Controllability and Observability of Sylvester Matrix Dynamical Systems on Time Scales

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ABSTRACT. In this paper, we obtain solution for the first order matrix dynamical system and also we provide set of necessary and sufficient conditions for complete controllability and complete observability of the Sylvester matrix dynamical system.

1. Introduction

The importance of Sylvester matrix and Lyapunov matrix differential equations and their occurrence in a number of areas of applied mathematics such as control systems, dynamic programming, optimal filters, quantum mechanics and systems engineering etc., are well known. The two main objectives of this paper are therefore (1) to develop the theory and methods to solve dynamical system on time scales (2) to explore the techniques of controllability and observability . In this paper we mainly focus our attention to first order Sylvester matrix dynamical systems of the form

$$(1.1) \qquad X^{\Delta}(t) = A(t)X(t) + X(t)B(t) + \mu(t)A(t)X(t)B(t) + C(t)U(t)D^*(t),$$
$$X(t_0) = X_0$$

(1.2) $Y(t) = K(t)X(t)L^*(t),$

where $X(t)$ is an $n \times n$ matrix, $U(t)$ is $m \times n$ input piecewise rd-continuous matrix called control and $Y(t)$ is $r \times n$ output rd-continuous matrix. Here $A(t)$, $B(t)$, $C(t)$ and $K(t)$ are $n \times n$, $n \times n$, $n \times m$ and $r \times n$ rd-continuous matrices respectively. $D(t)$, $L(t)$ are rd-continuous matrices of order $n \times n$. X^Δ is the generalized delta derivative of X,(Definition 1.10 of [2]) and t is from a time scale T, which is a non-empty closed subset of $\mathbb{R}$ and μ is a graininess function. When $B = A^*$ (* denotes the transpose of matrix) equation (1.1) is called matrix Lyapunov dynamical system on time scales. Many authors [4, 11], were obtained complete controllability and complete observability criteria for similar systems of the type (1.1) and (1.2) with $B(t) = 0, D(t)$ and $L(t)$ are identity matrices and $X(t)$ is a vector.

If the time scale $T = \mathbb{R}$, then $\mu = 0$, the system (1.1) becomes Sylvester matrix dynamical system of the form

(1.3) $X^\Delta(t) = A(t)X(t) + X(t)B(t) + C(t)U(t)D^*(t).$

If the time scale $T = Z$, then $\mu = 1$, the system (1.1) become Sylvester matrix delta-difference system of the form

(1.4) $\Delta X(t) = A(t)X(t) + X(t)B(t) + A(t)X(t)B(t) + C(t)U(t)D^*(t),$

where $\Delta X(t) = X(t+1) - X(t)$. In the above system, if we put $A(t) = A_1(t) - I_n$ and $B(t) = B_1(t) - I_n$ then system (1.4) becomes Sylvester matrix difference system of the form

(1.5) $X(t+1) = A_1(t)X(t)B_1(t) + C(t)U(t)D^*(t).$

Therefore, study of behavior of controllability of Sylvester matrix system (1.1) unify the study of (1.3), (1.4), (1.5) and extended to matrix dynamic systems on time scales. The analytical, numerical solutions and control aspects of Sylvester matrix differential system (1.3) was studied by Fausett [5]. The existence and uniqueness, controllability and observability of matrix delta-difference system (1.4) were studied by Murty [10]. The calculus of time scales was initiated by Stefan Hilger [6] in order to create a theory that can unify discrete and continuous analysis. The study of dynamic equations on time scales, is an area of mathematics that has recently received a lot of attention and sheds new light on the discrepancies between continuous differential equations and discrete difference equations. It also prevents one from proving a result twice, once for differential equations and once for difference equations. The general idea, which is the main goal of Bohner and Peterson's excellent introductory text [2, 3] is to prove a result for a dynamic equation where the domain of the unknown function is so-called time scale. If $T = \mathbb{R}$, the general result obtained yields the same result concerning an ordinary differential equation. If $T = Z$, the general result is the same result one would obtain concerning a difference equation. This paper is well organized as follows:

In section 2 we study some basic properties of time scales, Kronecker product of matrices and develop preliminary results by converting the given problem into a Kronecker product problem. The solution to the corresponding initial value problem obtained in terms of two transition matrices of the systems $X^\Delta(t) = A(t)X(t)$ and $X^\Delta(t) = B^*(t)X(t)$ by using the standard technique of variation of parameters [8].

In Section 3 we address the necessary and sufficient conditions for complete controllability and complete observability under certain smoothness conditions.

2. Preliminaries

A great deal of work has been done since 1988, unifying the theory of differential equations and the theory of difference equations by establishing the corresponding results in time scale setting. For more detailed information refer the books [2] and [3].

Definition 2.1. A nonempty closed subset of $\mathbb{R}$ is called a time scale. It is denoted by T. By an interval we mean the intersection of the given interval with a time scale. For $t < \sup T$ and $t > \inf T$, define the forward jump operator σ, and backward jump operator ρ, respectively by

$$\sigma(t) = \inf\{s \in T, \ s > t\} \in T$$
$$\rho(t) = \sup\{s \in T, \ s < t\} \in T$$

For all $t, r \in T$ if $\sigma(t) = t$, t is said to be right dense, (otherwise t is said to be right scattered) and $\rho(r) = r$, r is said to be left dense, (otherwise r is said to be left scattered). The graininess function $\mu(t) : T[0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 2.2. A function $x : T \to \mathbb{R}$ is right dense continuous (rd-continuous) if it is continuous at every right dense point $t \in \mathbb{R}$ and its left hand limit exists at each left dense point $t \in \mathbb{R}$.

Definition 2.3. A mapping $f : T \to X$, where X is a banach space, is called rd-continuous if

(i) It is continuous at each right dense $t \in T$.

(ii) At each left dense point the left side limit $f(t^-)$ exists.

Definition 2.4. $F : T^k \to \mathbb{R}$ is called an antiderivative of $f : T^k \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in T^k$. We then define the integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Theorem 2.5. *Assume $f : T \to \mathbb{R}$ is a function and let $t \in T^k$. Then we have the following:*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then $f(t)$ is differentiable at t with*
$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(iii) *If t is right-dense, then f is differentiable at t if and only if the limit*
$$\lim_{t \to s} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case
$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(iv) *If f is differentiable at t, then*
$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

Definition 2.6. A function $f : T \to \mathbb{R}$ is called rd-continuous provided it is continuous at right dense points in T and its left sided limits exists (finite) at left dense points in T. The set rd-continuous functions $f : T \to \mathbb{R}$ will be denoted by
$$C_{rd} = C_{rd}(T) = C_{rd}(T, \mathbb{R}).$$

The set of functions $f : T \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by
$$C_{rd}^{1} = C_{rd}^{1}(T) = C_{rd}^{1}(T, \mathbb{R}).$$

Results 2.7. *$A, B \in \mathbb{R}$ are matrix-valued functions on T, then*

(i) $\phi_0(t, s) \equiv I$ *and* $\phi_A(t, t) \equiv I$

(ii) $\phi_A(\sigma(t), s) \equiv (I + \mu(t)A(t) \quad \phi_A(t, s);$

(iii) $\phi_A^{-1}(t, s) \equiv \phi_{\ominus A^*}^*(t, s);$

(iv) $\phi_A(t, s) = \phi_A^{-1}(s, t) = \phi_{\ominus A^*}^*(s, t)$

(v) $\phi_A(t, s)\phi_A(s, r) = \phi_A(t, r)$

(vi) $\phi_A(t, s)\phi_B(t, s) = \phi_{A \oplus B}(t, s)$ *if $\phi_A(t, s)$ and $B(t)$ commute*

Theorem 2.8.([2]) *Let $A \in \mathbb{R}$ be an $n \times n$ matrix-valued function on T and suppose that $f : T \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in T$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^\Delta(t) = A(t)y(t) + f(t), y(t_0) = y_0$$

has a unique solution $y : T \to \mathbb{R}$. Moreover, this solution is given by

$$y(t) = \phi_A(t, t_0)y_0 + \int_{t_0}^{t} \phi_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Now we present some properties and results for Kronecker products which are useful for studying Kronecker product matrix dynamical systems on time scales. Kronecker product is also known as a direct product or a tensor product is a concept having its origin in group theory and has important applications in particle physics. This technique has been successfully applied in various fields of matrix theory.

Definition 2.9. Let $A \in \mathbb{C}^{m \times n}(\mathbb{R}^{m \times n})$ and $B \in \mathbb{C}^{p \times q}(\mathbb{R}^{p \times q})$, then the Kronecker product of A and B written as $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ ... & ... & ... & ... \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{bmatrix}$$

is an $mp \times nq$ matrix and is in $\mathbb{C}^{mp \times nq}(\mathbb{R}^{mp \times nq})$.

The Kronecker product has the following properties and rules [1].

(1) $(A \otimes B)^* = A^* \otimes B^*$

(2) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (provided A and B are invertible)

(3) $(A \otimes B)^- = A^- \otimes B^-$ (A^- is the generalized inverse of A)

(4) The mixed product rule: $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ provided the dimensions of the matrices are such that the various expressions exist.

(5) $\|A \otimes B\| = \|A\| \, \|B\|$, where norm of A is defined by $\|A\| = \max_{i,j} |a_{ij}|$.

(6) There exists a zero element $O_{mn} = O_m \otimes O_n$

(7) There exists a unit element $I_{mn} = I_m \otimes I_n$

(8) $Vec(YAB) = (B^* \otimes A)VecY$

(9) If A and B are matrices both of order $n \times n$, then

    (i) $Vec(AX) = (I_n \otimes A)VecX$

    (ii) $Vec(XA) = (A^* \otimes I_n)VecX$

Definition 2.10. Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, we denote

$$\hat{A} = \text{Vec } A = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.n} \end{bmatrix}, \quad \text{where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad i \leq j \leq n.$$

Definition 2.11. Let A and B are rd-continuous matrices on time scale T, then

$$(A \otimes B)^{\Delta}(\mathrm{t}) = A^{\Delta}(t) \otimes \mathrm{B}(\mathrm{t}) + \mathrm{A}(\sigma(\mathrm{t})) \otimes B^{\Delta}(t)$$

A^{Δ} is the generalized delta derivative of A, t is from a time scale T, which is a known non-empty closed subset of $\mathbb{R}$.

Now by applying the Vec operator to the Δ-differentiable matrix dynamical system (1.1) also the output equation (1.2) and using Kronecker product properties, we have

$$(2.1) \qquad\qquad Z^{\Delta}(t) = G(t)Z(t) + [D \otimes C]\hat{U}(t); Z(\mathrm{t}_0) = Z_0$$

$$(2.2) \qquad\qquad\qquad \hat{Y}(t) = [L \otimes K]Z(t),$$

where $Z(t) = VecX(t)$, $\hat{U}(t) = VecU(t)$, $\hat{Y}(t) = VecY(t)$ and

$$G(t) = [B^* \otimes I + I \otimes A + \mu(t)(B^* \otimes A)]$$

is a $n^2 \times n^2$ matrix. Let $A(t)$ and $B(t)$ be regressive and rd-continuous.

From the definition of Kronecker product $G : T^k \to \mathbb{R}^{n^2 \times n^2}$ is regressive and rd-continuous. System (2.1) and (2.2) is called the Kronecker product system associated with (1.1) and (1.2).

Remark 2.12. It is easily seen that, if $X(t)$ is the solution of (1.1) then $VecX(t) = Z(t)$ is the solution of (2.1) and vice-versa.

Now we confine our attention to corresponding homogeneous matrix dynamical system on time scales (2.1) given by

$$(2.3) \qquad\qquad\qquad Z^{\Delta}(t) = G(t)Z(t)$$

Lemma 2.13. *Let $\phi_1(t, s)$ and $\phi_2(t, s)$ are denote state transition matrices of the systems $X^{\Delta}(t) = A(t)X(t)$ and $X^{\Delta}(t) = B^*(t)X(t)$ respectively. Then the matrix $\phi(t, s)$ defined by*

$$(2.4) \qquad\qquad\qquad \phi(t, s) = \phi_2(t, s) \otimes \phi_1(t, s)$$

is the state transition matrix of (2.3) and every solution of (2.3) is of the form $Z(t) = \phi(t,s)\Psi$ (where Ψ is any constant matrix of order n^2).

Proof. Consider

$$
\begin{aligned}
\phi^\Delta(t,s) &= \phi_2^\Delta(t,s) \otimes \phi_1(t,s) + \phi_2(\sigma(t),s) \otimes \phi_1^\Delta(t,s) \\
&= B^*\phi_2(t,s) \otimes \phi_1(t,s) + (1 + \mu(t)B^*)\phi_2(t,s) \otimes A\phi_1(t,s) \\
&= (B^* \otimes I_n)(\phi_2(t,s) \otimes \phi_1(t,s)) + (\phi_2(t,s) \otimes (A\phi_1(t,s)) \\
&\quad + \mu(t)B^*\phi_2(t,s)) \otimes (A\phi_1(t,s)) \\
&= [(B^* \otimes I_n) + (I_n \otimes A) + \mu(t)(B^* \otimes A)](\phi_2(t,s) \otimes \phi_1(t,s)) \\
&= G\phi(t,s),
\end{aligned}
$$

also

$$
\phi(t,t) = \phi_2(t,t) \otimes \phi_1(t,t) = I_n \otimes I_n = I_{n^2}.
$$

Hence $\phi(t,s)$ is the transition matrix of (2.3). Moreover it can be easily seen that $Z(t)$ is a solution of (2.3) and every solution of (2.3) is of this form. $\qquad\square$

Theorem 2.14. ([7]) *Let $\phi(t,s) = \phi_2(t,s) \otimes \phi_1(t,s)$ be a transition matrix of (2.3), then the unique solution of (2.1), subject to the initial condition $Z(t_0) = Z_0$ is*

$$
(2.5) \qquad Z(t) = \phi(t,t_0)[Z_0 + \int_{t_0}^{t} \phi(t_0, \sigma(s))(D \otimes C)(s)\hat{U}(s)\Delta s].
$$

3. Main Results

In this section, we prove necessary and sufficient conditions for complete controllability and complete observability of the matrix dynamical systems on time scales (2.1) and (2.2).

Definition 3.1. The Δ-differential systems S_1 given by (2.1) and (2.2) is said to be completely controllable if for t_0, any initial state $Z(t_0) = Z_0$ and any given final state Z_f there exists a finite time $t_1 > t_0$ and a control $\hat{U}(t)$, $t_0 \leq t_1$ such that $Z(t_1) = Z_f$.

Theorem 3.2. *The time scale dynamical system S_1 is completely controllable on the closed interval $J = [t_0, t_1]$ if and only if the $n^2 \times n^2$ symmetric controllability matrix*

$$
(3.1) \qquad V(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \sigma(s))(D \otimes C)(s)(D \otimes C)^*(s)\phi^*(t_0, \sigma(s))\Delta s,
$$

where $\phi(t,s)$ is defined in (2.4), is non-singular. In this case the control

$$
(3.2) \qquad \hat{U}(t) = -(D \otimes C)^*(t)\phi^*(t_0, \sigma(s))V^{-1}(t_0, t_1)\{Z_0 - \phi(t_0, t_1)Z_f\}
$$

defined on $t_0 \leq t_1$, *transfers* $Z(t_0) = Z_0$ *to* $Z(t_1) = Z_f$.

Proof. Suppose that $V(t_0, t_1)$ is non-singular, then the control defined by (3.2) exists. Now substituting (3.2) in (2.5) with $t = t_1$, we have

$$Z(t_1) = \phi(t_1, t_0)[Z_0 - \int_{t_0}^{t_1} \phi(t_0, \sigma(s))(D \otimes C)(s)(D \otimes C)^*(s)\phi^*(t_0, \sigma(s))$$

$$\times V^{-1}(t_0, t_1)\{Z_0 - \phi(t_0, t_1)Z_f\}\Delta s] = Z_f.$$

Hence the dynamical system S_1 is completely controllable.

Conversely, suppose that the dynamical system S_1 is completely controllable on J, then we have to show that $V(t_0, t_1)$ is non singular. Then there exists a non zero $n^2 \times 1$ vector α such that

$$\alpha^* V(t_0, t_1)\alpha = \int_{t_0}^{t_1} \alpha^* \phi(t_0, \sigma(s))(D \otimes C)(s)(D \otimes C)^*(s)\phi^*(t_0, \sigma(s))\alpha\Delta s$$

$$= \int_{t_0}^{t_1} \theta^*(\sigma(s), t_0)\theta(\sigma(s), t_0)\Delta s$$

$$(3.3) \qquad\qquad \alpha^* V(t_0, t_1)\alpha = \int_{t_0}^{t_1} \|\theta\|^2 \Delta s \geq 0$$

, where $\theta = (D \otimes C)^*(s)\phi^*(t_0, \sigma(s))\alpha$. From (3.3) $V(t_0, t_1)$ is positive semi definite. Suppose that there exists some $\beta \neq \mathbf{0}$ (zero vector) such that $\beta^* V(t_0, t_1)\beta = 0$, then from (3.3) with $\theta = \eta$, when $\alpha = \beta$ implies

$$\int_{t_0}^{t_1} \|\eta\|^2 \Delta s = 0.$$

Using the properties of norm, we have

$$(3.4) \qquad\qquad \eta(\sigma(s), t_0) = \mathbf{0}, \;\; t_0 \leq t \leq t_1.$$

Since S_1 is completely controllable, so there exists a control $\hat{U}(t)$ making $Z(t_1) = \mathbf{0}$ if $Z(t_0) = \beta$. Hence from (2.5), we have

$$\beta = -\int_{t_0}^{t_1} \phi(t_0, \sigma(s))(D \otimes C)(s)\hat{U}(s)\Delta s.$$

Consider

$$\|\beta\|^2 = \beta^*\beta = -\int_{t_0}^{t_1} \hat{U}^*(s)(D \otimes C)^*(s)\phi^*(t_0, \sigma(s))\beta\Delta s$$

$$= -\int_{t_0}^{t_1} \hat{U}^*(s)\eta(\sigma(s), t_0)\Delta s = 0.$$

Hence $\beta = \mathbf{0}$, which is a contradiction to our assumption. Thus $V(t_0, t_1)$ is positive definite and is therefore non-singular. $\square$

We now turn our attention to the concept of observability on a timescale dynamical system.

Definition 3.3. The timescale dynamical system (2.1), (2.2) is completely observable on $J = [t_0, t_1]$ if for any time t_0 and any initial state $Z(t_0) = Z_0$ there exists a finite time $t_1 > t_0$ such that the knowledge of $\hat{U}(t)$ and $\hat{Y}(t)$ for $t_0 \leq t_1$ suffices to determine Z_0 uniquely.

Now we present a necessary and sufficient condition for the system (2.1), (2.2) to be completely observable.

Theorem 3.4. *The system S_1 is completely observable on J if and only if the $n^2 \times n^2$ symmetric observability matrix*

$$(3.5) \qquad W(t_0, t_1) = \int_{t_0}^{t_1} \phi^*(s, t_0)(L \otimes K)^*(s)(L \otimes K)(s)\phi(s, t_0)\Delta s$$

is non singular.

Proof. Suppose that $W(t_0, t_1)$ is non-singular. It is simpler to consider the case of zero input, and it does not entail any loss of generality. Since the concept is not altered in the presence of a known input signal. Implies $\hat{Y}(t) = (L \otimes K)Z(t)$ since from $Z(t) = \phi(t, t_0)Z_0$, we have

$$(3.6) \qquad \qquad \hat{Y}(t) = (L \otimes K)\phi(t, t_0)Z_0$$

multiplying (3.6) on the left by $\phi^*(t, t_0)(L \otimes K)^*(t)$ and integrating from t_0 to t_1 we obtain

$$\int_{t_0}^{t_1} \phi^*(s, t_0)(L \otimes K)^*(s)\hat{Y}(s)\Delta s = W(t_0, t_1)Z_0$$

since $W(t_0, t_1)$ is non singular, Z_0 can be determined uniquely. Hence the dynamical system S_1 is completely observable.
Conversely, suppose that the dynamical system S_1 is completely observable. Then

we prove that $W(t_0, t_1)$ is non singular. Since $W(t_0, t_1)$ is symmetric, we can construct the quadratic form

$$(3.7) \qquad \alpha^* W(t_0, t_1)\alpha = \int\limits_{t_0}^{t_1} \alpha^* \phi^*(s, t_0)(L \otimes K)^*(s)(L \otimes K)(s)\phi(t_0, \sigma(s))\alpha \Delta s$$

$$= \int\limits_{t_0}^{t_1} \|\eta(s, t_0)\|^2 \, \Delta s \geq 0,$$

where α is an arbitrary column $n^2 \times 1$ vector and $\eta(s, t_0) = (L \otimes K)(s)\phi(t_0, \sigma(s))\alpha$. From (3.7) $W(t_0, t_1)$ is positive semi definite. Suppose that there exists some $\beta \neq \mathbf{0}$ such that $\beta^* W(t_0, t_1)\beta = 0$, then from (3.7) with $\eta = \theta$ when $\alpha = \beta$, implies

$$\int\limits_{t_0}^{t_1} \|\theta(s, t_0)\|^2 \, \Delta s = 0 \Rightarrow \theta(\mathrm{s}, \mathrm{t}_0) = \mathbf{0}, \ \ \mathrm{t}_0 \leq \mathrm{s} \leq \mathrm{t}_1$$

$$\Rightarrow (L \otimes K)(s)\phi(t_0, \sigma(s))\beta = \mathbf{0}, \ \ \mathrm{t}_0 \leq \mathrm{s} \leq \mathrm{t}_1.$$

From (3.6), this implies that when $Z_0 = \beta$, the out put is identically zero throughout the interval, so that Z_0 can not be determined from the knowledge of $\hat{Y}(t)$. This contradicts the supposition that S_1 is completely observable. Hence $W(t_0, t_1)$ is positive definite, therefore it is non singular. This completes the proof. $\qquad \square$

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