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Recursion Formulas for Exton's triple Hypergeometric Functions

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Abstract. This paper continues the study of recursion formulas of multivariable hypergeometric functions. Earlier, in [4], the authors have given the recursion formulas for three variable Lauricella functions, Srivastava's triple hypergeometric functions and k–variable Lauricella functions. Further, in [5], we have obtained recursion formulas for the general triple hypergeometric function. We present here the recursion formulas for Exton's triple hypergeometric functions.

1. Introduction

Recursion formulas for multivariable hypergeometric functions have received considerable attention recently. In this direction, Opps, Saad and Srivastava [2] and Wang [8] have studied the recursion relations of Appell functions. The authors, in [4], have started a systematic study of recursion formulas of multivariable hypergeometric functions including fourteen three variable Lauricella functions, three Srivastava's triple hypergeometric functions and four k–variable Lauricella functions. In [5], we have obtained recursion formulas for the general triple hypergeometric function [6]. These results unify and generalize the results in [4] for the three variable hypergeometric function. In this paper, we obtain recursion formulas for Exton's triple hypergeometric functions [1].

Exton [1] gave the following hypergeometric functions of three variables, denoted by $X_1, X_2, \ldots, X_{20}$. We recall their definition below:

(1.1)
$$X_1(a_1, a_2; c_1, c_2; x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+2m_2+m_3}(a_2)_{m_3}}{(c_1)_{m_2+m_3}(c_2)_{m_1}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

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(1.2)

$$X_2(a_1,\, a_2;\, c_1,\, c_2,\, c_3;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+2m_2+m_3}\,(a_2)_{m_3}}{(c_1)_{m_1}(c_2)_{m_2}(c_3)_{m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.3)

$$X_3(a_1,\, a_2;\, c_1,\, c_2;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3}\,(a_2)_{m_2+m_3}}{(c_1)_{m_1+m_2}(c_2)_{m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.4)

$$X_4(a_1,\, a_2;\, c_1,\, c_2,\, c_3;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3}\,(a_2)_{m_2+m_3}}{(c_1)_{m_1}(c_2)_{m_2}(c_3)_{m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.5)

$$X_5(a_1,\, a_2,\, a_3;\, c;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3}\,(a_2)_{m_2}\,(a_3)_{m_3}}{(c)_{m_1+m_2+m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.6)

$$X_6(a_1,\, a_2,\, a_3;\, c_1,\, c_2;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3}\,(a_2)_{m_2}\,(a_3)_{m_3}}{(c_1)_{m_1+m_2}(c_2)_{m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.7)

$$X_7(a_1,\, a_2,\, a_3;\, c_1,\, c_2;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3}\,(a_2)_{m_2}\,(a_3)_{m_3}}{(c_1)_{m_2+m_3}(c_2)_{m_1}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.8)

$$X_8(a_1, a_2, a_3; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2+m_3}(a_2)_{m_2}(a_3)_{m_3}}{(c_1)_{m_1}(c_2)_{m_2}(c_3)_{m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.9)

$$X_9(a_1,\, a_2;\, c;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2}\,(a_2)_{m_2+2m_3}}{(c)_{m_1+m_2+m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.10)

$$X_{10}(a_1,\, a_2;\, c_1,\, c_2;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2}\,(a_2)_{m_2+2m_3}}{(c_1)_{m_1+m_2}(c_2)_{m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.11)

$$X_{11}(a_1,\, a_2;\, c_1,\, c_2;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2}\,(a_2)_{m_2+2m_3}}{(c_1)_{m_1+m_3}(c_2)_{m_2}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.12)

$$X_{12}(a_1,\, a_2;\, c_1,\, c_2,\, c_3;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2}\,(a_2)_{m_2+2m_3}}{(c_1)_{m_1}(c_2)_{m_2}(c_3)_{m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.13)

$$X_{13}(a_1,\, a_2,\, a_3;\, c;\, x_1, x_2, x_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{2m_1+m_2}\,(a_2)_{m_2+m_3}(a_3)_{m_3}}{(c)_{m_1+m_2+m_3}} \prod_{i=1}^{3} \frac{{x_i}^{m_i}}{m_i!},$$

(1.14)

$$X_{14}(a_1,a_2,a_3;c_1,c_2;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a_1)_{2m_1+m_2}(a_2)_{m_2+m_3}(a_3)_{m_3}}{(c_1)_{m_1+m_2}(c_2)_{m_3}}\prod_{i=1}^{3}\frac{{x_i}^{m_i}}{m_i!},$$

(1.15)

$$X_{15}(a_1,a_2,a_3;c_1,c_2;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a_1)_{2m_1+m_2}(a_2)_{m_2+m_3}(a_3)_{m_3}}{(c_1)_{m_2+m_3}(c_2)_{m_1}}\prod_{i=1}^{3}\frac{{x_i}^{m_i}}{m_i!},$$

(1.16)

$$X_{16}(a_1,a_2,a_3;c_1,c_2;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a_1)_{2m_1+m_2}(a_2)_{m_2+m_3}(a_3)_{m_3}}{(c_1)_{m_1+m_3}(c_2)_{m_2}}\prod_{i=1}^{3}\frac{{x_i}^{m_i}}{m_i!},$$

(1.17)

$$X_{17}(a_1,a_2,a_3;c_1,c_2,c_3;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a_1)_{2m_1+m_2}(a_2)_{m_2+m_3}(a_3)_{m_3}}{(c_1)_{m_1}(c_2)_{m_2}(c_3)_{m_3}}\prod_{i=1}^{3}\frac{{x_i}^{m_i}}{m_i!},$$

(1.18)

$$X_{18}(a_1,a_2,a_3,a_4;c;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a_1)_{2m_1+m_2}(a_2)_{m_2}(a_3)_{m_3}(a_4)_{m_3}}{(c)_{m_1+m_2+m_3}}\prod_{i=1}^{3}\frac{{x_i}^{m_i}}{m_i!},$$

(1.19)

$$X_{19}(a_1,a_2,a_3,a_4;c_1,c_2;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a_1)_{2m_1+m_2}(a_2)_{m_2}(a_3)_{m_3}(a_4)_{m_3}}{(c_1)_{m_2+m_3}(c_2)_{m_1}}\prod_{i=1}^{3}\frac{{x_i}^{m_i}}{m_i!},$$

(1.20)

$$X_{20}(a_1,a_2,a_3,a_4;c_1,c_2;x_1,x_2,x_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a_1)_{2m_1+m_2}(a_2)_{m_2}(a_3)_{m_3}(a_4)_{m_3}}{(c_1)_{m_1+m_3}(c_2)_{m_2}}\prod_{i=1}^{3}\frac{{x_i}^{m_i}}{m_i!}.$$

Here, $(a)_m$ is the Pochhammer symbol [3, 7]. For convergence of the above listed twenty Exton's triple hypergeometric functions, see [6].

The paper is organised as follows. There are 20 further sections, one for each of the twenty Exton's triple hypergeometric functions. Following abbreviated notations are used in the paper. We, for example, write X_1 for the series $X_1(a_1,\, a_2;\, c_1,\, c_2;\, x_1, x_2, x_3)$, $X_1(a_1{+}n)$ for the series $X_1(a_1{+}n, a_2;\, c_1,\, c_2;\, x_1, x_2, x_3)$ and $X_1(a_1+n,\, c_1+n_1)$ stands for the series $X_1(a_1+n,\, a_2;\, c_1+n_1,\, c_2;\, x_1, x_2, x_3)$, etc. Throughout n denotes a non-negative integer.

2. Recursion Formulas for X_1

Theorem 2.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_1 :*

$$X_1(a_1+n) \;=\; X_1+\frac{2\,x_1}{c_2}\sum_{n_1=1}^{n}(a_1+n_1)\,X_1(a_1+1+n_1,\,c_2+1)$$

$$+\frac{2\,x_2}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,X_1(a_1+1+n_1,\,c_1+1)$$

$$+\frac{a_2\,x_3}{c_1}\sum_{n_1=1}^{n}X_1(a_1+n_1,\,a_2+1,c_1+1), \tag{2.1}$$

$$X_1(a_1-n)\;=\;X_1-\frac{2\,x_1}{c_2}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_1(a_1+2-n_1,\,c_2+1)$$

$$-\frac{2\,x_2}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_1(a_1+2-n_1,\,c_1+1)$$

$$-\frac{a_2\,x_3}{c_1}\sum_{n_1=1}^{n}X_1(a_1+1-n_1,\,a_2+1\,,c_1+1). \tag{2.2}$$

and

$$X_1(a_2+n)\;=\;X_1+\frac{a_1\,x_3}{c_1}\sum_{n_1=1}^{n}X_1(a_1+1,\,a_2+n_1,\,c_1+1), \tag{2.3}$$

$$X_1(a_2-n)\;=\;X_1-\frac{a_1\,x_3}{c_1}\sum_{n_1=1}^{n}X_1(a_1+1,\,a_2+1-n_1,\,c_1+1), \tag{2.4}$$

Results (2.3) and (2.4) can also be represented as follows

$$X_1(a_2+n)\;=\;\sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}x_3^{n_1}}{(c_1)_{n_1}}X_1(a_1+n_1,\,a_2+n_1,\,c_1+n_1), \tag{2.5}$$

$$X_1(a_2-n)\;=\;\sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-x_3)^{n_1}}{(c_1)_{n_1}}X_1(a_1+n_1,\;c_1+n_1). \tag{2.6}$$

Proof. From the definition of Exton's triple hypergeometric function X_1 and the relation

$$(a_1+1)_{2m_1+2m_2+m_3}=(a_1)_{2m_1+2m_2+m_3}\Big(1+\frac{2m_1}{a_1}+\frac{2m_2}{a_1}+\frac{m_3}{a_1}\Big) \tag{2.7}$$

we obtain the following contiguous relation:

$$X_1(a_1+1)\;=\;X_1+\frac{2x_1}{c_2}(a_1+1)\,X_1(a_1+2,\,c_2+1)+\frac{2x_2}{c_1}(a_1+1)\,X_1(a_1+2,\,c_1+1)$$

$$+\frac{a_2x_3}{c_1}\,X_1(a_1+1,\,a_2+1,\,c_1+1). \tag{2.8}$$

To obtain contiguous relation for $X_1(a_1+2)$, we replace $a_1 \to a_1+1$ in (2.8) and simplify. This leads to

$$\begin{aligned} X_1(a_1+2) = X_1 &+ \frac{2x_1}{c_2}\sum_{n_1=1}^{2}(a_1+n_1)\,X_1(a_1+n_1+1,\,c_2+1) \\ &+ \frac{2x_2}{c_1}\sum_{n_1=1}^{2}(a_1+n_1)\,X_1(a_1+n_1+1,\,c_1+1) \\ &+ \frac{a_2x_3}{c_1}\sum_{n_1=1}^{2}X_1(a_1+n_1,\,a_2+1,\,c_1+1). \end{aligned} \tag{2.9}$$

Iterating this process n times, we obtain (2.1). For the proof of (2.2), replace the parameter $a_1 \to a_1-1$ in (2.8). This gives

$$\begin{aligned} X_1(a_1-1) \;=\; X_1 &- \frac{2x_1}{c_2}a_1\,X_1(a_1+1\,,c_2+1) - \frac{2x_2}{c_1}a_1\,X_1(a_1+1,\,c_1+1) \\ &- \frac{a_2x_3}{c_1}\,X_1(a_2+1,\,c_1+1). \end{aligned} \tag{2.10}$$

Iteratively, we get (2.2).

The recursion formulas (2.3) and (2.4) can be proved in similar manner. Now, we give the proof of (2.5).

Using the definition of Exton's triple hypergeometric function X_1 and the relation

$$(a_2+1)_{m_3} = (a_2)_{m_3}\Big(1+\frac{m_3}{a_2}\Big) \tag{2.11}$$

we get

$$X_1(a_2+1) = X_1 + \frac{a_1x_3}{c_1}\,X_1(a_1+1,\,a_2+1,\,c_1+1). \tag{2.12}$$

Replace $a_2 \to a_2+1$ in (2.12) to get

$$\begin{aligned} X_1(a_2+2) &= X_1 + \frac{a_1x_3}{c_1}\,X_1(a_1+1,\,a_2+1,\,c_1+1) \\ &+ \frac{a_1x_3}{c_1}\Big[X_1(a_1+1,\,a_2+1,\,c_1+1) + \frac{(a_1+1)x_3}{(c_1+1)}\,X_1(a_1+2,\,a_2+2,\,c_1+2)\Big]. \end{aligned} \tag{2.13}$$

Simplifying, we get

$X_1(a_2+2)$

$$= X_1 + \frac{2a_1x_3}{c_1} X_1(a_1+1,\, a_2+1,\, c_1+1) + \frac{(a_1)_2\, x_3^2}{(c_1)_2} X_1(a_1+2,\, a_2+2,\, c_1+2) \tag{2.14}$$

Iterating this process n times, we get (2.5). Proof of (2.6) is similar. □

By using Pascal's identity the recursion formulas (2.5) and (2.6) can also be proved by induction method.

Theorem 2.2. *The following recursion formulas hold true for the denominator parameters* $c_1,\, c_2$ *of the Exton's triple hypergeometric function* X_1 *:*

$$\begin{aligned} X_1(c_1-n) = X_1\, &+ (a_1)_2\, x_2 \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)} X_1(a_1+2,\, c_1+2-n_1) \\ &+ a_1\, a_2\, x_3 \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)} X_1(a_1+1,\, a_2+1,\, c_1+2-n_1), \end{aligned} \tag{2.15}$$

$$X_1(c_2-n) = X_1\, + (a_1)_2\, x_1 \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)} X_1(a_1+2,\, c_2+2-n_1), \tag{2.16}$$

The above result can also be expressed as

$$X_1(c_2-n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{x_1^{n_1}(a_1)_{2n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}} X_1(a_1+2n_1,\, c_2+n_1). \tag{2.17}$$

Proof. Using the definition of Exton's triple hypergeometric function X_1 and the relation

$$\frac{1}{(c_1-1)_{m_2+m_3}} = \frac{1}{(c_1)_{m_2+m_3}} \left(1 + \frac{m_2}{c_1-1} + \frac{m_3}{c_1-1}\right) \tag{2.18}$$

we have

$$\begin{aligned} X_1(c_1-1) = X_1 &+ \frac{(a_1)_2\, x_2}{c_1(c_1-1)} X_1(a_1+2,\, c_1+1) \\ &+ \frac{a_1a_2x_3}{c_1(c_1-1)} X_1(a_1+1,\, a_2+1,\, c_1+1). \end{aligned} \tag{2.19}$$

Using this contiguous relation to the X_1 with the parameter $c_1 - n$ for n times, we get result (2.15). Second recursion formula (2.16) can be proved in a similar manner.

Now, we give the proof of (2.17). Apply the definition of X_1 and the relation

$$\frac{1}{(c_2-1)_{m_1}} = \frac{1}{(c_2)_{m_1}}\Big(1+\frac{m_1}{c_2-1}\Big) \tag{2.20}$$

to get

$$X_1(c_2-1) = X_1 + \frac{(a_1)_2 x_1}{c_2(c_2-1)} X_1(a_1+2,\, c_2+1) \tag{2.21}$$

Replacing $c_2 \to c_2 - 1$ in (2.21) and simplifying leads to

$$\begin{aligned} X_1(c_2-2) = X_1 &+ \frac{2(a_1)_2\, x_1}{c_2(c_2-2)} X_1(a_1+2,\, c_2+1) \\ &+ \frac{(a_1)_4\, x_1^2}{(c_2)_2(c_2-2)_2} X_1(a_1+4,\, c_2+2) \end{aligned} \tag{2.22}$$

Iteratively, we get (2.17). □

Now, we present the recursion formulas for other Exton's triple hypergeometric functions. We omit the proof of the given below theorems.

3. Recursion Formulas for X_2

Theorem 3.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_2 :*

$$\begin{aligned} X_2(a_1+n) \;=\; X_2 &+ \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,X_2(a_1+1+n_1,\, c_1+1) \\ &+ \frac{2\,x_2}{c_2}\sum_{n_1=1}^{n}(a_1+n_1)\,X_2(a_1+1+n_1,\, c_2+1) \\ &+ \frac{a_2\,x_3}{c_3}\sum_{n_1=1}^{n} X_2(a_1+n_1,\, a_2+1,\, c_3+1), \end{aligned} \tag{3.1}$$

$$\begin{aligned} X_2(a_1-n) \;=\; X_2 &- \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_2(a_1+2-n_1,\, c_1+1) \\ &- \frac{2\,x_2}{c_2}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_2(a_1+2-n_1,\, c_2+1) \\ &- \frac{a_2\,x_3}{c_3}\sum_{n_1=1}^{n} X_2(a_1+1-n_1,\, a_2+1,\, c_3+1), \end{aligned} \tag{3.2}$$

$$X_2(a_2+n) \;=\; X_2 + \frac{a_1\,x_3}{c_3}\sum_{n_1=1}^{n} X_2(a_1+1,\, a_2+n_1,\, c_3+1), \tag{3.3}$$

$$X_2(a_2 - n) \;=\; X_2 - \frac{a_1\, x_3}{c_3} \sum_{n_1=1}^{n} X_2(a_1+1,\, a_2+1-n_1,\, c_3+1). \tag{3.4}$$

Theorem 3.2. *The following recursion formulas hold true for the denominator parameter a_2 of the Exton's triple hypergeometric function X_2 :*

$$X_2(a_2 + n) \;=\; \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_1)_{n_1} x_3^{n_1}}{(c_3)_{n_1}} X_2(a_1+n_1,\, a_2+n_1,\, c_3+n_1), \tag{3.5}$$

$$X_2(a_2 - n) \;=\; \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_1)_{n_1} (-x_3)^{n_1}}{(c_3)_{n_1}} X_2(a_1+n_1,\; c_3+n_1). \tag{3.6}$$

Theorem 3.3. *The following recursion formulas hold true for the denominator parameters c_i, $i = 1, 2$, and c_3 of the Exton's triple hypergeometric function X_2 :*

(3.7)

$$X_2(c_i - n) = X_2 \; + (a_1)_2\, x_i \sum_{n_1=1}^{n} \frac{1}{(c_i - n_1)(c_i + 1 - n_1)}\, X_2(a_1+2,\, c_i+2-n_1),$$

(3.8)

$$X_2(c_i - n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{x_i^{n_1} (a_1)_{2n_1}}{(c_i)_{n_1} (c_i - n)_{n_1}}\, X_2(a_1+2n_1,\, c_i+n_1),$$

(3.9)

$$X_2(c_3 - n) = X_2 \; + a_1 a_2\, x_3 \sum_{n_1=1}^{n} \frac{1}{(c_3 - n_1)(c_3 + 1 - n_1)}\, X_2(a_1+1, a_2+1,\, c_3+2-n_1),$$

(3.10)

$$X_2(c_3 - n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{x_3^{n_1} (a_1)_{n_1} (a_2)_{n_1}}{(c_3)_{n_1} (c_3 - n)_{n_1}}\, X_2(a_1+n_1, a_2+n_1\, c_3+n_1).$$

4. Recursion Formulas for X_3

Theorem 4.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_3 :*

$$\begin{aligned} X_3(a_1 + n) \;=\; X_3 &+ \frac{2\, x_1}{c_1} \sum_{n_1=1}^{n} (a_1+n_1)\, X_3(a_1+1+n_1,\, c_1+1) \\ &+ \frac{a_2\, x_2}{c_1} \sum_{n_1=1}^{n} X_3(a_1+n_1,\, a_2+1,\, c_1+1) \\ &+ \frac{a_2\, x_3}{c_2} \sum_{n_1=1}^{n} X_3(a_1+n_1,\, a_2+1, c_2+1), \end{aligned} \tag{4.1}$$

$$X_3(a_1-n) \;=\; X_3 - \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_3(a_1+2-n_1,\,c_1+1)$$
$$-\frac{a_2\,x_2}{c_1}\sum_{n_1=1}^{n}X_3(a_1+1-n_1,\,a_2+1,\,c_1+1)$$
$$-\frac{a_2\,x_3}{c_2}\sum_{n_1=1}^{n}X_3(a_1+1-n_1,\,a_2+1,c_2+1), \tag{4.2}$$

$$X_3(a_2+n) = X_3 + \frac{a_1\,x_2}{c_1}\sum_{n_1=1}^{n}X_3(a_1+1,\,a_2+n_1,\,c_1+1)$$
$$+\frac{a_1\,x_3}{c_2}\sum_{n_1=1}^{n}X_3(a_1+1,\,a_2+n_1,\,c_2+1), \tag{4.3}$$

$$X_3(a_2-n) = X_3 - \frac{a_1\,x_2}{c_1}\sum_{n_1=1}^{n}X_3(a_1+1,\,a_2+1-n_1,\,c_1+1)$$
$$-\frac{a_1\,x_3}{c_2}\sum_{n_1=1}^{n}X_3(a_1+1,\,a_2+1-n_1,\,c_2+1). \tag{4.4}$$

Theorem 4.2. *The following recursion formulas hold true for the numerator parameter a_2 of the Exton's triple hypergeometric function X_3 :*

$$X_3(a_2+n) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{N_2}\,x_2^{n_1}\,x_3^{n_2}}{(c_1)_{n_1}(c_2)_{n_2}}$$
$$\times\; X_3(a_1+N_2,\,a_2+N_2,\,c_1+n_1,\,c_2+n_2), \tag{4.5}$$
$$X_3(a_2-n) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{N_2}(-x_2)^{n_1}(-x_3)^{n_2}}{(c_1)_{n_1}(c_2)_{n_2}}$$
$$\times\; X_3(a_1+N_2,\,c_1+n_1,\,c_2+n_2), \tag{4.6}$$

where $N_2 = n_1+n_2$.

Theorem 4.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_3 :*

$$X_3(c_1-n) = X_3 \;+ (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_3(a_1+2,\,c_1+2-n_1)$$
$$+\;a_1\,a_2\,x_2\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_3(a_1+1,\,a_2+1,\,c_1+2-n_1), \tag{4.7}$$

$X_3(c_2-n)$

(4.8)

$$= X_3 + a_1 a_2 x_3 \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)} X_3(a_1+1,\, a_2+1,\, c_2+2-n_1),$$

(4.9)

$$X_3(c_2-n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{x_3^{n_1}(a_1)_{n_1}(a_2)_{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}} X_3(a_1+n_1,\, a_2+n_1,\, c_2+n_1).$$

5. Recursion Formulas for X_4

Theorem 5.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_4 :*

$$X_4(a_1+n) = X_4 + \frac{2\,x_1}{c_1} \sum_{n_1=1}^{n} (a_1+n_1)\, X_4(a_1+1+n_1,\, c_1+1)$$

(5.1)

$$+ \frac{a_2\,x_2}{c_2} \sum_{n_1=1}^{n} X_4(a_1+n_1,\, a_2+1,\, c_2+1) + \frac{a_2\,x_3}{c_3} \sum_{n_1=1}^{n} X_4(a_1+n_1,\, a_2+1, c_3+1),$$

$$X_4(a_1-n) = X_4 - \frac{2\,x_1}{c_1} \sum_{n_1=1}^{n} (a_1-n_1+1)\, X_4(a_1+2-n_1,\, c_1+1)$$

$$- \frac{a_2\,x_2}{c_2} \sum_{n_1=1}^{n} X_4(a_1+1-n_1,\, a_2+1,\, c_2+1)$$

(5.2)

$$- \frac{a_2\,x_3}{c_3} \sum_{n_1=1}^{n} X_4(a_1+1-n_1,\, a_2+1, c_3+1),$$

$$X_4(a_2+n) = X_4 + \frac{a_1\,x_2}{c_2} \sum_{n_1=1}^{n} X_4(a_1+1,\, a_2+n_1,\, c_2+1)$$

(5.3)

$$+ \frac{a_1\,x_3}{c_3} \sum_{n_1=1}^{n} X_4(a_1+1,\, a_2+n_1,\, c_3+1),$$

$$X_4(a_2-n) = X_4 - \frac{a_1\,x_2}{c_2} \sum_{n_1=1}^{n} X_4(a_1+1,\, a_2+1-n_1,\, c_2+1)$$

(5.4)

$$- \frac{a_1\,x_3}{c_3} \sum_{n_1=1}^{n} X_4(a_1+1,\, a_2+1-n_1,\, c_3+1).$$

Theorem 5.2. *The following recursion formulas hold true for the numerator parameter a_2 of the Exton's triple hypergeometric function X_4 :*

$$X_4(a_2+n) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{N_2}\,x_2^{n_1}x_3^{n_2}}{(c_2)_{n_1}(c_3)_{n_2}}$$
$$\times\; X_4(a_1+N_2,\, a_2+N_2,\, c_2+n_1,\, c_3+n_2), \tag{5.5}$$

$$X_4(a_2-n) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{N_2}(-x_2)^{n_1}(-x_3)^{n_2}}{(c_2)_{n_1}(c_3)_{n_2}}$$
$$\times\; X_4(a_1+N_2,\, c_2+n_1,\, c_3+n_2), \tag{5.6}$$

where $N_2 = n_1+n_2$.

Theorem 5.3. *The following recursion formulas hold true for the denominator parameters $c_1,\, c_i;\, i=2,3$ of the Exton's triple hypergeometric function X_4 :*

(5.7)
$$X_4(c_1-n) = X_4 + (a_1)_2\, x_1 \sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_4(a_1+2,\, c_1+2-n_1),$$

(5.8)
$$X_4(c_1-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_1^{n_1}(a_1)_{2n_1}}{(c_1)_{n_1}(c_1-n)_{n_1}}\, X_4(a_1+2n_1,\, c_1+n_1),$$

$X_4(c_i-n)$

(5.9)
$$= X_4 + a_1a_2\, x_i \sum_{n_1=1}^{n}\frac{1}{(c_i-n_1)(c_i+1-n_1)}\, X_4(a_1+1,\, a_2+1,\, c_i+2-n_1),$$

(5.10)
$$X_4(c_i-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_i^{n_1}(a_1)_{n_1}(a_2)_{n_1}}{(c_i)_{n_1}(c_i-n)_{n_1}}\, X_4(a_1+n_1,\, a_2+n_1,\, c_i+n_1).$$

6. Recursion Formulas for X_5

Theorem 6.1. *The following recursion formulas hold true for the numerator parameters $a_1, a_i;\, i=2,3,$ of the Exton's triple hypergeometric function X_5 :*

$X_5(a_1+n)$

$$= X_5 + \frac{2\,x_1}{c}\sum_{n_1=1}^{n}(a_1+n_1)\, X_5(a_1+1+n_1,\, c+1)$$

$$
+\frac{a_2\,x_2}{c}\sum_{n_1=1}^{n} X_5(a_1+n_1,\, a_2+1,\, c+1)+\frac{a_3\,x_3}{c}\sum_{n_1=1}^{n} X_5(a_1+n_1,\, a_3+1, c+1), \tag{6.1}
$$

$$
\begin{aligned}
X_5(a_1-n) \;=\; X_5 &- \frac{2\,x_1}{c}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_5(a_1+2-n_1,\, c+1)\\
&- \frac{a_2\,x_2}{c}\sum_{n_1=1}^{n} X_5(a_1+1-n_1,\, a_2+1,\, c+1)\\
&- \frac{a_3\,x_3}{c}\sum_{n_1=1}^{n} X_5(a_1+1-n_1,\, a_3+1, c+1),
\end{aligned} \tag{6.2}
$$

$$
X_5(a_i+n) \;=\; X_5+\frac{a_1\,x_i}{c}\sum_{n_1=1}^{n} X_5(a_1+1,\, a_i+n_1,\, c+1), \tag{6.3}
$$

$$
X_5(a_i-n) \;=\; X_5-\frac{a_1\,x_i}{c}\sum_{n_1=1}^{n} X_5(a_1+1,\, a_i+1-n_1,\, c+1), \tag{6.4}
$$

Theorem 6.2. *The following recursion formulas hold true for the numerator parameters a_i ; $i=2,3$, of the Exton's triple hypergeometric function X_5 :*

$$
X_5(a_i+n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}x_i^{n_1}}{(c)_{n_1}}X_5(a_1+n_1,\, a_i+n_1,\, c+n_1), \tag{6.5}
$$

$$
X_5(a_i-n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-x_i)^{n_1}}{(c)_{n_1}}X_5(a_1+n_1,\; c+n_1). \tag{6.6}
$$

Theorem 6.3. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_5 :*

$$
\begin{aligned}
X_5(c-n) = X_5 &+ (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}\,X_5(a_1+2,\, c+2-n_1)\\
&+ a_1\,a_2\,x_2\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}\,X_5(a_1+1,\, a_2+1,\, c+2-n_1)\\
&+ a_1\,a_3\,x_3\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}\,X_5(a_1+1,\, a_3+1,\, c+2-n_1).
\end{aligned} \tag{6.7}
$$

7. Recursion Formulas for X_6

Theorem 7.1. *The following recursion formulas hold true for the numerator parameters a_1, a_i; $i = 2, 3$, of the Exton's triple hypergeometric function X_6 :*

$$\begin{aligned} X_6(a_1+n) \;=\; & X_6 + \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,X_6(a_1+1+n_1,\,c_1+1) \\ & + \frac{a_2\,x_2}{c_1}\sum_{n_1=1}^{n} X_6(a_1+n_1,\,a_2+1,\,c_1+1) \\ & + \frac{a_3\,x_3}{c_2}\sum_{n_1=1}^{n} X_6(a_1+n_1,\,a_3+1, c_2+1), \end{aligned} \tag{7.1}$$

$$\begin{aligned} X_6(a_1-n) \;=\; & X_6 - \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_6(a_1+2-n_1,\,c_1+1) \\ & - \frac{a_2\,x_2}{c_1}\sum_{n_1=1}^{n} X_6(a_1+1-n_1,\,a_2+1,\,c_1+1) \\ & - \frac{a_3\,x_3}{c_2}\sum_{n_1=1}^{n} X_6(a_1+1-n_1,\,a_3+1, c_2+1), \end{aligned} \tag{7.2}$$

$$X_6(a_i+n) \;=\; X_6 + \frac{a_1\,x_i}{c_{i-1}}\sum_{n_1=1}^{n} X_6(a_1+1,\,a_i+n_1,\,c_{i-1}+1), \tag{7.3}$$

$$X_6(a_i-n) \;=\; X_6 - \frac{a_1\,x_i}{c_{i-1}}\sum_{n_1=1}^{n} X_6(a_1+1,\,a_i+1-n_1,\,c_{i-1}+1). \tag{7.4}$$

Theorem 7.2. *The following recursion formulas hold true for the numerator parameters a_i; $i = 2, 3$, of the Exton's triple hypergeometric function X_6 :*

$$X_6(a_i+n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}x_i^{n_1}}{(c_{i-1})_{n_1}}\,X_6(a_1+n_1,\,a_i+n_1,\,c_{i-1}+n_1), \tag{7.5}$$

$$X_6(a_i-n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-x_i)^{n_1}}{(c_{i-1})_{n_1}}\,X_6(a_1+n_1,\; c_{i-1}+n_1). \tag{7.6}$$

Theorem 7.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_6 :*

$$\begin{aligned} X_6(c_1-n) = X_6 \; & + (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_6(a_1+2,\,c_1+2-n_1) \\ & + \,a_1\,a_2\,x_2\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_6(a_1+1,\,a_2+1,\,c_1+2-n_1), \end{aligned} \tag{7.7}$$

$X_6(c_2-n)$

(7.8)

$$= X_6 + a_1 a_3\, x_3 \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)}\, X_6(a_1+1,\, a_3+1,\, c_2+2-n_1),$$

(7.9)

$$X_6(c_2-n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{x_3^{n_1}(a_1)_{n_1}(a_3)_{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}}\, X_6(a_1+n_1,\, a_3+n_1,\, c_2+n_1).$$

8. Recursion Formulas for X_7

Theorem 8.1. *The following recursion formulas hold true for the numerator parameters a_1, a_i; $i = 2, 3$, of the Exton's triple hypergeometric function X_7 :*

$$\begin{aligned} X_7(a_1+n) \;=\; & X_7 + \frac{2\,x_1}{c_2} \sum_{n_1=1}^{n} (a_1+n_1)\, X_7(a_1+1+n_1,\, c_2+1) \\ & + \frac{a_2\,x_2}{c_1} \sum_{n_1=1}^{n} X_7(a_1+n_1,\, a_2+1,\, c_1+1) \\ & + \frac{a_3\,x_3}{c_1} \sum_{n_1=1}^{n} X_7(a_1+n_1,\, a_3+1, c_1+1), \end{aligned} \tag{8.1}$$

$$\begin{aligned} X_7(a_1-n) \;=\; & X_7 - \frac{2\,x_1}{c_2} \sum_{n_1=1}^{n} (a_1+1-n_1)\, X_7(a_1+2-n_1,\, c_2+1) \\ & - \frac{a_2\,x_2}{c_1} \sum_{n_1=1}^{n} X_7(a_1+1-n_1,\, a_2+1,\, c_1+1) \\ & - \frac{a_3\,x_3}{c_1} \sum_{n_1=1}^{n} X_7(a_1+1-n_1,\, a_3+1, c_1+1), \end{aligned} \tag{8.2}$$

$$X_7(a_i+n) \;=\; X_7 + \frac{a_1\,x_i}{c_1} \sum_{n_1=1}^{n} X_7(a_1+1,\, a_i+n_1,\, c_1+1), \tag{8.3}$$

$$X_7(a_i-n) \;=\; X_7 - \frac{a_1\,x_i}{c_1} \sum_{n_1=1}^{n} X_7(a_1+1,\, a_i+1-n_1,\, c_1+1). \tag{8.4}$$

Theorem 8.2. *The following recursion formulas hold true for the numerator parameters a_i ; $i = 2, 3$, of the Exton's triple hypergeometric function X_7 :*

$$X_7(a_i+n) \;=\; \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_1)_{n_1} x_i^{n_1}}{(c_1)_{n_1}}\, X_7(a_1+n_1,\, a_i+n_1,\, c_1+n_1), \tag{8.5}$$

(8.6) $$X_7(a_i - n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_1)_{n_1}(-x_i)^{n_1}}{(c_1)_{n_1}} X_7(a_1 + n_1,\ c_1 + n_1).$$

Theorem 8.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_7 :*

$$X_7(c_1 - n)$$

$$= X_7 + a_1\, a_2\, x_2 \sum_{n_1=1}^{n} \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_7(a_1 + 1,\ a_2 + 1,\ c_1 + 2 - n_1)$$

(8.7)

$$+ a_1\, a_3\, x_3 \sum_{n_1=1}^{n} \frac{1}{(c_1 - n_1)(c_1 + 1 - n_1)} X_7(a_1 + 1,\ a_3 + 1,\ c_1 + 2 - n_1),$$

(8.8)

$$X_7(c_2 - n) = X_7 + (a_1)_2\, x_1 \sum_{n_1=1}^{n} \frac{1}{(c_2 - n_1)(c_2 + 1 - n_1)} X_7(a_1 + 2,\ c_2 + 2 - n_1),$$

(8.9)

$$X_7(c_2 - n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{x_1^{n_1}(a_1)_{2n_1}}{(c_2)_{n_1}(c_2 - n)_{n_1}} X_7(a_1 + 2n_1,\ c_2 + n_1).$$

9. Recursion Formulas for X_8

Theorem 9.1. *The following recursion formulas hold true for the numerator parameters a_1, a_i; $i = 2, 3$, of the Exton's triple hypergeometric function X_8 :*

$$X_8(a_1 + n) = X_8 + \frac{2\, x_1}{c_1} \sum_{n_1=1}^{n} (a_1 + n_1)\, X_8(a_1 + 1 + n_1,\ c_1 + 1)$$

$$+ \frac{a_2\, x_2}{c_2} \sum_{n_1=1}^{n} X_8(a_1 + n_1,\ a_2 + 1,\ c_2 + 1)$$

(9.1) $$+ \frac{a_3\, x_3}{c_3} \sum_{n_1=1}^{n} X_8(a_1 + n_1,\ a_3 + 1, c_3 + 1),$$

$$X_8(a_1 - n) = X_8 - \frac{2\, x_1}{c_1} \sum_{n_1=1}^{n} (a_1 + 1 - n_1)\, X_8(a_1 + 2 - n_1,\ c_1 + 1)$$

$$- \frac{a_2\, x_2}{c_2} \sum_{n_1=1}^{n} X_8(a_1 + 1 - n_1,\ a_2 + 1,\ c_2 + 1)$$

(9.2) $$- \frac{a_3\, x_3}{c_3} \sum_{n_1=1}^{n} X_8(a_1 + 1 - n_1,\ a_3 + 1, c_3 + 1),$$

$$X_8(a_i+n) \;=\; X_8 + \frac{a_1\,x_i}{c_i}\sum_{n_1=1}^{n} X_8(a_1+1,\, a_i+n_1,\, c_i+1), \tag{9.3}$$

$$X_8(a_i-n) \;=\; X_8 - \frac{a_1\,x_i}{c_i}\sum_{n_1=1}^{n} X_8(a_1+1,\, a_i+1-n_1,\, c_i+1). \tag{9.4}$$

Theorem 9.2. *The following recursion formulas hold true for the numerator parameters a_i; $i = 2, 3$, of the Exton's triple hypergeometric function X_8 :*

$$X_8(a_i+n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1} x_i^{n_1}}{(c_i)_{n_1}}\, X_8(a_1+n_1,\, a_i+n_1,\, c_i+n_1), \tag{9.5}$$

$$X_8(a_i-n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-x_i)^{n_1}}{(c_i)_{n_1}}\, X_8(a_1+n_1,\; c_i+n_1). \tag{9.6}$$

Theorem 9.3. *The following recursion formulas hold true for the denominator parameters c_1, c_i, $i = 2, 3$, of the Exton's triple hypergeometric function X_8 :*

(9.7)

$$X_8(c_1-n) = X_8 \,+ (a_1)_2\, x_1 \sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_8(a_1+2,\, c_1+2-n_1),$$

(9.8)

$$X_8(c_1-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_1^{n_1}(a_1)_{2n_1}}{(c_1)_{n_1}(c_1-n)_{n_1}}\, X_8(a_1+2n_1,\, c_1+n_1),$$

$X_8(c_i-n)$

(9.9)

$$= X_8 \,+ a_1\, a_i\, x_i \sum_{n_1=1}^{n}\frac{1}{(c_i-n_1)(c_i+1-n_1)}\, X_8(a_1+1,\, a_i+1,\, c_i+2-n_1),$$

(9.10)

$$X_8(c_i-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(a_i)_{n_1} x_i^{n_1}}{(c_i)_{n_1}(c_i-n)_{n_1}}\, X_8(a_1+n_1,\, a_i+n_1,\, c_i+n_1).$$

10. Recursion Formulas for X_9

Theorem 10.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_9 :*

$$X_9(a_1+n) \;=\; X_9 + \frac{2\,x_1}{c}\sum_{n_1=1}^{n}(a_1+n_1)\, X_9(a_1+1+n_1,\, c+1)$$

$$+\frac{a_2\,x_2}{c}\sum_{n_1=1}^{n} X_9(a_1+n_1,\,a_2+1,\,c+1), \tag{10.1}$$

$$X_9(a_1-n) \;=\; X_9-\frac{2\,x_1}{c}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_9(a_1+2-n_1,\,c+1)$$
$$-\frac{a_2\,x_2}{c}\sum_{n_1=1}^{n} X_9(a_1+1-n_1,\,a_2+1,\,c+1), \tag{10.2}$$

$$X_9(a_2+n) \;=\; X_9+\frac{a_1\,x_2}{c}\sum_{n_1=1}^{n} X_9(a_2+n_1,\,a_1+1,\,c+1)$$
$$+\frac{2\,x_3}{c}\sum_{n_1=1}^{n}(a_2+n_1)\,X_9(a_2+1+n_1,\,c+1), \tag{10.3}$$

$$X_9(a_2-n) \;=\; X_9-\frac{a_1\,x_2}{c}\sum_{n_1=1}^{n} X_9(a_2+1-n_1,\,a_1+1,\,c+1)$$
$$-\frac{2\,x_3}{c}\sum_{n_1=1}^{n}(a_2+1-n_1)\,X_9(a_2+2-n_1,\,c+1). \tag{10.4}$$

Theorem 10.2. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_9 :*

$$\begin{aligned} X_9(c-n) = X_9\; &+ (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}\,X_9(a_1+2,\,c+2-n_1)\\ &+ a_1\,a_2\,x_2\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}\,X_9(a_1+1,\,a_2+1,\,c+2-n_1)\\ &+ (a_2)_2\,x_3\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}\,X_9(a_2+2,\,c+2-n_1). \end{aligned} \tag{10.5}$$

11. Recursion Formulas for X_{10}

Theorem 11.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2 of the Exton's triple hypergeometric function X_{10} :*

$$X_{10}(a_1+n) \;=\; X_{10}+\frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,X_{10}(a_1+1+n_1,\,c_1+1)$$
$$+\frac{a_2\,x_2}{c_1}\sum_{n_1=1}^{n} X_{10}(a_1+n_1,\,a_2+1,\,c_1+1), \tag{11.1}$$

$$X_{10}(a_1 - n) = X_{10} - \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_{10}(a_1+2-n_1,\,c_1+1)$$
$$- \frac{a_2\,x_2}{c_1}\sum_{n_1=1}^{n} X_{10}(a_1+1-n_1,\,a_2+1,\,c_1+1), \tag{11.2}$$

$$X_{10}(a_2 + n) = X_{10} + \frac{a_1\,x_2}{c_1}\sum_{n_1=1}^{n} X_{10}(a_2+n_1,\,a_1+1,\,c_1+1)$$
$$+ \frac{2\,x_3}{c_2}\sum_{n_1=1}^{n}(a_2+n_1)\,X_{10}(a_2+1+n_1,\,c_2+1), \tag{11.3}$$

$$X_{10}(a_2 - n) = X_{10} - \frac{a_1\,x_2}{c_1}\sum_{n_1=1}^{n} X_{10}(a_2+1-n_1,\,a_1+1,\,c_1+1)$$
$$- \frac{2\,x_3}{c_2}\sum_{n_1=1}^{n}(a_2+1-n_1)\,X_{10}(a_2+2-n_1,\,c_2+1). \tag{11.4}$$

Theorem 11.2. *The following recursion formulas hold true for the denominator parameters* $c_1,\,c_2$ *of the Exton's triple hypergeometric function* X_{10} :

$$X_{10}(c_1 - n) = X_{10} + (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{10}(a_1+2,\,c_1+2-n_1)$$
(11.5)
$$+ a_1\,a_2\,x_2\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{10}(a_1+1,\,a_2+1,\,c_1+2-n_1),$$
(11.6)
$$X_{10}(c_2 - n) = X_{10} + (a_2)_2\,x_3\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)}\,X_{10}(a_2+2,\,c_2+2-n_1),$$
(11.7)
$$X_{10}(c_2 - n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_3^{n_1}(a_2)_{2n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}}\,X_{10}(a_2+2n_1,\,c_2+n_1).$$

12. Recursion Formulas for X_{11}

Theorem 12.1. *The following recursion formulas hold true for the numerator parameters* $a_1,\,a_2$ *of the Exton's triple hypergeometric function* X_{11} :

$$X_{11}(a_1 + n) = X_{11} + \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,X_{11}(a_1+1+n_1,\,c_1+1)$$

$$+\frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{11}(a_1+n_1,\,a_2+1,\,c_2+1), \tag{12.1}$$

$$X_{11}(a_1-n) = X_{11}-\frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_{11}(a_1+2-n_1,\,c_1+1)$$
$$-\frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{11}(a_1+1-n_1,\,a_2+1,\,c_2+1), \tag{12.2}$$

$$X_{11}(a_2+n) = X_{11}+\frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{11}(a_2+n_1,\,a_1+1,\,c_2+1)$$
$$+\frac{2\,x_3}{c_1}\sum_{n_1=1}^{n}(a_2+n_1)\,X_{11}(a_2+1+n_1,\,c_1+1), \tag{12.3}$$

$$X_{11}(a_2-n) = X_{11}-\frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{11}(a_2+1-n_1,\,a_1+1,\,c_2+1)$$
$$-\frac{2\,x_3}{c_1}\sum_{n_1=1}^{n}(a_2+1-n_1)\,X_{11}(a_2+2-n_1,\,c_1+1). \tag{12.4}$$

Theorem 12.2. *The following recursion formulas hold true for the denominator parameters $c_1,\,c_2$ of the Exton's triple hypergeometric function X_{11} :*

$$X_{11}(c_1-n) = X_{11} + (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{11}(a_1+2,\,c_1+2-n_1)$$
$$+ (a_2)_2\,x_3\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{11}(a_2+2,\,c_1+2-n_1), \tag{12.5}$$

$$X_{11}(c_2-n) = X_{11} + a_1\,a_2x_2\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)}\,X_{11}(a_1+1,\,a_2+1,\,c_2+2-n_1), \tag{12.6}$$

$$X_{11}(c_2-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_2^{n_1}(a_1)_{n_1}(a_2)_{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}}\,X_{11}(a_1+n_1,\,a_2+n_1,\,c_2+n_1). \tag{12.7}$$

13. Recursion Formulas for X_{12}

Theorem 13.1. *The following recursion formulas hold true for the numerator pa-*

rameters a_1, a_2 of the Exton's triple hypergeometric function X_{12} :

$$\begin{aligned} X_{12}(a_1+n) \;=\; & X_{12} + \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,X_{12}(a_1+1+n_1,\,c_1+1) \\ & + \frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{12}(a_1+n_1,\,a_2+1,\,c_2+1), \end{aligned} \tag{13.1}$$

$$\begin{aligned} X_{12}(a_1-n) \;=\; & X_{12} - \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_{12}(a_1+2-n_1,\,c_1+1) \\ & - \frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{12}(a_1+1-n_1,\,a_2+1,\,c_2+1), \end{aligned} \tag{13.2}$$

$$\begin{aligned} X_{12}(a_2+n) \;=\; & X_{12} + \frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{12}(a_2+n_1,\,a_1+1,\,c_2+1) \\ & + \frac{2\,x_3}{c_3}\sum_{n_1=1}^{n}(a_2+n_1)\,X_{12}(a_2+1+n_1,\,c_3+1), \end{aligned} \tag{13.3}$$

$$\begin{aligned} X_{12}(a_2-n) \;=\; & X_{12} - \frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{12}(a_2+1-n_1,\,a_1+1,\,c_2+1) \\ & - \frac{2\,x_3}{c_3}\sum_{n_1=1}^{n}(a_2+1-n_1)\,X_{12}(a_2+2-n_1,\,c_3+1). \end{aligned} \tag{13.4}$$

Theorem 13.2. *The following recursion formulas hold true for the denominator parameters c_1, c_2 and c_3 of the Exton's triple hypergeometric function X_{12} :*

$$\begin{aligned} & X_{12}(c_1-n) \\ & = X_{12} + (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{12}(a_1+2,\,c_1+2-n_1), \end{aligned} \tag{13.5}$$

$$\begin{aligned} & X_{12}(c_2-n) \\ & = X_{12} + a_1\,a_2 x_2\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)}\,X_{12}(a_1+1,\,a_2+1,\,c_2+2-n_1), \end{aligned} \tag{13.6}$$

$$\begin{aligned} & X_{12}(c_3-n) \\ & = X_{12} + (a_2)_2\,x_3\sum_{n_1=1}^{n}\frac{1}{(c_3-n_1)(c_3+1-n_1)}\,X_{12}(a_2+2,\,c_3+2-n_1). \end{aligned} \tag{13.7}$$

Theorem 13.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 and c_3 of the Exton's triple hypergeometric function X_{12} :*

$$X_{12}(c_1-n)=\sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_1^{n_1}(a_1)_{2n_1}}{(c_1)_{n_1}(c_1-n)_{n_1}}\,X_{12}(a_1+2n_1,\,c_1+n_1), \tag{13.8}$$

$$X_{12}(c_2-n)=\sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_2^{n_1}(a_1)_{n_1}(a_2)_{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}}\,X_{12}(a_1+n_1,\,a_2+n_1,\,c_2+n_1), \tag{13.9}$$

$$X_{12}(c_3-n)=\sum_{n_1=0}^{n}\binom{n}{n_1}\frac{x_3^{n_1}(a_2)_{2n_1}}{(c_3)_{n_1}(c_3-n)_{n_1}}\,X_{12}(a_2+2n_1,\,c_3+n_1). \tag{13.10}$$

14. Recursion Formulas for X_{13}

Theorem 14.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{13} :*

$$\begin{aligned} X_{13}(a_1+n) &= X_{13}+\frac{2\,x_1}{c}\sum_{n_1=1}^{n}(a_1+n_1)\,X_{13}(a_1+1+n_1,\,c+1)\\ &\quad+\frac{a_2\,x_2}{c}\sum_{n_1=1}^{n}X_{13}(a_1+n_1,\,a_2+1,\,c+1), \end{aligned} \tag{14.1}$$

$$\begin{aligned} X_{13}(a_1-n) &= X_{13}-\frac{2\,x_1}{c}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_{13}(a_1+2-n_1,\,c+1)\\ &\quad-\frac{a_2\,x_2}{c}\sum_{n_1=1}^{n}X_{13}(a_1+1-n_1,\,a_2+1,\,c+1). \end{aligned} \tag{14.2}$$

$$\begin{aligned} X_{13}(a_2+n) &= X_{13}+\frac{a_1\,x_2}{c}\sum_{n_1=1}^{n}X_{13}(a_2+n_1,\,a_1+1,\,c+1)\\ &\quad+\frac{a_3\,x_3}{c}\sum_{n_1=1}^{n}X_{13}(a_2+n_1,\,a_3+1,\,c+1), \end{aligned} \tag{14.3}$$

$$\begin{aligned} X_{13}(a_2-n) &= X_{13}-\frac{a_1\,x_2}{c}\sum_{n_1=1}^{n}X_{13}(a_2+1-n_1,\,a_1+1,\,c+1)\\ &\quad-\frac{a_3\,x_3}{c}\sum_{n_1=1}^{n}X_{13}(a_2+1-n_1,\,a_3+1,\,c+1), \end{aligned} \tag{14.4}$$

$$X_{13}(a_3+n) = X_{13}+\frac{a_2\,x_3}{c}\sum_{n_1=1}^{n}X_{13}(a_3+n_1,\,a_2+1,\,c+1), \tag{14.5}$$

$$X_{13}(a_3 - n) = X_{13} - \frac{a_2\, x_3}{c} \sum_{n_1=1}^{n} X_{13}(a_3 + 1 - n_1,\, a_2 + 1,\, c + 1). \tag{14.6}$$

Theorem 14.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{13} :*

$$\begin{aligned} X_{13}(a_2 + n) &= \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} x_2^{n_1} x_3^{n_2}}{(c)_{N_2}} \\ &\quad \times X_{13}(a_1 + n_1,\, a_2 + N_2,\, a_3 + n_2,\, c + N_2), \end{aligned} \tag{14.7}$$

$$\begin{aligned} X_{13}(a_2 - n) &= \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c)_{N_2}} \\ &\quad \times X_{13}(a_1 + n_1,\, a_3 + n_2,\, c + N_2), \end{aligned} \tag{14.8}$$

$$X_{13}(a_3 + n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c)_{n_1}} X_{13}(a_3 + n_1,\, a_2 + n_1,\, c + n_1), \tag{14.9}$$

$$X_{13}(a_3 - n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c)_{n_1}} X_{13}(\, a_2 + n_1,\, c + n_1), \tag{14.10}$$

where $N_2 = n_1 + n_2$.

Theorem 14.3. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_{13} :*

$$\begin{aligned} X_{13}(c - n) &= X_{13} + (a_1)_2\, x_1 \sum_{n_1=1}^{n} \frac{1}{(c - n_1)(c + 1 - n_1)} X_{13}(a_1 + 2,\, c + 2 - n_1) \\ &\quad + a_1\, a_2\, x_2 \sum_{n_1=1}^{n} \frac{1}{(c - n_1)(c + 1 - n_1)} X_{13}(a_1 + 1,\, a_2 + 1,\, c + 2 - n_1) \\ &\quad + a_2\, a_3 x_3 \sum_{n_1=1}^{n} \frac{1}{(c - n_1)(c + 1 - n_1)} X_{13}(a_2 + 1,\, a_3 + 1,\, c + 2 - n_1). \end{aligned} \tag{14.11}$$

15. Recursion Formulas for X_{14}

Theorem 15.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{14} :*

$$\begin{aligned} X_{14}(a_1 + n) &= X_{14} + \frac{2\, x_1}{c_1} \sum_{n_1=1}^{n} (a_1 + n_1)\, X_{14}(a_1 + 1 + n_1,\, c_1 + 1) \\ &\quad + \frac{a_2\, x_2}{c_1} \sum_{n_1=1}^{n} X_{14}(a_1 + n_1,\, a_2 + 1,\, c_1 + 1), \end{aligned} \tag{15.1}$$

$$X_{14}(a_1 - n) = X_{14} - \frac{2\,x_1}{c_1} \sum_{n_1=1}^{n} (a_1 + 1 - n_1)\, X_{14}(a_1 + 2 - n_1,\, c_1 + 1)$$

$$- \frac{a_2\,x_2}{c_1} \sum_{n_1=1}^{n} X_{14}(a_1 + 1 - n_1,\, a_2 + 1,\, c_1 + 1), \tag{15.2}$$

$$X_{14}(a_2 + n) = X_{14} + \frac{a_1\,x_2}{c_1} \sum_{n_1=1}^{n} X_{14}(a_2 + n_1,\, a_1 + 1,\, c_1 + 1)$$

$$+ \frac{a_3\,x_3}{c_2} \sum_{n_1=1}^{n} X_{14}(a_2 + n_1,\, a_3 + 1,\, c_2 + 1), \tag{15.3}$$

$$X_{14}(a_2 - n) = X_{14} - \frac{a_1\,x_2}{c_1} \sum_{n_1=1}^{n} X_{14}(a_2 + 1 - n_1,\, a_1 + 1,\, c_1 + 1)$$

$$- \frac{a_3\,x_3}{c_2} \sum_{n_1=1}^{n} X_{14}(a_2 + 1 - n_1,\, a_3 + 1,\, c_2 + 1), \tag{15.4}$$

$$X_{14}(a_3 + n) = X_{14} + \frac{a_2\,x_3}{c_2} \sum_{n_1=1}^{n} X_{14}(a_3 + n_1,\, a_2 + 1,\, c_2 + 1), \tag{15.5}$$

$$X_{14}(a_3 - n) = X_{14} - \frac{a_2\,x_3}{c_2} \sum_{n_1=1}^{n} X_{14}(a_3 + 1 - n_1,\, a_2 + 1,\, c_2 + 1). \tag{15.6}$$

Theorem 15.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{14} :*

$$X_{14}(a_2 + n) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} x_2^{n_1} x_3^{n_2}}{(c_1)_{n_1} (c_2)_{n_2}}$$

$$\times\, X_{14}(a_1 + n_1,\, a_2 + N_2,\, a_3 + n_2,\, c_1 + n_1,\, c_2 + n_2), \tag{15.7}$$

$$X_{14}(a_2 - n) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \binom{n}{n_1} \binom{n-n_1}{n_2} \frac{(a_1)_{n_1} (a_3)_{n_2} (-x_2)^{n_1} (-x_3)^{n_2}}{(c_1)_{n_1} (c_2)_{n_2}}$$

$$\times\, X_{14}(a_1 + n_1,\, a_3 + n_2,\, c_1 + n_1,\, c_2 + n_2), \tag{15.8}$$

$$X_{14}(a_3 + n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c_2)_{n_1}} X_{14}(a_3 + n_1,\, a_2 + n_1,\, c_2 + n_1), \tag{15.9}$$

$$X_{14}(a_3 - n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c_2)_{n_1}} X_{14}(\, a_2 + n_1,\, c_2 + n_1), \tag{15.10}$$

where $N_2 = n_1 + n_2$.

Theorem 15.3. *The following recursion formulas hold true for the denominator* *parameters c_1, c_2 of the Exton's triple hypergeometric function X_{14} :*

$$X_{14}(c_1-n) = X_{14} + (a_1)_2\, x_1 \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_{14}(a_1+2,\, c_1+2-n_1)$$
$$+a_1\, a_2\, x_2 \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_{14}(a_1+1,\, a_2+1,\, c_1+2-n_1), \tag{15.11}$$

$$X_{14}(c_2-n) = X_{14} + a_2\, a_3 x_3 \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)}$$
$$\times X_{14}(a_2+1,\, a_3+1,\, c_2+2-n_1), \tag{15.12}$$

$$X_{14}(c_2-n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1}(a_3)_{n_1} x_3^{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}}$$
$$\times X_{14}(\,a_2+n_1,\, a_3+n_1,\, c_2+n_1). \tag{15.13}$$

16. Recursion Formulas for X_{15}

Theorem 16.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{15} :*

$$X_{15}(a_1+n) = X_{15} + \frac{2\, x_1}{c_2} \sum_{n_1=1}^{n} (a_1+n_1)\, X_{15}(a_1+1+n_1,\, c_2+1)$$
$$+ \frac{a_2\, x_2}{c_1} \sum_{n_1=1}^{n} X_{15}(a_1+n_1,\, a_2+1,\, c_1+1), \tag{16.1}$$

$$X_{15}(a_1-n) = X_{15} - \frac{2\, x_1}{c_2} \sum_{n_1=1}^{n} (a_1+1-n_1)\, X_{15}(a_1+2-n_1,\, c_2+1)$$
$$- \frac{a_2\, x_2}{c_1} \sum_{n_1=1}^{n} X_{15}(a_1+1-n_1,\, a_2+1,\, c_1+1), \tag{16.2}$$

$$X_{15}(a_2+n) = X_{15} + \frac{a_1\, x_2}{c_1} \sum_{n_1=1}^{n} X_{15}(a_2+n_1,\, a_1+1,\, c_1+1)$$
$$+ \frac{a_3\, x_3}{c_1} \sum_{n_1=1}^{n} X_{15}(a_2+n_1,\, a_3+1,\, c_1+1), \tag{16.3}$$

$$X_{15}(a_2-n) = X_{15} - \frac{a_1\, x_2}{c_1} \sum_{n_1=1}^{n} X_{15}(a_2+1-n_1,\, a_1+1,\, c_1+1)$$

$$-\frac{a_3\,x_3}{c_1}\sum_{n_1=1}^{n} X_{15}(a_2+1-n_1,\,a_3+1,\,c_1+1), \tag{16.4}$$

$$X_{15}(a_3+n) = X_{15}+\frac{a_2\,x_3}{c_1}\sum_{n_1=1}^{n} X_{15}(a_3+n_1,\,a_2+1,\,c_1+1), \tag{16.5}$$

$$X_{15}(a_3-n) = X_{15}-\frac{a_2\,x_3}{c_1}\sum_{n_1=1}^{n} X_{15}(a_3+1-n_1,\,a_2+1,\,c_1+1). \tag{16.6}$$

Theorem 16.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{15} :*

$$X_{15}(a_2+n) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{n_1}(a_3)_{n_2}x_2^{n_1}x_3^{n_2}}{(c_1)_{N_2}} \times X_{15}(a_1+n_1,\,a_2+N_2,\,a_3+n_2,\,c_1+N_2), \tag{16.7}$$

$$X_{15}(a_2-n) = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{n_1}(a_3)_{n_2}(-x_2)^{n_1}(-x_3)^{n_2}}{(c_1)_{N_2}} \times X_{15}(a_1+n_1,\,a_3+n_2,\,c_1+N_2), \tag{16.8}$$

$$X_{15}(a_3+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_2)_{n_1}x_3^{n_1}}{(c_1)_{n_1}} X_{15}(a_3+n_1,\,a_2+n_1,\,c_1+n_1), \tag{16.9}$$

$$X_{15}(a_3-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_2)_{n_1}(-x_3)^{n_1}}{(c_1)_{n_1}} X_{15}(\,a_2+n_1,\,c_1+n_1), \tag{16.10}$$

where $N_2 = n_1+n_2$.

Theorem 16.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 of the Exton's triple hypergeometric function X_{15} :*

$$\begin{aligned} &X_{15}(c_1-n)\\ =\ & X_{15} + a_1\,a_2x_2\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)} X_{15}(a_1+1,\,a_2+1,\,c_1+2-n_1)\\ &+a_2\,a_3\,x_3\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)} X_{15}(a_2+1,\,a_3+1,\,c_1+2-n_1), \end{aligned} \tag{16.11}$$

$$X_{15}(c_2-n) = X_{15} + (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)} \times X_{15}(a_1+2,\,c_2+2-n_1), \tag{16.12}$$

$$X_{15}(c_2-n) = \sum_{n=0}^{n}\binom{n}{n_1}\frac{(a_1)_{2n_1}\,x_1^{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}} X_{15}(a_1+2n_1,\,c_2+n_1). \tag{16.13}$$

17. Recursion Formulas for X_{16}

Theorem 17.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 of the Exton's triple hypergeometric function X_{16} :*

$$\begin{aligned} X_{16}(a_1+n) \;=\; & X_{16} + \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,X_{16}(a_1+1+n_1,\,c_1+1) \\ & + \frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{16}(a_1+n_1,\,a_2+1,\,c_2+1), \end{aligned} \tag{17.1}$$

$$\begin{aligned} X_{16}(a_1-n) \;=\; & X_{16} - \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_{16}(a_1+2-n_1,\,c_1+1) \\ & - \frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{16}(a_1+1-n_1,\,a_2+1,\,c_2+1), \end{aligned} \tag{17.2}$$

$$\begin{aligned} X_{16}(a_2+n) \;=\; & X_{16} + \frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{16}(a_2+n_1,\,a_1+1,\,c_2+1) \\ & + \frac{a_3\,x_3}{c_1}\sum_{n_1=1}^{n} X_{16}(a_2+n_1,\,a_3+1,\,c_1+1), \end{aligned} \tag{17.3}$$

$$\begin{aligned} X_{16}(a_2-n) \;=\; & X_{16} - \frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{16}(a_2+1-n_1,\,a_1+1,\,c_2+1) \\ & - \frac{a_3\,x_3}{c_1}\sum_{n_1=1}^{n} X_{16}(a_2+1-n_1,\,a_3+1,\,c_1+1), \end{aligned} \tag{17.4}$$

$$X_{16}(a_3+n) \;=\; X_{16} + \frac{a_2\,x_3}{c_1}\sum_{n_1=1}^{n} X_{16}(a_3+n_1,\,a_2+1,\,c_1+1), \tag{17.5}$$

$$X_{16}(a_3-n) \;=\; X_{16} - \frac{a_2\,x_3}{c_1}\sum_{n_1=1}^{n} X_{16}(a_3+1-n_1,\,a_2+1,\,c_1+1). \tag{17.6}$$

Theorem 17.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{16} :*

$$\begin{aligned} X_{16}(a_2+n) \;=\; & \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{n_1}(a_3)_{n_2}x_2^{n_1}x_3^{n_2}}{(c_2)_{n_1}(c_1)_{n_2}} \\ & \times\, X_{16}(a_1+n_1,\,a_2+N_2,\,a_3+n_2,\,c_2+n_1,\,c_1+n_2), \end{aligned} \tag{17.7}$$

$$\begin{aligned} X_{16}(a_2-n) \;=\; & \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{n_1}(a_3)_{n_2}(-x_2)^{n_1}(-x_3)^{n_2}}{(c_2)_{n_1}(c_1)_{n_2}} \\ & \times\, X_{16}(a_1+n_1,\,a_3+n_2,\,c_2+n_1,\,c_1+n_2), \end{aligned} \tag{17.8}$$

$$(17.9)\qquad X_{16}(a_3+n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1} x_3^{n_1}}{(c_1)_{n_1}} X_{16}(a_2+n_1,\, a_3+n_1,\, c_1+n_1),$$

$$(17.10)\qquad X_{16}(a_3-n) = \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1} (-x_3)^{n_1}}{(c_1)_{n_1}} X_{16}(a_2+n_1,\, c_1+n_1),$$

where $N_2 = n_1 + n_2$.

Theorem 17.3. *The following recursion formulas hold true for the denominator parameters* c_1, c_2 *of the Exton's triple hypergeometric function* X_{16} :

$$\begin{aligned} X_{16}(c_1-n) &= X_{16} + (a_1)_2\, x_1 \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)} X_{16}(a_1+2,\, c_1+2-n_1) \\ &\quad + a_2\, a_3\, x_3 \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)} X_{16}(a_2+1,\, a_3+1,\, c_1+2-n_1), \end{aligned} \tag{17.11}$$

$$\begin{aligned} X_{16}(c_2-n) &= X_{16} + a_1 a_2\, x_2 \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)} \\ &\quad \times X_{16}(a_1+1,\, a_2+1,\, c_2+2-n_1), \end{aligned} \tag{17.12}$$

$$\begin{aligned} X_{16}(c_2-n) &= \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_1)_{n_1} (a_2)_{n_1} x_2^{n_1}}{(c_2)_{n_1} (c_2-n)_{n_1}} \\ &\quad \times X_{16}(a_1+n_1,\, a_2+n_1,\, c_2+n_1). \end{aligned} \tag{17.13}$$

18. Recursion Formulas for X_{17}

Theorem 18.1. *The following recursion formulas hold true for the numerator parameters* a_1, a_2 *and* a_3 *of the Exton's triple hypergeometric function* X_{17} :

$$\begin{aligned} X_{17}(a_1+n) &= X_{17} + \frac{2\, x_1}{c_1} \sum_{n_1=1}^{n} (a_1+n_1)\, X_{17}(a_1+1+n_1,\, c_1+1) \\ &\quad + \frac{a_2\, x_2}{c_2} \sum_{n_1=1}^{n} X_{17}(a_1+n_1,\, a_2+1,\, c_2+1), \end{aligned} \tag{18.1}$$

$$\begin{aligned} X_{17}(a_1-n) &= X_{17} - \frac{2\, x_1}{c_1} \sum_{n_1=1}^{n} (a_1+1-n_1)\, X_{17}(a_1+2-n_1,\, c_1+1) \\ &\quad - \frac{a_2\, x_2}{c_2} \sum_{n_1=1}^{n} X_{17}(a_1+1-n_1,\, a_2+1,\, c_2+1), \end{aligned} \tag{18.2}$$

$$
\begin{aligned}
X_{17}(a_2+n) &= X_{17} + \frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{17}(a_2+n_1,\, a_1+1,\, c_2+1) \\
&\quad + \frac{a_3\,x_3}{c_3}\sum_{n_1=1}^{n} X_{17}(a_2+n_1,\, a_3+1,\, c_3+1),
\end{aligned} \tag{18.3}
$$

$$
\begin{aligned}
X_{17}(a_2-n) &= X_{17} - \frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{17}(a_2+1-n_1,\, a_1+1,\, c_2+1) \\
&\quad - \frac{a_3\,x_3}{c_3}\sum_{n_1=1}^{n} X_{17}(a_2+1-n_1,\, a_3+1,\, c_3+1),
\end{aligned} \tag{18.4}
$$

$$
X_{17}(a_3+n) = X_{17} + \frac{a_2\,x_3}{c_3}\sum_{n_1=1}^{n} X_{17}(a_3+n_1,\, a_2+1,\, c_3+1), \tag{18.5}
$$

$$
X_{17}(a_3-n) = X_{17} - \frac{a_2\,x_3}{c_3}\sum_{n_1=1}^{n} X_{17}(a_3+1-n_1,\, a_2+1,\, c_3+1). \tag{18.6}
$$

Theorem 18.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 of the Exton's triple hypergeometric function X_{17} :*

$$
\begin{aligned}
X_{17}(a_2+n) &= \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{n_1}(a_3)_{n_2}x_2^{n_1}x_3^{n_2}}{(c_2)_{n_1}(c_3)_{n_2}} \\
&\quad \times X_{17}(a_1+n_1,\, a_2+N_2,\, a_3+n_2,\, c_2+n_1,\, c_3+n_2),
\end{aligned} \tag{18.7}
$$

$$
\begin{aligned}
X_{17}(a_2-n) &= \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\frac{(a_1)_{n_1}(a_3)_{n_2}(-x_2)^{n_1}(-x_3)^{n_2}}{(c_2)_{n_1}(c_3)_{n_2}} \\
&\quad \times X_{17}(a_1+n_1,\, a_3+n_2,\, c_2+n_1,\, c_3+n_2),
\end{aligned} \tag{18.8}
$$

$$
X_{17}(a_3+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_2)_{n_1}x_3^{n_1}}{(c_3)_{n_1}} X_{17}(a_2+n_1,\, a_3+n_1,\, c_3+n_1), \tag{18.9}
$$

$$
X_{17}(a_3-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_2)_{n_1}(-x_3)^{n_1}}{(c_3)_{n_1}} X_{17}(\,a_2+n_1,\, c_3+n_1), \tag{18.10}
$$

where $N_2 = n_1 + n_2$.

Theorem 18.3. *The following recursion formulas hold true for the denominator parameters c_1, c_2 and c_3 of the Exton's triple hypergeometric function X_{17} :*

$$
\begin{aligned}
X_{17}(c_1-n) &= X_{17} + (a_1)_2\, x_1 \sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)} \\
&\quad \times X_{17}(a_1+2,\, c_1+2-n_1),
\end{aligned} \tag{18.11}
$$

$$X_{17}(c_2-n) = X_{17} + a_1\,a_2\,x_2 \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)} \times X_{17}(a_1+1,\, a_2+1,\, c_2+2-n_1), \tag{18.12}$$

$$X_{17}(c_3-n) = X_{17} + a_2 a_3\,x_3 \sum_{n_1=1}^{n} \frac{1}{(c_3-n_1)(c_3+1-n_1)} \times X_{17}(a_2+1,\, a_3+1,\, c_3+2-n_1). \tag{18.13}$$

Theorem 18.4. *The following recursion formulas hold true for the denominator parameters c_1, c_2 and c_3 of the Exton's triple hypergeometric function X_{17} :*

$$X_{17}(c_1-n) = \sum_{n=0}^{n} \binom{n}{n_1} \frac{(a_1)_{2n_1}\,x_1^{n_1}}{(c_1)_{n_1}(c_1-n)_{n_1}}\, X_{17}(a_1+2n_1,\, c_1+n_1), \tag{18.14}$$

$$X_{17}(c_2-n) = \sum_{n=0}^{n} \binom{n}{n_1} \frac{(a_1)_{n_1}(a_2)_{n_1}x_2^{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}}\, X_{17}(a_1+n_1,\, a_2+n_1,\, c_2+n_1), \tag{18.15}$$

$$X_{17}(c_3-n) = \sum_{n=0}^{n} \binom{n}{n_1} \frac{(a_2)_{n_1}(a_3)_{n_1}x_3^{n_1}}{(c_3)_{n_1}(c_3-n)_{n_1}}\, X_{17}(a_2+n_1,\, a_3+n_1,\, c_3+n_1). \tag{18.16}$$

19. Recursion Formulas for X_{18}

Theorem 19.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{18} :*

$$X_{18}(a_1+n) = X_{18} + \frac{2\,x_1}{c} \sum_{n_1=1}^{n} (a_1+n_1)\, X_{18}(a_1+1+n_1,\, c+1) + \frac{a_2\,x_2}{c} \sum_{n_1=1}^{n} X_{18}(a_1+n_1,\, a_2+1,\, c+1), \tag{19.1}$$

$$X_{18}(a_1-n) = X_{18} - \frac{2\,x_1}{c} \sum_{n_1=1}^{n} (a_1+1-n_1)\, X_{18}(a_1+2-n_1,\, c+1) - \frac{a_2\,x_2}{c} \sum_{n_1=1}^{n} X_{18}(a_1+1-n_1,\, a_2+1,\, c+1), \tag{19.2}$$

$$X_{18}(a_2+n) = X_{18} + \frac{a_1\,x_2}{c} \sum_{n_1=1}^{n} X_{18}(a_2+n_1,\, a_1+1,\, c+1), \tag{19.3}$$

$$X_{18}(a_2-n) = X_{18} - \frac{a_1\,x_2}{c} \sum_{n_1=1}^{n} X_{18}(a_2+1-n_1,\, a_1+1,\, c+1), \tag{19.4}$$

$$X_{18}(a_3+n) = X_{18} + \frac{a_4\,x_3}{c}\sum_{n_1=1}^{n} X_{18}(a_3+n_1,\, a_4+1,\, c+1), \tag{19.5}$$

$$X_{18}(a_3-n) = X_{18} - \frac{a_4\,x_3}{c}\sum_{n_1=1}^{n} X_{18}(a_3+1-n_1,\, a_4+1,\, c+1), \tag{19.6}$$

$$X_{18}(a_4+n) = X_{18} + \frac{a_3\,x_3}{c}\sum_{n_1=1}^{n} X_{18}(a_3+1,\, a_4+n_1,\, c+1), \tag{19.7}$$

$$X_{18}(a_4-n) = X_{18} - \frac{a_3\,x_3}{c}\sum_{n_1=1}^{n} X_{18}(a_3+1,\, a_4+1-n_1,\, c+1). \tag{19.8}$$

Theorem 19.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{18} :*

$$X_{18}(a_2+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}x_2^{n_1}}{(c)_{n_1}} X_{18}(a_1+n_1,\, a_2+n_1,\, c+n_1), \tag{19.9}$$

$$X_{18}(a_2-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-x_2)^{n_1}}{(c)_{n_1}} X_{18}(a_1+n_1,\, c+n_1), \tag{19.10}$$

$$X_{18}(a_3+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_4)_{n_1}x_3^{n_1}}{(c)_{n_1}} X_{18}(a_3+n_1,\, a_4+n_1,\, c+n_1), \tag{19.11}$$

$$X_{18}(a_3-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_4)_{n_1}(-x_3)^{n_1}}{(c)_{n_1}} X_{18}(\, a_4+n_1,\, c+n_1), \tag{19.12}$$

$$X_{18}(a_4+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_3)_{n_1}x_3^{n_1}}{(c)_{n_1}} X_{18}(a_3+n_1,\, a_4+n_1,\, c+n_1), \tag{19.13}$$

$$X_{18}(a_4-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_3)_{n_1}(-x_3)^{n_1}}{(c)_{n_1}} X_{18}(\, a_3+n_1,\, c+n_1). \tag{19.14}$$

Theorem 19.3. *The following recursion formulas hold true for the denominator parameter c of the Exton's triple hypergeometric function X_{18} :*

$$\begin{aligned} X_{18}(c-n) \;=\;& X_{18} + (a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)} X_{18}(a_1+2,\, c+2-n_1) \\ &+a_1\,a_2\,x_2\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)} X_{18}(a_1+1,\, a_2+1,\, c+2-n_1) \\ &+a_3\,a_4 x_3\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)} X_{18}(a_3+1,\, a_4+1,\, c+2-n_1). \end{aligned} \tag{19.15}$$

20. Recursion Formulas for X_{19}

Theorem 20.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{19} :*

$$X_{19}(a_1+n) = X_{19} + \frac{2\,x_1}{c_2}\sum_{n_1=1}^{n}(a_1+n_1)\,X_{19}(a_1+1+n_1,\,c_2+1) + \frac{a_2\,x_2}{c_1}\sum_{n_1=1}^{n}X_{19}(a_1+n_1,\,a_2+1,\,c_1+1), \tag{20.1}$$

$$X_{19}(a_1-n) = X_{19} - \frac{2\,x_1}{c_2}\sum_{n_1=1}^{n}(a_1+1-n_1)\,X_{19}(a_1+2-n_1,\,c_2+1) - \frac{a_2\,x_2}{c_1}\sum_{n_1=1}^{n}X_{19}(a_1+1-n_1,\,a_2+1,\,c_1+1), \tag{20.2}$$

$$X_{19}(a_2+n) = X_{19} + \frac{a_1\,x_2}{c_1}\sum_{n_1=1}^{n}X_{19}(a_2+n_1,\,a_1+1,\,c_1+1), \tag{20.3}$$

$$X_{19}(a_2-n) = X_{19} - \frac{a_1\,x_2}{c_1}\sum_{n_1=1}^{n}X_{19}(a_2+1-n_1,\,a_1+1,\,c_1+1), \tag{20.4}$$

$$X_{19}(a_3+n) = X_{19} + \frac{a_4\,x_3}{c_1}\sum_{n_1=1}^{n}X_{19}(a_3+n_1,\,a_4+1,\,c_1+1), \tag{20.5}$$

$$X_{19}(a_3-n) = X_{19} - \frac{a_4\,x_3}{c_1}\sum_{n_1=1}^{n}X_{19}(a_3+1-n_1,\,a_4+1,\,c_1+1), \tag{20.6}$$

$$X_{19}(a_4+n) = X_{19} + \frac{a_3\,x_3}{c_1}\sum_{n_1=1}^{n}X_{19}(a_3+1,\,a_4+n_1,\,c_1+1), \tag{20.7}$$

$$X_{19}(a_4-n) = X_{19} - \frac{a_3\,x_3}{c_1}\sum_{n_1=1}^{n}X_{19}(a_3+1,\,a_4+1-n_1,\,c_1+1). \tag{20.8}$$

Theorem 20.2. *The following recursion formulas hold true for the numerator parameters a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{19} :*

$$X_{19}(a_2+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}x_2^{n_1}}{(c_1)_{n_1}}\,X_{19}(a_1+n_1,\,a_2+n_1,\,c_1+n_1), \tag{20.9}$$

$$X_{19}(a_2-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-x_2)^{n_1}}{(c_1)_{n_1}}\,X_{19}(a_1+n_1,\,c_1+n_1), \tag{20.10}$$

$$X_{19}(a_3+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_4)_{n_1}x_3^{n_1}}{(c_1)_{n_1}}\,X_{19}(a_3+n_1,\,a_4+n_1,\,c_1+n_1), \tag{20.11}$$

$$X_{19}(a_3-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_4)_{n_1}(-x_3)^{n_1}}{(c_1)_{n_1}} X_{19}(a_4+n_1,\, c_1+n_1), \tag{20.12}$$

$$X_{19}(a_4+n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_3)_{n_1}x_3^{n_1}}{(c_1)_{n_1}} X_{19}(a_3+n_1,\, a_4+n_1,\, c_1+n_1), \tag{20.13}$$

$$X_{19}(a_4-n) = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_3)_{n_1}(-x_3)^{n_1}}{(c_1)_{n_1}} X_{19}(a_3+n_1,\, c_1+n_1). \tag{20.14}$$

Theorem 20.3. *The following recursion formulas hold true for the denominator parameters $c_1,\, c_2$ of the Exton's triple hypergeometric function X_{19} :*

$$\begin{aligned} X_{19}(c_1-n) &= X_{19} \\ &+ a_1a_2x_2\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}X_{19}(a_1+1,\, a_2+1,\, c_1+2-n_1) \\ &+ a_3a_4x_3\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}X_{19}(a_3+1,\, a_4+1,\, c_1+2-n_1), \end{aligned} \tag{20.15}$$

$$\begin{aligned} X_{19}(c_2-n) &= X_{19} + (a_1)_2\, x_1\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)} \\ &\quad \times X_{19}(a_1+2,\, c_2+2-n_1), \end{aligned} \tag{20.16}$$

$$X_{19}(c_2-n) = \sum_{n=0}^{n}\binom{n}{n_1}\frac{(a_1)_{2n_1}\, x_1^{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}} X_{19}(a_1+2n_1,\, c_2+n_1). \tag{20.17}$$

21. Recursion Formulas for X_{20}

Theorem 21.1. *The following recursion formulas hold true for the numerator parameters a_1, a_2, a_3 and a_4 of the Exton's triple hypergeometric function X_{20} :*

$$\begin{aligned} X_{20}(a_1+n) &= X_{20} + \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\, X_{20}(a_1+1+n_1,\, c_1+1) \\ &\quad + \frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{20}(a_1+n_1,\, a_2+1,\, c_2+1), \end{aligned} \tag{21.1}$$

$$\begin{aligned} X_{20}(a_1-n) &= X_{20} - \frac{2\,x_1}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\, X_{20}(a_1+2-n_1,\, c_1+1) \\ &\quad - \frac{a_2\,x_2}{c_2}\sum_{n_1=1}^{n} X_{20}(a_1+1-n_1,\, a_2+1,\, c_2+1), \end{aligned} \tag{21.2}$$

$$(21.3)\qquad X_{20}(a_2+n) \;=\; X_{20}+\frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{20}(a_2+n_1,\,a_1+1,\,c_2+1),$$

$$(21.4)\qquad X_{20}(a_2-n) \;=\; X_{20}-\frac{a_1\,x_2}{c_2}\sum_{n_1=1}^{n} X_{20}(a_2+1-n_1,\,a_1+1,\,c_2+1),$$

$$(21.5)\qquad X_{20}(a_3+n) \;=\; X_{20}+\frac{a_4\,x_3}{c_1}\sum_{n_1=1}^{n} X_{20}(a_3+n_1,\,a_4+1,\,c_1+1),$$

$$(21.6)\qquad X_{20}(a_3-n) \;=\; X_{20}-\frac{a_4\,x_3}{c_1}\sum_{n_1=1}^{n} X_{20}(a_3+1-n_1,\,a_4+1,\,c_1+1),$$

$$(21.7)\qquad X_{20}(a_4+n) \;=\; X_{20}+\frac{a_3\,x_3}{c_1}\sum_{n_1=1}^{n} X_{20}(a_3+1,\,a_4+n_1,\,c_1+1),$$

$$(21.8)\qquad X_{20}(a_4-n) \;=\; X_{20}-\frac{a_3\,x_3}{c_1}\sum_{n_1=1}^{n} X_{20}(a_3+1,\,a_4+1-n_1,\,c_1+1).$$

Theorem 21.2. *The following recursion formulas hold true for the numerator parameters* a_2, a_3 *and* a_4 *of the Exton triple hypergeometric function* X_{20} :

$$(21.9)\qquad X_{20}(a_2+n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}x_2^{n_1}}{(c_2)_{n_1}}\,X_{20}(a_1+n_1,\,a_2+n_1,\,c_2+n_1),$$

$$(21.10)\qquad X_{20}(a_2-n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-x_2)^{n_1}}{(c_2)_{n_1}}\,X_{20}(a_1+n_1,\,c_2+n_1),$$

$$(21.11)\qquad X_{20}(a_3+n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_4)_{n_1}x_3^{n_1}}{(c_1)_{n_1}}\,X_{20}(a_3+n_1,\,a_4+n_1,\,c_1+n_1),$$

$$(21.12)\qquad X_{20}(a_3-n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_4)_{n_1}(-x_3)^{n_1}}{(c_1)_{n_1}}\,X_{20}(\,a_4+n_1,\,c_1+n_1),$$

$$(21.13)\qquad X_{20}(a_4+n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_3)_{n_1}x_3^{n_1}}{(c_1)_{n_1}}\,X_{20}(a_3+n_1,\,a_4+n_1,\,c_1+n_1),$$

$$(21.14)\qquad X_{20}(a_4-n) \;=\; \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_3)_{n_1}(-x_3)^{n_1}}{(c_1)_{n_1}}\,X_{20}(\,a_3+n_1,\,c_1+n_1).$$

Theorem 21.3. *The following recursion formulas hold true for the denominator parameters* c_1, c_2 *of the Extonʼs triple hypergeometric function* X_{20} :

$$\begin{aligned} X_{20}(c_1-n) \;&=\; X_{20}+(a_1)_2\,x_1\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{20}(a_1+2,\,c_1+2-n_1)\\ &\quad+\;a_3\,a_4\,x_3\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}X_{20}(a_3+1,\,a_4+1,c_1+2-n_1), \end{aligned}\qquad(21.15)$$

$$
\begin{aligned}
X_{20}(c_2-n) &= X_{20} + a_1 a_2\, x_2 \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)} \\
&\quad \times X_{20}(a_1+1,\, a_2+1,\, c_2+2-n_1),
\end{aligned}
\tag{21.16}
$$

$$
\begin{aligned}
X_{20}(c_2-n) &= \sum_{n_1=0}^{n} \binom{n}{n_1} \frac{(a_1)_{n_1}(a_2)_{n_1} x_2^{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}} \\
&\quad \times X_{20}(a_1+n_1,\, a_2+n_1,\, c_2+n_1).
\end{aligned}
\tag{21.17}
$$

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