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Some Finite Integrals Involving Srivastava's Polynomials and the Aleph Function

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Abstract. In this paper, we establish certain integrals involving Srivastava's Polynomials [5] and Aleph Function ([8], [10]). On account of general nature of the functions and polynomials involved in the integrals, our results provide interesting unifications and generalizations of a large number of new and known results, which may find useful applications in the field of science and engineering. To illustrate, we have recorded some special cases of our main results which are also sufficiently general and unified in nature and are of interest in themselves.

1. Introduction

The Aleph function, introduced by Südland et al. ([8]; see also [10]), is defined in terms of Mellin Barnes type integrals as:

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left\{ z \left| \begin{array}{l} (a_j, A_j)_{1,n}; [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i;r} \\ (b_j, B_j)_{1,m}; [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{array} \right. \right\}$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) z^{-s} ds \tag{1.1}$$

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where $\omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} - B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + A_{ji}s)} \tag{1.2}$$

The integration path $L = L_{\omega\gamma\infty},\ \gamma\epsilon\ \Re$ extends from $\gamma - \omega\infty\ to\ \gamma + \omega\infty$, and is such that the poles of $\Gamma(1 - a_j - A_j s),\ j = 1, \ldots, n$, do not coincide with the poles of $\Gamma(b_j + B_j s), j = 1, \ldots, m$. The parameters $p_i\ and\ q_i$ are non-negative integers satisfying the conditions $0 \le n \le p_i,\ 0 \le m \le q_i, \tau_i > 0$ for $i = 1, \ldots, r$. Also, $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. An empty product in (2) is interpreted as unity. The existence conditions for (1) are:

$$\varphi_i > 0, |\arg z\ | < \frac{\pi}{2}\varphi_i \qquad ;\quad i = 1, \ldots, r \tag{1.3}$$

$$\varphi_i \ge 0, |\arg z\ | < \frac{\pi}{2}\varphi_i \quad and \quad \Re\{\zeta_i\} + 1 < 0 \tag{1.4}$$

where

$$\varphi_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_i\left(\sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} B_{ji}\right) \tag{1.5}$$

$$\zeta_i = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_i\left(\sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=n+1}^{p_i} a_{ji}\right) + \frac{1}{2}(p_i - q_i) \quad ;\quad i = 1, \ldots, r \tag{1.6}$$

The Srivastava's Polynomials [5] occurring in the present paper is defined as:

$$S^{m_1,m_2,\ldots,m_\lambda}_{n_1,n_2,\ldots,n_\lambda}[x] = \sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_\lambda=0}^{[n_\lambda/m_\lambda]}\prod_{g=1}^{\lambda}\frac{(-n_g)_{m_g l_g}}{l_g!}A_{n_g,l_g}x^{l_g} \tag{1.7}$$

where n_g &$m_g(g = 1, 2, \ldots, \lambda)$ are arbitrary positive integers and the coefficients $A_{n_g,l_g}(n_g, l_g \ge 0)$ are arbitrary constants, real or complex. For the present study, we shall require the following results due to [3] and [1] respectively, pertaining to the Jacobi Polynomials $P_\eta^{(\alpha,\beta)}(x)$ [2]:

$$P_\mu^{(\alpha,\beta)}(t + y)\,P_\mu^{(\alpha,\beta)}(t - y) = \frac{(-1)^\mu(1+\alpha)_\mu(1+\beta)_\mu}{(\mu!)^2}\times$$

$$\sum_{\eta=0}^{\mu}\frac{(-\mu)_\eta(1+\alpha+\beta+\mu)_\eta}{(1+\alpha)_\eta(1+\beta)_\eta}P_\eta^{(\alpha,\beta)}(x)\,t^\eta \tag{1.8}$$

and

$$\frac{1}{y}(1-t+y)^{-\alpha}(1-t+y)^{-\beta} = 2^{-\alpha-\beta}\sum_{\eta=0}^{\infty} P_{\eta}^{(\alpha,\beta)}(x)\, t^{\eta} \tag{1.9}$$

where

$$y = \left(1-2xt+t^2\right)^{1/2} \tag{1.10}$$

2. Main Integrals

In this section, we establish the following three integrals:

First Integral

$$I_1 = \int_0^t (1-x)^{\rho-1}(1+x)^{\sigma-1} S_{n_1,n_2,\ldots,n_\lambda}^{m_1,m_2,\ldots,m_\lambda}[w(1-x)^u(1+x)^v]$$

$$\times \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left\{ z(1-x)^h(1+x)^k \,\middle|\, \begin{matrix} (a_j,A_j)_{1,n};[\tau_i\,(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m};[\tau_i\,(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right\} dx$$

$$= 2^{\rho+\sigma-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_\lambda=0}^{[n_\lambda/m_\lambda]}\prod_{g=1}^{\lambda}\frac{(-n_g)_{m_g l_g}}{l_g!}A_{n_g,l_g}w^{l_g}2^{(u+v)l_g}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left\{ z\; 2^{(h+k)} \,\middle|\, \begin{matrix} (1-\rho-ul_g,h)\,,(1-\sigma-vl_g,k)\,,(a_j,A_j)_{1,n};[\tau_i\,(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m};[\tau_i\,(b_{ji},B_{ji})]_{m+1,q_i;r},(1-\rho-\sigma-(u+v)\,l_g,h+k) \end{matrix} \right\} \tag{2.11}$$

The above integral converges under the conditions (3), (4) and (i) $\rho \geq 1,\ \sigma \geq 1; u \geq 0, v \geq 0; h \geq 0,\ k \geq 0$ (Not both h and k be zero simultaneously)
(ii) $Re\,(\rho) + h \min\left[Re\left(\frac{b_j}{B_j}\right)\right] > 0$ and (iii) $Re\,(\sigma) + k \min\left[Re\left(\frac{b_j}{B_j}\right)\right] > 0$

Proof. On using the representations (1.1) and (1.7) in the left hand side of the integral (1.11), and then on interchanging the order of summations and the s - integral, we get

$$I_1 = \int_0^t (1-x)^{\rho-1}(1+x)^{\sigma-1}$$

$$\times \left\{ \frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)z^{-s}(1-x)^{-hs}(1+x)^{-ks}ds \right.$$

$$\left. \times \sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_\lambda=0}^{[n_\lambda/m_\lambda]}\prod_{g=1}^{\lambda}\frac{(-n_g)_{m_g l_g}}{l_g!}A_{n_g,l_g}[w(1-x)^u(1+x)^v]^{l_g} \right\} dx$$

Now, on interchanging the order of x- and s- integrals,which is permissible under the conditions stated,and then on using [7], p. 314, eqn. (3) we easily arrive at the desired result after a little simplification. □

To establish the second and third integrals we use the following Lemma derived from (2.11):

Lemma Under the conditions derived from those stated with (1.11), the following integral formula holds:

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_{\eta}^{(\alpha,\beta)}(x)$$

$$\times\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left\{z(1-x)^{h}(1+x)^{k}\;\middle|\;\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right\}dx$$

$$=2^{\rho+\sigma-1}\sum_{l=0}^{\eta}\frac{(-\eta)_l}{l!}\begin{pmatrix}\eta+\alpha\\\eta\end{pmatrix}\frac{(\alpha+\beta+\eta+1)_l}{(\alpha+1)_l}$$

(2.12)

$$\times\aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left\{z.2^{(h+k)}\;\middle|\;\begin{matrix}(1-\rho-l,h),(1-\sigma,k),(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(1-\rho-\sigma-l,h+k)\end{matrix}\right\}$$

Proof. On setting $\lambda=1,n_1=\eta;m_1=1;\omega=\frac{1}{2};u=1,v=0$ and $A_{\eta,l}=\begin{pmatrix}\eta+\alpha\\\eta\end{pmatrix}\frac{(\alpha+\beta+\eta+1)_l}{(\alpha+1)_l}$ in (11), the Srivastava's polynomial occurring therein reduces to$S_{\eta}^{1}\left[\frac{1-x}{2}\right]$, which in view of [4, p. 68, eq. (4.3.2)] is expressed in terms of Jacobi Polynomials. Then, after a little simplification, we readily arrive at (1.12).□

Second Integral

$$I_2=\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_{\mu}^{(\alpha,\beta)}(t+y)\,P_{\mu}^{(\alpha,\beta)}(t-y)$$

$$\times\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left\{z(1-x)^{h}(1+x)^{k}\;\middle|\;\begin{matrix}(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right\}dx$$

$$=2^{\rho+\sigma-1}\frac{(-1)^{\mu}\Gamma(1+\alpha+\mu)\Gamma(1+\beta+\mu)}{(\mu!)^2}$$

$$\times\sum_{\eta=0}^{\mu}\sum_{l=0}^{\eta}\frac{(-\mu)_{\eta}(1+\alpha+\beta+\mu)_{\eta}}{\Gamma(1+\alpha+\eta)\Gamma(1+\beta+\eta)}\frac{(-\eta)_l}{l!}\begin{pmatrix}\eta+\alpha\\\eta\end{pmatrix}\frac{(\alpha+\beta+\eta+1)_l}{(\alpha+1)_l}$$

(2.13)

$$\times\aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left\{z.2^{(h+k)}\;\middle|\;\begin{matrix}(1-\rho-l,h),(1-\sigma,k),(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(1-\rho-\sigma-l,h+k)\end{matrix}\right\}$$

The integral (2.13) converges under the conditions derived from those mentioned with (2.11).

Proof. We use (1.8) in the LHS of (2.13) and then interchange the order of summation and integration in the expression so obtained. Now by virtue of the Lemma we readily arrive at the desired result. □

Third Integral

$$I_3 = \int_{-1}^{1} (1-x)^{\rho-1}(1+x)^{\sigma-1}\frac{1}{y}(1-\mathrm{t}+\mathrm{y})^{-\alpha}(1+\mathrm{t}+\mathrm{y})^{-\beta}$$

$$\times \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left\{ z(1-x)^h(1+x)^k \,\middle|\, \begin{matrix} (a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right\} dx$$

$$= 2^{-\alpha-\beta+\rho+\sigma-1}\sum_{\eta=0}^{\mu}\sum_{l=0}^{\eta}\frac{(-\eta)_l}{l!}\begin{pmatrix} \eta+\alpha \\ \eta \end{pmatrix}\frac{(\alpha+\beta+\eta+1)_l}{(\alpha+1)_l}$$

(2.14)

$$\times \aleph^{m,n+2}_{p_i+2,q_i+1,\tau_i;r}\left\{ z.2^{(h+k)} \,\middle|\, \begin{matrix} (1-\rho-l,h),(1-\sigma,k),(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(1-\rho-\sigma-l,h+k) \end{matrix} \right\}$$

The integral (2.14) converges under the conditions derived from those mentioned with (2.11).

Proof. We use (1.9) in the LHS of (2.14) and then interchange the order of summation and integration in the expression so obtained. Now by virtue of the Lemma we readily arrive at the desired result. □

3. Special Cases

(i) On taking $\lambda = 1$ in (2.11), we get the following integral in terms of General class of polynomials $S^m_n[x]$ [5]:

$$I_4 = \int_0^t (1-x)^{\rho-1}(1+x)^{\sigma-1}S^m_n[w(1-x)^u(1+x)^v]$$

$$\times \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left\{ z(1-x)^h(1+x)^k \,\middle|\, \begin{matrix} (a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right\} dx$$

$$= 2^{\rho+\sigma-1}\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}w^l 2^{(u+v)l}$$

(3.15)

$$\times \aleph^{m,n+2}_{p_i+2,q_i+1,\tau_i;r}\left\{ z2^{(h+k)} \,\middle|\, \begin{matrix} (1-\rho-ul,h),(1-\sigma-vl,k),(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(1-\rho-\sigma-(u+v)l,h+k) \end{matrix} \right\}$$

The integral (3.15) converges under the conditions derived from those mentioned with (2.11).

(ii) On setting $\tau_i = 1$ *and* $r = 1$ in the main integrals, we obtain the following three integrals in terms of Fox's H-Function [6].

$$I_5 = \int_0^t (1-x)^{\rho-1}(1+x)^{\sigma-1}S^{m_1,m_2,\ldots,m_\lambda}_{n_1,n_2,\ldots,n_\lambda}[w(1-x)^u(1+x)^v]$$

$$\times H_{p,q}^{m,n}\left\{z(1-x)^h(1+x)^k \,\middle|\, \begin{array}{c}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{array}\right\}dx$$

$$= 2^{\rho+\sigma-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_\lambda=0}^{[n_\lambda/m_\lambda]}\prod_{g=1}^{\lambda}\frac{(-n_g)_{m_g l_g}}{l_g!}A_{n_g,l_g}w^{l_g}2^{(u+v)l_g}$$

$$\times H_{p+2,q+1}^{m,n+2}\left\{z2^{(h+k)} \,\middle|\, \begin{array}{c}(1-\rho-ul_g,h),(1-\sigma-vl_g,k),(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q},(1-\rho-\sigma-(u+v)l_g,h+k)\end{array}\right\} \tag{3.16}$$

$$I_6=\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_\mu^{(\alpha,\beta)}(t+y)\,P_\mu^{(\alpha,\beta)}(t-y)$$

$$\times H_{p,q}^{m,n}\left\{z(1-x)^h(1+x)^k \,\middle|\, \begin{array}{c}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{array}\right\}dx$$

$$=2^{\rho+\sigma-1}\frac{(-1)^\mu\Gamma(1+\alpha+\mu)\Gamma(1+\beta+\mu)}{(\mu!)^2}$$

$$\times\sum_{\eta=0}^{\mu}\sum_{l=0}^{\eta}\frac{(-\mu)_\eta(1+\alpha+\beta+\mu)_\eta}{\Gamma(1+\alpha+\eta)\Gamma(1+\beta+\eta)}\frac{(-\eta)_l}{l!}\begin{pmatrix}\eta+\alpha\\\eta\end{pmatrix}\frac{(\alpha+\beta+\eta+1)_l}{(\alpha+1)_l}$$

$$\times H_{p+2,q+1}^{m,n+2}\left\{z.2^{(h+k)} \,\middle|\, \begin{array}{c}(1-\rho-l,h),(1-\sigma,k),(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q},(1-\rho-\sigma-l,h+k)\end{array}\right\} \tag{3.17}$$

and

$$I_7=\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}\frac{1}{y}(1-\mathrm{t}+\mathrm{y})^{-\alpha}(1+\mathrm{t}+\mathrm{y})^{-\beta}$$

$$\times H_{p,q}^{m,n}\left\{z(1-x)^h(1+x)^k \,\middle|\, \begin{array}{c}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{array}\right\}dx$$

$$=2^{-\alpha-\beta+\rho+\sigma-1}\sum_{\eta=0}^{\mu}\frac{(-\eta)_l}{l!}\begin{pmatrix}\eta+\alpha\\\eta\end{pmatrix}\frac{(\alpha+\beta+\eta+1)_l}{(\alpha+1)_l}$$

$$\times H_{p+2,q+1}^{m,n+2}\left\{z.2^{(h+k)} \,\middle|\, \begin{array}{c}(1-\rho-l,h),(1-\sigma,k),(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q},(1-\rho-\sigma-l,h+k)\end{array}\right\} \tag{3.18}$$

Integrals (16), (17) and (18) converges under the conditions derived from those mentioned with (11), (13) and (14) respectively.

(iii) On taking $\tau_i=1$ in the main integrals, we get the results due to Agarwal et al. [9].

On account of being general and unified in nature, the results established here yield a large number of known and new results involving simpler functions on suitable specifications of the parameters involved.

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