that $\xi(g(AW, W)) = W\mu$, which together with (5.4) and (6.10) yields

$$\xi(g(AW, W)) = \frac{1}{\mu}\{2(\lambda-\alpha) - g(AW, W)\}W\alpha.$$

Substituting this and (4.5) into (6.14), we find

$$\frac{\alpha}{2}\xi(g(A^2W, W)) = \left\{\frac{c}{\mu}g(AW, W) + \mu\alpha + \frac{\mu}{\alpha}(\mu^2 - c)\right\} W\alpha. \tag{6.17}$$

On the other hand, we have

$$\frac{1}{2}(X(g(A^2W,W))) = g((\nabla_X A)W, AW) + g(A^2W, \nabla_X W),$$

which implies

$$\begin{aligned}(6.18)\quad \frac{1}{2}\alpha(X(g(A^2W,W))) =&\alpha g((\nabla_W A)X, AW) + 2c\alpha u(X) - cg(AW, \nabla_X W)\\ &+ cu(AX) + \alpha(\alpha + g(AW,W))u(AX),\end{aligned}$$

where we have used (2.6), (2.11) and (4.3).

By the way, putting $X = AW$ in (4.12) and making use of (2.6) and (6.14), we obtain

$$\begin{aligned}&\alpha(\nabla_W A)AW\\ &= -(W\alpha)A^2W\\ &\quad + (\alpha + g(AW,W))\left\{-\left(\alpha + \frac{c}{\alpha}\right)AU - \frac{c}{\alpha}(\lambda+\alpha)U + \mu\nabla\mu\right\}\\ &\quad + \frac{1}{\mu\alpha}\left\{(\mu^2+c)g(AW,W) + \alpha\mu^2\right\}(W\alpha)A\xi\\ &\quad - \frac{c}{\alpha}g(AW,W)\{AU + (\lambda-\alpha)U\},\end{aligned}$$

which implies

$$\alpha g((\nabla_W A)AW, \xi) = \frac{1}{\mu}\{\alpha\mu^2 + (\mu^2+c)g(AW,W)\}$$

because of (2.6) and (4.11). If we replace X by ξ in (6.16) and make use of (4.11), (4.18) and (6.17), then we obtain

$$(\mu^2 - c - \alpha g(AW,W))(W\alpha) = 0$$

because of $\lambda - \alpha \neq 0$.

Now, suppose that $W\alpha \neq 0$ on Ω. Then since $\lambda \neq \alpha$, we have $\alpha g(AW,W) = \mu^2 - c$, which together with (3.2) and (4.7) gives $\alpha g(AU,U) = -c\mu^2$. From this and (4.6) we verify that $\alpha^2 g(A^2U,U) = c^2\mu^2$. Using the last two equations it is seen that $||\alpha AU + cU||^2 = 0$ and hence $\alpha AU + cU = 0$. Thus, (3.5) is reduced to $\mu\phi AW = (\lambda - \alpha - c/\alpha)U$, which shows that $AW = \mu\xi + g(AW,W)W$ on this subset. According to Lemma 5.2, we have $\Omega = \emptyset$, and hence $W\alpha = 0$ on Ω. Thus, it is clear that $W\mu = 0$, $\xi\alpha = 0$, $\xi h = 0$ and $\xi\lambda = 0$, where we have used (4.11), (5.6), (6.2), (6.9), (6.10) and (6.11). Since (3.2), $W\alpha = 0$ and $W\mu = 0$, we have $W\lambda = 0$. Hence Lemma 6.2 is proved. □

Because of Lemma 6.2, we can write (6.4) as

$$\begin{aligned}&\alpha(h-\lambda)\left(\nabla\alpha - \frac{U\alpha}{\mu^2}U\right)\\ &= 4(\mu^2+c)\{AU - (\lambda - \alpha - g(AW,W))U\},\end{aligned}$$

which tells us that

$$\frac{1}{4}\alpha(h-\lambda)\nabla\alpha = (\mu^2+c)AU + \theta U, \tag{6.19}$$

where the function θ is defined by

$$\mu^2\theta = \frac{\alpha(h-\lambda)}{4}(U\alpha) - (\mu^2+c)g(AU,U).$$

We also have from (5.4)

$$\mu\nabla\mu - (\lambda-\alpha)\nabla\alpha = (2\lambda-\alpha)AU + \rho U, \tag{6.20}$$

where we have put

$$\rho = \left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW,W) - (\lambda-\alpha)^2 + c. \tag{6.21}$$

Remark 6.3. *$\mu^2 + c \neq 0$ on Ω.*

If not, then we have $\mu^2+c=0$ and hence μ is constant on this subset. So (6.19) and (6.20) are reduced respectively to

$$\mu^2\nabla\alpha = (U\alpha)U,$$
$$(\lambda-\alpha)\nabla\alpha + (2\lambda-\alpha)AU + \{c-(\lambda-\alpha)^2\}U = 0$$

because of Lemma 5.2. Combining these two equations, we obtain

$$(2\lambda-\alpha)AU = \left\{(\lambda-\alpha)^2 - c - \frac{U\alpha}{\alpha}\right\}U.$$

Suppose that $2\lambda-\alpha=0$ on this subset. Then, the equation $\mu^2+c=0$ becomes $\alpha^2-2c=0$, a contradiction. Thus we have $2\lambda-\alpha\neq 0$. Owing to Remark 5.1 and Lemma 5.2, above equation produces a contradiction. Hence $\mu^2+c\neq 0$ on Ω is proved.

Lemma 6.4. *$(2\lambda-\alpha)\theta = (\mu^2+c)\rho$ on Ω.*

Proof. From (6.17) and (6.18) we have

$$\frac{1}{4}\alpha(h-\lambda)(2\lambda-\alpha)\nabla\alpha - (\mu^2+c)\left\{\frac{1}{2}\nabla\mu^2 - (\lambda-\alpha)\nabla\alpha\right\}$$
$$= \{(2\lambda-\alpha)\theta - (\mu^2+c)\rho\}U.$$

Using the same method as that used to derive (6.7) from (6.6), we can deduce from this that

$$(2\lambda-\alpha)(\xi\theta)U + \{(2\lambda-\alpha)\theta - (\mu^2+c)\rho\}(\nabla_\xi U + \mu AW) = 0,$$

where, we have used (2.10), (6.1) and Lemma 6.2. If we take the inner product with U to this and make use of $\xi\mu = 0$, then we get $(2\lambda - \alpha)\xi\theta = 0$ and hence

$$\{(2\lambda - \alpha)\theta - (\mu^2 + c)\rho\}(\nabla_\xi U + \mu AW) = 0.$$

If $(2\lambda - \alpha)\theta - (\mu^2 + c)\rho \neq 0$ on Ω, then we have

$$\nabla_\xi U + \mu AW = 0.$$

We discuss our arguments on such a place. Using (3.7), the last equation can be written as

$$\phi\nabla\alpha = 2\mu AW + (2\alpha - 3\lambda)A\xi + \alpha\lambda\xi.$$

Applying this by ϕ and taking account of (3.5) and Lemma 6.2, we obtain

$$\nabla\alpha = -2AU + \lambda U. \tag{6.22}$$

Combining this to (6.19), we obtain

$$\left\{\mu^2 + c + \frac{1}{2}\alpha(h - \lambda)\right\} AU = \left\{\frac{1}{4}\alpha\lambda(h - \lambda) - \theta\right\} U.$$

Because of Remark 5.1 and Lemma 5.2, we conclude that $\mu^2 + c + (1/2)\alpha(h - \lambda) = 0$. Hence it follows from (6.1) that μ is constant. Thus, (6.20) reformed as

$$(\lambda - \alpha)\nabla\alpha = (\alpha - 2\lambda)AU - \rho U,$$

which together with (6.22) implies that $\alpha AU = \{\lambda(\alpha - \lambda) - \rho\}U$. Therefore we verify that $(2\lambda - \alpha)\theta - \rho(\mu^2 + c) = 0$ by virtue of Remark 5.1 and Lemma 5.2. This completes the proof. □

Lemma 6.5 *Let* span$\{\xi, W\}$ *be the linear subspace spanned by* ξ *and* W*. Then there exists* $P \in$ span$\{\xi, W\}$ *such that*

$$g(AW, \nabla_X U) = \frac{c}{\alpha}w(A^2X) - \left\{\mu^2 + \left(\lambda - \alpha + \frac{c}{\alpha}\right) g(AW, W)\right\} w(AX) + g(P, X).$$

Proof. Putting $Y = AW$ in (5.9) and using (3.6), (4.3), (6.13) and Lemma 6.2, we find

$$\begin{aligned}
&\mu du(X, AW) \\
&= \frac{2c}{\alpha}\mu\{g(A^2W, W)w(X) - g(AW, W)w(AX)\} \\
&\quad + \eta(AX)g(\phi\nabla\mu, AW) - \mu(\alpha + g(AW, W))g(\phi\nabla\mu, X) \\
&\quad + \alpha\{g(A^2W, W)\eta(AX) - \mu(\alpha + g(AW, W))w(AX)\} \\
&\quad + \{(\mu^2 + c)g(AW, W) + \alpha\mu^2 - c\alpha + 2c\lambda\}(g(AW, W)\eta(X) - \mu w(X)),
\end{aligned}$$

which enables us to obtain

$$\begin{aligned}
&g(AW,\nabla_X U)-g(\nabla_{AW}U,X)\\
&=-\alpha\left(\alpha+g(AW,W)+\frac{2c}{\alpha^2}g(AW,W)\right)w(AX)\\
&\quad-(\alpha+g(AW,W))g(\phi\nabla\mu,X)+g(P_1,X),
\end{aligned}$$

for some $P_1\in\operatorname{span}\{\xi,W\}$. If we replace X by AW in (4.23) and make use of (3.5), (4.3), (6.14) and Lemma 6.2, then we get

$$\begin{aligned}
&g(\nabla_X U,AW)+g(\nabla_{AW}U,X)\\
&=2cw(AX)+2\alpha w(A^2X)-2w(A^3X)\\
&\quad+\left(\mu+\frac{\mu}{\alpha}g(AW,W)\right)\{(3\lambda-2\alpha)\eta(AX)-2\mu w(AX)\\
&\quad-\alpha\lambda\eta(X)+g(\phi\nabla\alpha,X)\}\\
&\quad+\Big\{\mu(3\lambda-2\alpha)(\alpha+g(AW,W))-2\mu g(AW,W)-\alpha\lambda\mu\\
&\quad-\frac{1}{\mu}g(AU+(\lambda-\alpha)U,\nabla\alpha)\Big\}(\eta(X)+\tau w(X))-2c\mu\eta(X),
\end{aligned}$$

which shows that

$$\begin{aligned}
&g(\nabla_X U,AW)+g(\nabla_{AW}U,X)\\
&=-2w(A^3X)+2\alpha w(A^2X)+2cw(AX)\\
&\quad-2(\lambda-\alpha)(\alpha+g(AW,W))w(AX)\\
&\quad+\frac{\mu}{\alpha}(\alpha+g(AW,W))g(\phi\nabla\alpha,X)+g(P_2,X),
\end{aligned}$$

for some $P_2\in\operatorname{span}\{\xi,W\}$. Adding to the last two equations, we obtain

$$\begin{aligned}
2g(AW,\nabla_X U)=&-2w(A^3X)+2\alpha w(A^2X)+2cw(AX)\\
&-2(\lambda-\alpha)(\alpha+g(AW,W))w(AX)\\
&-\alpha\left(\alpha+g(AW,W)+\frac{2c}{\alpha^2}g(AW,W)\right)w(AX)\\
&-(\alpha+g(AW,W))\left(\phi\nabla\mu-\frac{\mu}{\alpha}\phi\nabla\alpha\right)\\
&+g(P_3,X)
\end{aligned}$$

for some $P_3\in\operatorname{span}\{\xi,W\}$.

By the way, applying (6.20) by ϕ, and using (2.8) and (3.4), we find

$$\phi\nabla\mu-\frac{\mu}{\alpha}\phi\nabla\alpha=(2\lambda-\alpha)\{-AW+\mu\xi+(\lambda-\alpha)W\}-\rho W. \tag{6.23}$$

Because of (4.3), we have

$$A^3W = -\frac{c}{\alpha}A^2W + (\lambda-\alpha)(\alpha+g(AW,W))AW$$
$$+\mu\left(\alpha+\frac{c}{\alpha}+g(AW,W)\right)A\xi.$$

Combining the last three equations, we obtain

$$g(AW,\nabla_X U)$$
$$=\frac{c}{\alpha}w(A^2X)-\left\{\mu^2+\left(\lambda-\alpha+\frac{c}{\alpha}\right)g(AW,W)\right\}w(AX)+g(P_4,X)$$

for some $P_4\in\text{span}\{\xi,W\}$. The completes the proof. □

Remark 6.6. *$W\rho=0$ on Ω.*

In fact, we have

$$W(g(AW,W))=g((\nabla_W A)W,W)+2g(AW,\nabla_W W),$$

which together with (4.13) and Lemma 6.2 yields

$$W(g(AW,W))=2g(AW,\nabla_W W).$$

However, if we take the inner product with AW to (4.19) and make use of Lemma 6.2 and (6.14), then we obtain $g(AW,\nabla_W W)=0$. So we have $W(g(AW,W))=0$, which connected to (6.21) and Lemma 6.2 gives $W\rho=0$.

7. Proof of the Main Theorem

We will continue our discussions under the same assumptions as those in Section 6. Taking the inner product X to (6.20) and differentiating covariantly, we have

$$\begin{aligned}&(Y\mu)(X\mu)+\mu(Y(X\mu))-(Y\lambda-Y\alpha)(X\alpha)-(\lambda-\alpha)(Y(X\alpha))\\&=(2(Y\lambda)-Y\alpha)u(AX)\\&\quad+(2\lambda-\alpha)(g((\nabla_Y A)U,X)+g(A\nabla_Y U,X))\\&\quad+(Y\rho)u(X)+\rho g(\nabla_Y U,X)+g((2\lambda-\alpha)AU+\rho U,\nabla_Y X).\end{aligned}$$

Taking the skew-symmetric part of this and using (2.4), we find

$$\begin{aligned}&(X\lambda)(Y\alpha)-(Y\lambda)(X\alpha)\\&+(2(X\lambda)-X\alpha)u(AY)-(2(Y\lambda)-Y\alpha)u(AX)\\&=c\mu(2\lambda-\alpha)(\eta(Y)w(X)-\eta(X)w(Y))\\&\quad+(2\lambda-\alpha)(g(A\nabla_Y U,X)-g(A\nabla_X U,Y))\\&\quad+(Y\rho)u(X)-(X\rho)u(Y)+\rho(g(\nabla_Y U,X)-g(\nabla_X U,Y)),\end{aligned}\tag{7.1}$$

where we have used (2.4) and (2.8). Differentiating (6.21) covariantly and taking the inner product ξ to this, it follows from Lemma 6.2 that $\xi\rho = 0$. Putting $Y = \xi$ in (7.1) and using (2.6) and $\xi\rho = 0$, we find

$$\begin{aligned} &c\mu(2\lambda-\alpha)w(X) - (2\lambda-\alpha)\{g(\alpha\xi+\mu W, \nabla_X U) + g(\nabla_\xi U, AX)\} \\ &- \rho(g(\nabla_X U, \xi) - g(\nabla_\xi U, X)) = 0, \end{aligned}$$

or using (2.10), (5.7) and Lemma 6.2,

$$(2\lambda-\alpha)A\nabla_\xi U + \rho\nabla_\xi U + \mu\rho AW \in \operatorname{span}\{\xi, W\}. \tag{7.2}$$

If we put $Y = W$ in (7.1) and take account of Lemma 6.2 and Remark 6.6, then we have

$$\begin{aligned} &(2\lambda-\alpha)\{g(\nabla_X U, AW) - g(A\nabla_W U, X) + c\mu\eta(X)\} \\ &+ \rho(g(\nabla_X U, W) - g(\nabla_W U, X)) = 0. \end{aligned} \tag{7.3}$$

By the way, putting $Y = W$ in (5.9), we have

$$\begin{aligned} &g(\nabla_X U, W) - g(\nabla_W U, X) \\ &= -\left(\alpha + \frac{2c}{\alpha}\right) w(AX) - g(\phi\nabla\mu, X) + g(P_5, X) \end{aligned}$$

for some $P_5 \in \operatorname{span}\{\xi, W\}$, which together with Lemma 6.5 and (7.3) implies that

$$\begin{aligned} &(2\lambda-\alpha)\left\{\frac{c}{\alpha}A^2W - \left(\mu^2 + \left(\lambda-\alpha+\frac{c}{\alpha}\right)g(AW,W)\right)AW - A\nabla_W U\right\} \\ &-\rho\left\{\left(\alpha+\frac{2c}{\alpha}\right)AW + \phi\nabla\mu\right\} \in \operatorname{span}\{\xi, W\}. \end{aligned}$$

It follows from this and (4.14) that

$$\begin{aligned} &(2\lambda-\alpha)A\phi\nabla\mu + \rho\phi\nabla\mu \\ &+ (2\lambda-\alpha)\left\{\frac{c}{\alpha}A^2W + \left(\lambda-\alpha+\frac{c}{\alpha}\right)g(AW,W)AW\right\} \\ &+ \rho\left(\alpha+\frac{2c}{\alpha}\right)AW \in \operatorname{span}\{\xi, W\}. \end{aligned}$$

If we take account of (4.3), (6.21) and (6.23), then the last equation can be written as

$$\begin{aligned} &\frac{\mu}{\alpha}(2\lambda-\alpha)A\phi\nabla\mu + \rho\phi\nabla\mu + (2\lambda-\alpha)^2\left(\lambda-\alpha+\frac{c}{\alpha}\right)AW \\ &+(2\lambda-\alpha)\left\{\frac{c}{\alpha}A^2W + ((\lambda-\alpha)^2 - c)AW\right\} \\ &+\rho\left(\alpha+\frac{2c}{\alpha}\right)AW \in \operatorname{span}\{\xi, W\}. \end{aligned} \tag{7.4}$$

On the other hand, from (3.7) we have

$$A\nabla_\xi U = \mu(3\lambda - 2\alpha)AW - 3\mu A^2W + 2\mu^2 A\xi + A\phi\nabla\alpha,$$

where we have used (2.6). Substituting this into (7.2), we find

$$\begin{aligned}&(2\lambda-\alpha)A\phi\nabla\alpha + \rho\phi\nabla\alpha - 2\mu\rho AW\\ &-(2\lambda-\alpha)\mu\{3A^2W + (2\alpha-3\lambda)AW\} \in \operatorname{span}\{\xi, W\}.\end{aligned}$$

Combining this to (7.4), we obtain

$$\begin{aligned}&(\lambda-\alpha)\left\{-\frac{\rho}{\mu}\phi\nabla\alpha + 2\rho AW + (2\lambda-\alpha)(3A^2W + (2\alpha-3\lambda)AW)\right\}\\ &+\rho\phi\nabla\mu + (2\lambda-\alpha)^2\left(\lambda-\alpha+\frac{c}{\alpha}\right)AW + \frac{c}{\alpha}(2\lambda-\alpha)A^2W\\ &+(2\lambda-\alpha)\{(\lambda-\alpha)^2 - c\}AW + \rho\left(\alpha + \frac{2c}{\alpha}\right)AW \in \operatorname{span}\{\xi, W\},\end{aligned}$$

which together with (4.3) and (6.23) implies that

$$\{2\rho\alpha - (2\lambda-\alpha)(\mu^2+c)\}AW \in \operatorname{span}\{\xi, W\},$$

that is,

$$\{2\rho\alpha - (2\lambda-\alpha)(\mu^2+c)\}(AW - \mu\xi - g(AW,W)) = 0.$$

According to Lemma 5.2, we see that

$$2\rho\alpha = (2\lambda-\alpha)(\mu^2+c). \tag{7.5}$$

From this fact and Lemma 6.4, we see that $2\alpha\theta = (\mu^2+c)^2$ by virtue of $2\lambda-\alpha \neq 0$. Thus, (6.19) is reduced to

$$\kappa\nabla\alpha = 2\alpha AU + (\mu^2+c)U \tag{7.6}$$

with the aid of Remark 6.3, where we have put

$$\kappa = \frac{\alpha^2(h-\lambda)}{2(\mu^2+c)}.$$

Differentiating this covariantly and taking the inner product with ξ, it follows from (6.1) and Lemma 6.2 that $\xi\kappa = 0$.

As in the same method as that used from (6.6) to drive (6.7), we can deduce from (7.6) that

$$\begin{aligned}&2\alpha g(A\nabla_\xi U, X) + (\mu^2+c)g(\nabla_\xi U, X)\\ &= \mu\{-2c\alpha w(X) - 2\alpha^2 w(AX) - (\mu^2+c)w(AX) + 2\alpha g(\nabla_X U, W)\},\end{aligned}$$

which together with (5.7) implies that

$$2\alpha A\nabla_\xi U + (\mu^2 + c)\nabla_\xi U + \mu(\mu^2 + c)AW \in \operatorname{span}\{\xi, W\}. \tag{7.7}$$

On the other hand, applying (7.6) by ϕ and using (2.6) and (3.3), we find

$$\frac{\kappa}{\mu}\phi\nabla\alpha = -2\alpha AW + (\mu^2 + c)W + 2\alpha\mu\xi,$$

which together with (4.3) yields

$$\frac{\kappa}{\mu}A\phi\nabla\alpha = (\mu^2 + c)AW - 2\mu g(AW, W)A\xi - 2c\mu\xi.$$

From Lemma 6.1 we have $\kappa \neq 0$ and hence combining the last two equations, it is verified that

$$2\alpha A\phi\nabla\alpha + (\mu^2 + c)\phi\nabla\alpha \in \operatorname{span}\{\xi, W\}. \tag{7.8}$$

By the way, applying (3.7) by A and using (4.3), we find

$$\begin{aligned}&2\alpha A\nabla_\xi U + (\mu^2 + c)\nabla_\xi U - 2\alpha\mu\left(3\lambda - 2\alpha + \frac{3c}{\alpha}\right)AW\\ &+ 3\mu(\mu^2 + c)AW - 2\alpha A\phi\nabla\alpha - (\mu^2 + c)\phi\nabla\alpha \in \operatorname{span}\{\xi, W\},\end{aligned}$$

which together with (7.7) and (7.8) gives

$$(2\mu^2 + \alpha^2 + 2c)(AW - \mu\xi - g(AW, W)) = 0.$$

Owing to Lemma 5.2, we see that $2\mu^2 + \alpha^2 + 2c = 0$, which implies that $2\mu\nabla\mu + \alpha\nabla\alpha = 0$. Hence (6.20) reformed as

$$\nabla\alpha + 2AU + \frac{\mu^2 + c}{\alpha}U = 0 \tag{7.9}$$

by virtue of $2\lambda - \alpha \neq 0$ on Ω, where we have used (7.5). Combining this to (6.19), we have

$$\left\{\mu^2 + c + \frac{1}{2}\alpha(h - \lambda)\right\}AU = \frac{1}{4}\{4\theta + (h - \lambda)(\mu^2 + c)\}U.$$

According to Remark 5.1, it follows that $\mu^2 + c + (1/2)\alpha(h - \lambda) = 0$, which together with (6.1) gives μ is constant and hence α is constant. Thus (7.9) becomes $AU = -\{(\mu^2 + c)/(2\alpha)\}U$, a contradiction by virtue of Remark 5.1.

Therefore we verify that $\Omega = \emptyset$, that is, $A\xi = \alpha\xi$ on M. Thus, from (2.18) we see that $R_\xi S = SR_\xi$. Hence from Theorem 1.2 ([9]) M is homogeneous real hypersurfaces of Type A.

Let M be of Type A. Then M always satisfies $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$. Since $\mathrm{Tr}A$ is constant and (2.16), it is easy to see that $\phi R_\xi = R_\xi\phi$ and $\mathrm{Tr}R_\xi$ is constant.

Consequently we conclude that

Theorem 7.1. *Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0, n \geq 3$ which satisfies $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ and TrR_ξ is constant. Then M holds $\phi R_\xi = R_\xi\phi$ if and only if $A\xi = 0$ or M is locally congruent to one of following:*

(I) *In cases that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(II) *In cases $M_n(c) = H_n\mathbb{C}$,*

(A_0) *a horosphere;*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;*

(A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$.*

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