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Behavior of Solutions of a Fourth Order Difference Equation

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Abstract. In this paper, we introduce an explicit formula for the solutions and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-3}}{b - cx_{n-1}x_{n-3}}, \qquad n = 0, 1, \ldots$$

where a, b, c are positive real numbers and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are real numbers.

1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [3, 7, 9, 10, 11, 12, 13, 14, 16, 17] and the references therein.

In [8], E.M. Elsayed determined the solutions to some difference equations. He obtained the solution to the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 - x_{n-1}x_{n-3}}, \quad n = 0, 1, ... \tag{1.1}$$

where the initial conditions are arbitrary nonzero positive real numbers. But he didn't point to any constraints on the initial conditions.

In fact, if we start with initial conditions $x_0 = 2, x_{-1} = 1, x_{-2} = 1, x_{-3} = 0.5$ in equation (1.1), then undefined value for x_3 will be obtained. Therefore, additional information about the initial conditions must be given for any solution of equation (1.1) to be well-defined.

In [4], M. Aloqeili discussed the stability properties and semicycle behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

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with real initial conditions and positive real number a.

In [1], we have discussed the oscillation, boundedness and the global behavior of all admissible solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B - Cx_n x_{n-2}}, \quad n = 0, 1, \ldots$$

where A, B, C are positive real numbers.

In [2], we have also discussed the oscillation, periodicity, boundedness and the global behavior of all admissible solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B - C\prod_{i=l}^{k} x_{n-2i}}, \quad n = 0, 1, \ldots$$

where A, B, C are positive real numbers.

In [5], the authors investigated the asymptotic behavior of solutions of the equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}}, \qquad n = 0, 1, \ldots$$

with positive parameters a and c, negative parameter b and nonnegative initial conditions.

In [6], they also used the explicit formula for the solutions of the equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}}, \qquad n = 0, 1, \ldots$$

with positive parameters and nonnegative initial conditions in investigating their behavior.

In [15], H. Sedaghat determined the global behavior of all solutions of the rational difference equations

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \quad x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \qquad n = 0, 1, \ldots$$

where $a, b > 0$.

In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-3}}{b - cx_{n-1} x_{n-3}}, \qquad n = 0, 1, \ldots \tag{1.2}$$

where a, b, c are positive real numbers and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are real numbers.

2. Solution of Equation (1.2)

We define $\alpha_i = x_{-2+i} x_{-4+i}$, $i = 1, 2$.

Theorem 2.1. *Let x_{-3}, x_{-2}, x_{-1} and x_0 be real numbers such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{b}{c\sum_{k=0}^{n}(\frac{a}{b})^k}$ for all $n \in \mathbb{N}$. If $a \neq b$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.2) is*

$$x_n = \begin{cases} x_{-3}\prod_{j=0}^{\frac{n-1}{4}} \frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c} & , n = 1, 5, 9, ... \\ x_{-2}\prod_{j=0}^{\frac{n-2}{4}} \frac{(\frac{b}{a})^{2j}\theta_2 - c}{(\frac{b}{a})^{2j+1}\theta_2 - c} & , n = 2, 6, 10, ... \\ x_{-1}\prod_{j=0}^{\frac{n-3}{4}} \frac{(\frac{b}{a})^{2j+1}\theta_1 - c}{(\frac{b}{a})^{2j+2}\theta_1 - c} & , n = 3, 7, 11, ... \\ x_{0}\prod_{j=0}^{\frac{n-4}{4}} \frac{(\frac{b}{a})^{2j+1}\theta_2 - c}{(\frac{b}{a})^{2j+2}\theta_2 - c} & , n = 4, 8, 12, ... \end{cases} \tag{2.1}$$

where $\theta_i = \frac{a-b+c\alpha_i}{\alpha_i}$, $\alpha_i = x_{-2+i}x_{-4+i}$, and $i = 1, 2$.

Proof. We can write the given solution as

$$x_{4m+1} = x_{-3}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}, \quad x_{4m+2} = x_{-2}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j}\theta_2 - c}{(\frac{b}{a})^{2j+1}\theta_2 - c},$$

$$x_{4m+3} = x_{-1}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j+1}\theta_1 - c}{(\frac{b}{a})^{2j+2}\theta_1 - c}, \quad x_{4m+4} = x_{0}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j+1}\theta_2 - c}{(\frac{b}{a})^{2j+2}\theta_2 - c}, \quad m = 0, 1, \ldots$$

It is easy to check the result when $m = 0$. Suppose that the result is true for $m > 0$. Then

$$\begin{aligned} x_{4(m+1)+1} =& \frac{ax_{4m+1}}{b - cx_{4m+1}x_{4m+3}} = \frac{ax_{-3}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}}{b - cx_{-3}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}x_{-1}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j+1}\theta_1 - c}{(\frac{b}{a})^{2j+2}\theta_1 - c}} \\ =& \frac{ax_{-3}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}}{b - cx_{-3}(\prod_{j=0}^{m}(\frac{b}{a})^{2j}\theta_1 - c)x_{-1}\prod_{j=0}^{m}\frac{1}{(\frac{b}{a})^{2j+2}\theta_1 - c}} \\ =& \frac{ax_{-3}\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}}{b - cx_{-1}x_{-3}(\theta_1 - c)(\frac{1}{(\frac{b}{a})^{2m+2}\theta_1 - c})} \\ =& \frac{ax_{-3}((\frac{b}{a})^{2m+2}\theta_1 - c)\prod_{j=0}^{m} \frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}}{b((\frac{b}{a})^{2m+2}\theta_1 - c) - c\alpha_1(\theta_1 - c)} \end{aligned}$$

$$=\frac{ax_{-3}((\frac{b}{a})^{2m+2}\theta_1 - c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1-c}{(\frac{b}{a})^{2j+1}\theta_1-c}}{b((\frac{b}{a})^{2m+2}\theta_1 - c) - c(a-b)}$$

$$=\frac{x_{-3}((\frac{b}{a})^{2m+2}\theta_1 - c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1-c}{(\frac{b}{a})^{2j+1}\theta_1-c}}{\frac{b}{a}((\frac{b}{a})^{2m+2}\theta_1 - c) - \frac{c}{a}(a-b)}$$

$$=x_{-3}\frac{(\frac{b}{a})^{2m+2}\theta_1 - c)}{((\frac{b}{a})^{2m+3}\theta_1 - c)}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}$$

$$=x_{-3}\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j}\theta_1 - c}{(\frac{b}{a})^{2j+1}\theta_1 - c}.$$

Similarly we can show that

$$x_{4(m+1)+2} = x_{-2}\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j}\theta_2 - c}{(\frac{b}{a})^{2j+1}\theta_2 - c}, \quad x_{4(m+1)+3} = x_{-1}\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j+1}\theta_1 - c}{(\frac{b}{a})^{2j+2}\theta_1 - c}$$

and

$$x_{4(m+1)+4} = x_0\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{2j+1}\theta_2 - c}{(\frac{b}{a})^{2j+2}\theta_2 - c}.$$

This completes the proof. □

3. Global Behavior of Equation (1.2)

In this section, we investigate the global behavior of equation (1.2) with $a \neq b$, using the explicit formula of its solution.

We can write the solution of equation (1.2) as

$$x_{4m+2t+i} = x_{-4+2t+i}\prod_{j=0}^{m}\zeta(j,t,i),$$

where $\zeta(j,t,i) = \frac{(\frac{b}{a})^{2j+t}\theta_i - c}{(\frac{b}{a})^{2j+t+1}\theta_i - c}$, $t \in \{0,1\}$ and $i \in \{1,2\}$.

Theorem 3.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation* (1.2) *such that for any $i \in \{1,2\}$, $\alpha_i \neq -\frac{b}{c\sum_{k=0}^{n}(\frac{a}{b})^k}$ for all $n \in \mathbb{N}$. If $\alpha_i = \frac{b-a}{c}$ for all $i \in \{1,2\}$, then $\{x_n\}_{n=-3}^{\infty}$ is periodic with prime period* 4.

Proof. Assume that $\alpha_i = \frac{b-a}{c}$ for all $i \in \{1,2\}$. Then $\theta_i = 0$ for all $i \in \{1,2\}$. Therefore,

$$x_{4m+2t+i} = x_{-4+2t+i}\prod_{j=0}^{m}\zeta(j,t,i) = x_{-4+2t+i}, \quad m = 0,1,...$$

This completes the proof. □

In the following Theorem, suppose that $\alpha_i \neq \frac{b-a}{c}$ for all $i \in \{1,2\}$.

Theorem 3.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (1.2) such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{b}{c\sum_{k=0}^{n}(\frac{a}{b})^k}$ for all $n \in \mathbb{N}$. Then the following statements are true.*

1. *If $a < b$, then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.*
2. *If $a > b$, then $\{x_n\}_{n=-3}^{\infty}$ converges to a period-4 solution.*

Proof.

1. If $a < b$, then $\zeta(j,t,i)$ converges to $\frac{a}{b} < 1$ as $j \to \infty$, for all $t \in \{0,1\}$ and $i \in \{1,2\}$. So, for every pair $(t,i) \in \{0,1\} \times \{1,2\}$ we have for a given $0 < \epsilon < 1$ that, there exists $j_0(t,i) \in \mathbb{N}$ such that, $\mid \zeta(j,t,i) \mid < \epsilon$ for all $j \geq j_0(t,i)$. If we set $j_0 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_0(t,i)$, then for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we get

$$\begin{aligned} \mid x_{4m+2t+i} \mid &= \mid x_{-4+2t+i} \mid \mid \prod_{j=0}^{m} \zeta(j,t,i) \mid \\ &= \mid x_{-4+2t+i} \mid \mid \prod_{j=0}^{j_0-1} \zeta(j,t,i) \mid \mid \prod_{j=j_0}^{m} \zeta(j,t,i) \mid \\ &< \mid x_{-4+2t+i} \mid \mid \prod_{j=0}^{j_0-1} \zeta(j,t,i) \mid \epsilon^{m-j_0+1}. \end{aligned}$$

   As m tends to infinity, the solution $\{x_n\}_{n=-3}^{\infty}$ converges to 0.

2. If $a > b$, then $\zeta(j,t,i) \to 1$ as $j \to \infty$, $t \in \{0,1\}$ and $i \in \{1,2\}$. This implies that, for every pair $(t,i) \in \{0,1\} \times \{1,2\}$, there exists $j_1(t,i) \in \mathbb{N}$ such that $\zeta(j,t,i) > 0$ for all $t \in \{0,1\}$ and $i \in \{1,2\}$. If we set $j_1 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_1(t,i)$, then for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we get

$$\begin{aligned} x_{4m+2t+i} &= x_{-4+2t+i} \prod_{j=0}^{m} \zeta(j,t,i) \\ &= x_{-4+2t+i} \prod_{j=0}^{j_1-1} \zeta(j,t,i) \exp\big(\sum_{j=j_1}^{m} \ln(\zeta(j,t,i))\big). \end{aligned}$$

   We shall test the convergence of the series $\sum_{j=j_1}^{\infty} \mid \ln(\zeta(j,t,i)) \mid$.
   Since for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we have $\lim_{j\to\infty} \mid \frac{\ln(\zeta(j+1,t,i)}{\ln(\zeta(j,t,i)} \mid = \frac{0}{0}$, using L'Hospital's rule we obtain

$$\lim_{j\to\infty} \mid \frac{\ln \zeta(j+1,t,i)}{\ln \zeta(j,t,i)} \mid = (\frac{b}{a})^2 < 1.$$

It follows from the ratio test that the series $\sum_{j=j_1}^{\infty} \mid \ln\zeta(j,t,i) \mid$ is convergent. This ensures that there are four positive real numbers μ_{ti}, $t \in \{0,1\}$ and $i \in \{1,2\}$ such that

$$\lim_{m\to\infty} x_{4m+2t+i} = \mu_{ti}, \quad t \in \{0,1\} \quad \text{and} \quad i \in \{1,2\}$$

where

$$\mu_{ti} = x_{-4+2t+i} \prod_{j=0}^{\infty} \frac{(\frac{b}{a})^{2j+t}\theta_i - c}{(\frac{b}{a})^{2j+t+1}\theta_i - c}, \quad t \in \{0,1\} \quad \text{and} \quad i \in \{1,2\}.$$

This completes the proof. □

Example (1) Figure 1. shows that if $a = 2$, $b = 3$, $c = 1$ $(a < b)$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.2) with initial conditions $x_{-3} = 0.2$, $x_{-2} = 2$, $x_{-1} = -2$ and $x_0 = 0.4$ converges to 0.

Example (2) Figure 2. shows that if $a = 3$, $b = 1$, $c = 0.8$ $(a > b)$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.2) with initial conditions $x_{-3} = 0.2$, $x_{-2} = 2$, $x_{-1} = -2$ and $x_0 = 0.4$ converges to a period-4 solution.

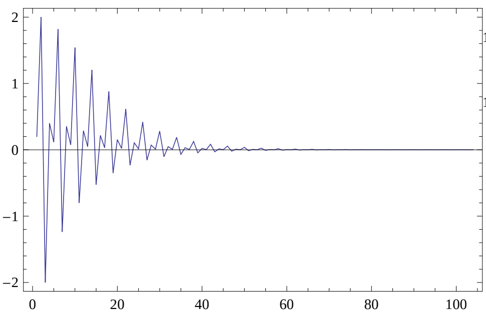

Figure 1: $x_{n+1} = \frac{2x_{n-3}}{3-x_{n-1}x_{n-3}}$

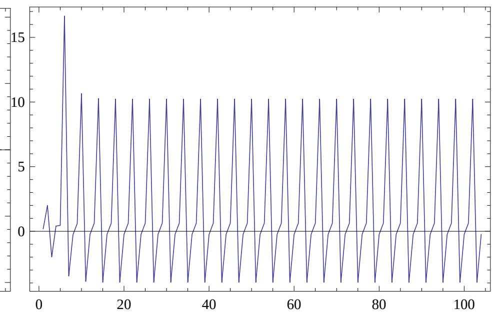

Figure 2: $x_{n+1} = \frac{3x_{n-3}}{1-0.8x_{n-1}x_{n-3}}$

4. Case $a = b$

In this section, we investigate the behavior of the solution of the difference equation

(3.1)
$$x_{n+1} = \frac{ax_{n-3}}{a - cx_{n-1}x_{n-3}}, \qquad n = 0, 1, \ldots$$

Theorem 4.1. *Let x_{-3}, x_{-2}, x_{-1} and x_0 be real numbers such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{a}{c(n+1)}$ for all $n \in \mathbb{N}$. Then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.1) is*

$$x_n = \begin{cases} x_{-3} \prod_{j=0}^{\frac{n-1}{4}} \frac{a-(2j)c\alpha_1}{a-(2j+1)c\alpha_1} & , n = 1, 5, 9, ... \\ x_{-2} \prod_{j=0}^{\frac{n-2}{4}} \frac{a-(2j)c\alpha_2}{a-(2j+1)c\alpha_2} & , n = 2, 6, 10, ... \\ x_{-1} \prod_{j=0}^{\frac{n-3}{4}} \frac{a-(2j+1)c\alpha_1}{a-(2j+2)c\alpha_1} & , n = 3, 7, 11, ... \\ x_0 \prod_{j=0}^{\frac{n-4}{4}} \frac{a-(2j+1)c\alpha_2}{a-(2j+2)c\alpha_2} & , n = 4, 8, 12, ... \end{cases} \tag{3.2}$$

Proof. The proof is similar to that of Theorem (2.1) and will be omitted. □

We can write the solution of equation (3.1) as

$$x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j,t,i),$$

where $\gamma(j,t,i) = \frac{a-(2j+t)c\alpha_i}{a-(2j+t+1)c\alpha_i}$, $t \in \{0,1\}$ and $i \in \{1,2\}$.

Theorem 4.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial solution of equation* (3.1) *such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{a}{c(n+1)}$ for all $n \in \mathbb{N}$. If $\alpha_i = 0$ for all $i \in \{1,2\}$, then $\{x_n\}_{n=-3}^{\infty}$ is periodic with prime period* 4.

Proof. Assume that $\alpha_i = 0$ for all $i \in \{1,2\}$. Then $\gamma(j,t,i) = 1$ for all $t \in \{0,1\}$ and $i \in \{1,2\}$. Therefore,

$$x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j,t,i) = x_{-4+2t+i}, \quad m = 0, 1, ...$$

This completes the proof. □

In the following Theorem, suppose that $\alpha_i \neq 0$ for all $i \in \{1,2\}$.

Theorem 4.3. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation* (3.1) *such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{a}{c(n+1)}$ for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=-3}^{\infty}$ converges to* 0.

Proof. It is clear that $\gamma(j,t,i) \to 1$ as $j \to \infty$, $t \in \{0,1\}$ and $i \in \{1,2\}$. This implies that, for every pair $(t,i) \in \{0,1\} \times \{1,2\}$ there exists $j_2(t,i) \in \mathbb{N}$ such that, $\gamma(j,t,i) > 0$ for all $t \in \{0,1\}$ and $i \in \{1,2\}$. If we set $j_2 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_2(t,i)$, then for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we get

$$\begin{aligned} x_{4m+2t+i} &= x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j,t,i) \\ &= x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j,t,i) \exp\left(-\sum_{j=j_2}^{m} \ln \frac{1}{\gamma(j,t,i)}\right). \end{aligned}$$

We shall show that $\sum_{j=j_2}^{\infty} \ln \frac{1}{\gamma(j,t,i)} = \sum_{j=j_2}^{\infty} \ln \frac{a-(2j+t+1)c\alpha_i}{a-(2j+t)c\alpha_i} = \infty$, by considering the series $\sum_{j=j_2}^{\infty} \frac{-c\alpha_i}{a-(2j+t)c\alpha_i}$. As

$$\lim_{j\to\infty} \frac{\ln(1/\gamma(j,t,i))}{-c\alpha_i/(a-(2j+t))c\alpha_i} = \lim_{j\to\infty} \frac{\ln((a-(2j+t+1)c\alpha_i)/(a-(2j+t)c\alpha_i))}{-c\alpha_i/(a-(2j+t)c\alpha_i)} = 1,$$

using the limit comparison test, we get $\sum_{j=j_2}^{\infty} \ln \frac{1}{\gamma(j,t,i)} = \infty$.Then

$$x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j,t,i) \exp(-\sum_{j=j_2}^{m} \ln \frac{1}{\gamma(j,t,i)})$$

converges to 0 as $m \to \infty$. Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to 0. □

5. Case $a = b = c$

In this section, we investigate the behavior of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 - x_{n-1}x_{n-3}}, \qquad n = 0, 1, \ldots \tag{3.3}$$

Theorem 5.1. *Let x_{-3}, x_{-2}, x_{-1} and x_0 be real numbers such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.3) is*

$$x_n = \begin{cases} x_{-3} \prod_{j=0}^{\frac{n-1}{4}} \frac{1-(2j)\alpha_1}{1-(2j+1)\alpha_1} & , n = 1, 5, 9, ... \\ x_{-2} \prod_{j=0}^{\frac{n-2}{4}} \frac{1-(2j)\alpha_2}{1-(2j+1)\alpha_2} & , n = 2, 6, 10, ... \\ x_{-1} \prod_{j=0}^{\frac{n-3}{4}} \frac{1-(2j+1)\alpha_1}{1-(2j+2)\alpha_1} & , n = 3, 7, 11, ... \\ x_0 \prod_{j=0}^{\frac{n-4}{4}} \frac{1-(2j+1)\alpha_2}{1-(2j+2)\alpha_2} & , n = 4, 8, 12, ... \end{cases} \tag{3.4}$$

Proof. The proof is similar to that of Theorem (2.1) and will be omitted. □

Theorem 5.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial solution of equation (3.3) such that for any $i \in \{1,2\}$, $\alpha_i \neq -\frac{b}{c\sum_{k=0}^{n}(\frac{a}{b})^k}$ for all $n \in \mathbb{N}$. If $\alpha_i = 0$ for all $i \in \{1,2\}$, then $\{x_n\}_{n=-3}^{\infty}$ is periodic with prime period 4.*

Proof. Assume that $\alpha_i = 0$ for all $i \in \{1,2\}$. Then

$$x_{4m+2t+i} = x_{-4+2t+i}, \quad m = 0, 1, ...$$

This completes the proof. □

In the following Theorem, suppose that $\alpha_i \neq 0$ for all $i \in \{1,2\}$.

Theorem 5.3. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (3.3) such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.*

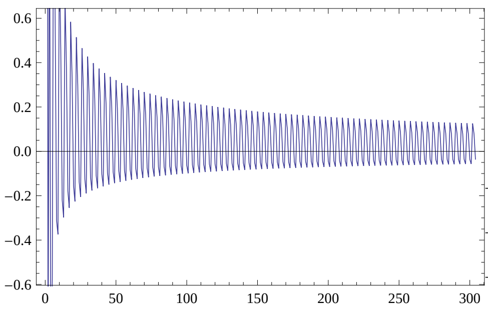

Figure 3: $x_{n+1} = \frac{x_{n-3}}{1-1.5x_{n-1}x_{n-3}}$

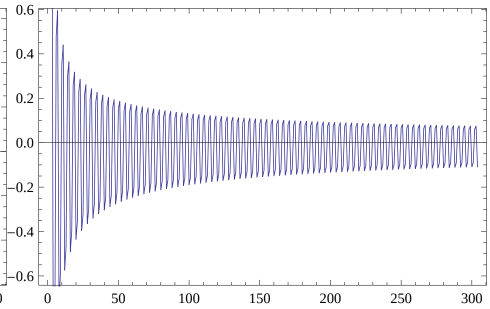

Figure 4: $x_{n+1} = \frac{x_{n-3}}{1-x_{n-1}x_{n-3}}$

Example (3) Figure 3. shows that if $a = b = 1$, $c = 1.5$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.1) with initial conditions$x_{-3} = 5$, $x_{-2} = -1$, $x_{-1} = 1.3$ and $x_0 = -1.1$ converges to 0.

Example (4) Figure 4. shows that if $a = b = c$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.3) with initial conditions $x_{-3} = 5$, $x_{-2} = 1$, $x_{-1} = 1.3$ and $x_0 = -1.1$ converges to 0.

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