

Positive Solutions for Three-point Boundary Value Problem of Nonlinear Fractional q -difference Equation

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ABSTRACT. In this paper, we investigate the existence and uniqueness of positive solutions for three-point boundary value problem of nonlinear fractional q -difference equation. Some existence and uniqueness results are obtained by applying some standard fixed point theorems. As applications, two examples are presented to illustrate the main results.

1. Introduction

As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc. involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, fractional differential equations have been of great interest; for example, see [2, 3, 4, 7, 16] and the references therein.

The q -difference calculus or quantum calculus is an old subject that was initially developed by Jackson [13, 14], basic definitions and properties of q -difference calculus can be found in the book mentioned in [15].

The fractional q -difference calculus had its origin in the works by Al-Salam [6] and Agarwal [1]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q -difference calculus were made, for example, q -analogues of the integral and differential fractional operators properties such as the q -Laplace transform, q -Taylor's formula, Mittag-Leffler function [5, 17, 18], just to mention some.

Recently, the question of the existence of solutions for fractional q -difference boundary value problems have aroused considerable attention. There have been some papers dealing with the existence and multiplicity of solutions or positive

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solutions for boundary value problems involving nonlinear fractional q -difference equations, such as the Krasnosel'skii fixed-point theorem, the Leggett-Williams fixed-point theorem, and the Schauder fixed-point theorem. For examples, see [8, 9] and the references therein.

El-Shahed and Hassan [10] studied the existence of positive solutions of the following q -difference boundary value problem

$$\begin{cases} -(D_q^2 u)(t) = a(t)f(u(t)), & 0 \leq t \leq 1, \\ \alpha u(0) - \beta D_q u(0) = 0, & \gamma u(1) - \delta D_q u(1) = 0. \end{cases}$$

Ferreira [11] and [12] considered the existence of positive solutions to nonlinear q -difference boundary value problems

$$\begin{cases} -(D_q^\alpha u)(t) = -f(t, u(t)), & 0 \leq t \leq 1, & 1 < \alpha \leq 2, \\ u(0) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} (D_q^\alpha u)(t) = -f(t, u(t)), & 0 \leq t \leq 1, & 2 < \alpha \leq 3, \\ u(0) = (D_q u)(0) = 0, & (D_q u)(1) = \beta \geq 0, \end{cases}$$

respectively.

In this paper, we investigate the existence and uniqueness results for the following nonlinear fractional q -difference equations with three-point boundary conditions

$$(1.1) \quad \begin{cases} (D_q^\alpha u)(t) + f(t, u(t)) = 0, & 0 \leq t \leq 1, & 1 < \alpha \leq 2, \\ u(0) = 0, & u(1) = \beta u(\xi), \end{cases}$$

where $0 < \beta \xi^{\alpha-1} < 1$, $0 < \xi < 1$, D_q^α is the fractional q -derivative of the Riemann-Liouville type of order α , and $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous function.

2. Preliminaries

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q -calculus theory to facilitate analysis of problem (1.1). These details can be found in the recent literature; see [15] and references therein.

Let $q \in (0, 1)$ and define

$$[a]_q = \frac{q^a - 1}{q - 1}, \quad a \in \mathbb{R}.$$

The q -analogue of the power $(a - b)^{(n)}$ with $n \in \mathbb{N}_0$ is

$$(a - b)^{(0)} = 1, \quad (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, \quad a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b=0$ then $a^{(\alpha)} = a^\alpha$. The q -gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q -derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x \rightarrow 0} (D_q f)(x),$$

and q -derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q -integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q , i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x=0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [15]. We now point out three formulas that will be used later (${}_i D_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \quad {}_t D_q (t-s)^{(\alpha)} = [a]_q (t-s)^{(\alpha-1)},$$

$$\left({}_x D_q \int_0^x f(x, t) d_q t \right) (x) = \int_0^x {}_x D_q f(x, t) d_q t + f(qx, x).$$

Denote that if $\alpha > 0$ and $a \leq b \leq t$, then $(t - a)^{(\alpha)} \geq (t - b)^{(\alpha)}$ [11].

Definition 2.1. ([19]) Let $\alpha \geq 0$ and f be function defined on $[0, 1]$. The fractional q -integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, \quad x \in [0, 1].$$

Definition 2.2. ([19]) The fractional q -derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α .

Lemma 2.1. ([19]) Let $\alpha, \beta \geq 0$ and f be a function defined on $[0, 1]$. Then the next formulas hold:

- (a) $(I_q^\beta I_q^\alpha f)(x) = I_q^{\alpha+\beta} f(x)$,
- (b) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.2. ([11]) Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Lemma 2.3. Let $y \in C[0, 1]$ and $1 < \alpha \leq 2$, the unique solution of

$$(2.1) \quad \begin{aligned} (D_q^\alpha u)(t) + y(t) &= 0, \quad 0 \leq t \leq 1, \\ u(0) &= 0, \quad u(1) = \beta u(\xi), \end{aligned}$$

is given by

$$u(t) = \int_0^1 G(t, qs) y(s) d_q s,$$

where

$$(2.2) \quad G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{(\alpha-1)} - \beta t^{\alpha-1}(\xi-s)^{(\alpha-1)} - (t-s)^{(\alpha-1)}(1-\beta\xi^{\alpha-1})}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \xi, \\ \frac{t^{\alpha-1}(1-s)^{(\alpha-1)} - (t-s)^{(\alpha-1)}(1-\beta\xi^{\alpha-1})}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 < \xi \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{(\alpha-1)} - \beta t^{\alpha-1}(\xi-s)^{(\alpha-1)}}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 \leq t \leq s \leq \xi \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{(\alpha-1)}}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 \leq t \leq s \leq 1, \xi \leq s. \end{cases}$$

Proof. At first, by Lemma 2.1 and Lemma 2.2, the equation (2.1) is equivalent to the integral equation

$$u(t) = -I_q^\alpha y(t) + B_1 t^{\alpha-1} + B_2 t^{\alpha-2}, \quad B_1, B_2 \in \mathbb{R},$$

that is,

$$u(t) = - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_qs + B_1 t^{\alpha-1} + B_2 t^{\alpha-2}.$$

By the boundary conditions $u(0) = 0$ and $u(1) = \beta u(\eta)$, we have

$$B_1 = \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)} y(s) d_qs - \int_0^\xi \frac{\beta(\xi-qs)^{(\alpha-1)}}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)} y(s) d_qs, \quad B_2 = 0.$$

Therefore, the solution $u(t)$ of boundary value problem (2.1) satisfies

$$\begin{aligned} u(t) = & - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_qs + \int_0^1 \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)}}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)} y(s) d_qs \\ & - \int_0^\xi \frac{\beta t^{\alpha-1}(\xi-qs)^{(\alpha-1)}}{(1-\beta\xi^{\alpha-1})\Gamma_q(\alpha)} y(s) d_qs = \int_0^1 G(t,qs) y(s) d_qs, \end{aligned}$$

where $G(t,s)$ is given by (2.2). The proof is completed. \square

Lemma 2.4. *The function $G(t,s)$ defined by (2.2) satisfies $G(t,qs) \geq 0$ for all $0 \leq s, t \leq 1$.*

Proof. We start by defining four functions as follows

$$\begin{aligned} g_1(t,s) &= t^{\alpha-1}(1-s)^{(\alpha-1)} - \beta t^{\alpha-1}(\xi-s)^{(\alpha-1)} - (t-s)^{(\alpha-1)}(1-\beta\xi^{\alpha-1}), \\ & \quad 0 \leq s \leq t \leq 1, \quad s \leq \xi, \\ g_2(t,s) &= t^{\alpha-1}(1-s)^{(\alpha-1)} - (t-s)^{(\alpha-1)}(1-\beta\xi^{\alpha-1}), \quad 0 < \xi \leq s \leq t \leq 1, \\ g_3(t,s) &= t^{\alpha-1}(1-s)^{(\alpha-1)} - \beta t^{\alpha-1}(\xi-s)^{(\alpha-1)}, \quad 0 \leq t \leq s \leq \xi \leq 1, \\ g_4(t,s) &= t^{\alpha-1}(1-s)^{(\alpha-1)}, \quad 0 \leq t \leq s \leq 1, \quad \xi \leq s. \end{aligned}$$

Firstly, we prove $g_1(t,qs) \geq 0$, $0 \leq s \leq t \leq 1$, $s \leq \xi$. In view of the fact that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$ [11], we get

$$\begin{aligned} g_1(t,qs) &= t^{\alpha-1}(1-qs)^{(\alpha-1)} - \beta t^{\alpha-1}(\xi-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)}(1-\beta\xi^{\alpha-1}) \\ &= t^{\alpha-1} \left((1-qs)^{(\alpha-1)} - \beta(\xi-qs)^{(\alpha-1)} - \left(1 - \frac{qs}{t}\right)^{(\alpha-1)} (1-\beta\xi^{\alpha-1}) \right) \\ &\geq t^{\alpha-1} \left((1-qs)^{(\alpha-1)} - \beta(\xi-qs)^{(\alpha-1)} - (1-qs)^{(\alpha-1)}(1-\beta\xi^{\alpha-1}) \right) \\ &= t^{\alpha-1} \left(\beta\xi^{\alpha-1}(1-qs)^{(\alpha-1)} - \beta(\xi-qs)^{(\alpha-1)} \right) \end{aligned}$$

$$\begin{aligned}
&= t^{\alpha-1} \left(\beta \xi^{\alpha-1} (1-qs)^{(\alpha-1)} - \beta \xi^{\alpha-1} \left(1 - q \frac{s}{\xi} \right)^{(\alpha-1)} \right) \\
&\geq \beta \xi^{\alpha-1} t^{\alpha-1} \left((1-qs)^{(\alpha-1)} - \left(1 - q \frac{s}{\xi} \right)^{(\alpha-1)} \right) = 0.
\end{aligned}$$

Therefore, $g_1(t, qs) \geq 0$, $0 \leq s \leq t \leq 1$, $s \leq \xi$. Similarly, with $0 < \beta \xi^{\alpha-1} < 1$, $0 < 1 - \beta \xi^{\alpha-1} < 1$, we have

$$\begin{aligned}
g_2(t, qs) &= t^{\alpha-1} (1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} (1 - \beta \xi^{\alpha-1}) \\
&\geq t^{\alpha-1} (1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)} \\
&= t^{\alpha-1} \left((1-qs)^{(\alpha-1)} - \left(1 - \frac{qs}{t} \right)^{(\alpha-1)} \right) \\
&\geq t^{\alpha-1} \left((1-qs)^{(\alpha-1)} - (1-qs)^{(\alpha-1)} \right) = 0, \\
g_3(t, qs) &= t^{\alpha-1} (1-qs)^{(\alpha-1)} - \beta t^{\alpha-1} (\xi-qs)^{(\alpha-1)} \\
&\geq t^{\alpha-1} \left((1-qs)^{(\alpha-1)} - \beta \xi^{\alpha-1} \left(1 - q \frac{s}{\xi} \right)^{(\alpha-1)} \right) \\
&\geq t^{\alpha-1} (1-qs)^{(\alpha-1)} (1 - \beta \xi^{\alpha-1}) \geq 0.
\end{aligned}$$

It is obvious that $g_4(t, qs) = t^{\alpha-1} (1-qs)^{(\alpha-1)} \geq 0$, $0 \leq t \leq s \leq 1$, $\xi \leq s$. Hence, $G(t, qs) \geq 0$ for all $0 \leq s, t \leq 1$. The proof is completed. \square

Let $\mathbb{C} = C([0, 1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ endowed with the norm defined by $\|u\| = \max_{t \in [0, 1]} |u(t)|$. Define the cone $\mathbb{P} \subset \mathbb{C}$ by

$$\mathbb{P} = \{u \in \mathbb{C} \mid u(t) \geq 0, \text{ for } t \in [0, 1]\}.$$

Lemma 2.5. *Let $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{C}$ be the operator defined by*

$$\begin{aligned}
\mathcal{T}u(t) &:= \int_0^1 G(t, qs) f(s, u(s)) d_q s = - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s \\
&\quad + \int_0^1 \frac{t^{\alpha-1} (1-qs)^{(\alpha-1)}}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha)} f(s, u(s)) d_q s - \int_0^\xi \frac{\beta t^{\alpha-1} (\xi-qs)^{(\alpha-1)}}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha)} f(s, u(s)) d_q s.
\end{aligned}$$

Then $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P}$ is completely continuous.

Proof. The operator $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P}$ is continuous in view of nonnegativeness and continuity of G and f . Let $\Omega \subset \mathbb{P}$ be bounded, i.e., there exists a positive constant $M > 0$ such that $\|u\| \leq M$, for all $u \in \Omega$. Let $K = \max_{0 \leq t \leq 1, 0 \leq u \leq M} |f(t, u)| + 1$, then, for all $u \in \Omega$, we have

$$|\mathcal{T}u(t)| = \left| \int_0^1 G(t, qs) f(s, u(s)) d_q s \right| \leq K \int_0^1 \max_{0 \leq t \leq 1} G(t, qs) d_q s.$$

Hence, $\mathcal{T}(\Omega)$ is bounded.

On the other hand, given $\epsilon > 0$, setting

$$\delta = \min \left\{ 1, \left(\frac{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)\epsilon}{K(2 - \beta\xi^{\alpha-1} + \beta\xi^\alpha)2^\alpha} \right)^{\frac{1}{\alpha-1}} \right\},$$

then, for each $u \in \Omega$, $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and $t_2 - t_1 < \delta$, one has $|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)| < \epsilon$. That is to say, $\mathcal{T}(\Omega)$ is equicontinuity. In fact,

$$\begin{aligned} & |\mathcal{T}u(t_2) - \mathcal{T}u(t_1)| \\ &= \left| - \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_qs + \int_0^1 \frac{t_2^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs \right. \\ &\quad - \int_0^\xi \frac{\beta t_2^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs + \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_qs \\ &\quad \left. - \int_0^1 \frac{t_1^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs + \int_0^\xi \frac{\beta t_1^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs \right| \\ &\leq \left| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_qs - \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_qs \right| \\ &\quad + \left| \int_0^1 \frac{t_2^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs - \int_0^1 \frac{t_1^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs \right| \\ &\quad + \left| \int_0^\xi \frac{\beta t_2^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs - \int_0^\xi \frac{\beta t_1^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} f(s, u(s)) d_qs \right| \\ &\leq \frac{K(t_2^\alpha - t_1^\alpha)}{\Gamma_q(\alpha + 1)} + \frac{K(1 + \beta\xi^\alpha)(t_2^{\alpha-1} - t_1^{\alpha-1})}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)}. \end{aligned}$$

In the following, we divide the proof into two cases.

Case 1. $\delta \leq t_1 < t_2 < 1$, with the use of mean value theorem,

$$t_2^{\alpha-1} - t_1^{\alpha-1} \leq \delta^{\alpha-2}(\alpha - 1)(t_2 - t_1) \leq (\alpha - 1)\delta^{\alpha-1}.$$

Case 2. $0 \leq t_1 < \delta, t_2 < 2\delta$. Then we have

$$t_2^{\alpha-1} - t_1^{\alpha-1} \leq t_2^{\alpha-1} < (2\delta)^{\alpha-1}.$$

Consequently, we have

$$\max\{t_2^{\alpha-1} - t_1^{\alpha-1}, t_2^\alpha - t_1^\alpha\} \leq 2^\alpha \delta^{\alpha-1}$$

and

$$|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)| \leq \frac{K(2 - \beta\xi^\alpha + \beta\xi^\alpha)2^\alpha}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)} \delta^{\alpha-1} \leq \epsilon.$$

By means of the Arzela-Ascoli theorem, we have that $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P}$ is completely continuous. The proof is complete. \square

3. Main Results

In this section, our objective is to give and prove our main results.

Theorem 3.1. *Suppose $f(t, u)$ satisfies*

$$(3.1) \quad 0 \leq \limsup_{u \rightarrow +\infty} \max_{t \in [0,1]} \frac{f(t, u)}{u} < (1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1).$$

Then the problem (1.1) has at least one positive solution.

Proof. By (3.1), taking into account the nonnegativity and continuity of f , there exist $C > 0$, $0 < M < (1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1)$ such that

$$0 \leq f(t, u) < Mu + C, \quad \text{for } t \in [0, 1], \quad u \in [0, +\infty).$$

Let

$$B_R = \left\{ u \in \mathbb{P} \left\| u - C \int_0^1 G(t, qs) d_qs \right\| \leq R \right\}$$

be a convex, bounded, and closed subset of the Banach space E . For $u \in B_R$, we have

$$\|u\| \leq C \left\| \int_0^1 G(t, qs) d_qs \right\| + R \leq R + \frac{C}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1)}$$

and

$$\begin{aligned} & \left| \mathcal{T}u(t) - C \int_0^1 G(t, qs) d_qs \right| \leq \int_0^1 G(t, qs) |f(t, u(s)) - C| d_qs \\ & \leq \max \left\{ \frac{M \|u\|}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1)}, \frac{C}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1)} \right\} \\ & \leq \max \left\{ \frac{M}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1)} \left(R + \frac{C}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1)} \right), \right. \\ & \quad \left. \frac{C}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1)} \right\} \leq R \end{aligned}$$

as long as

$$R \geq \frac{C}{(1 - \beta \xi^{\alpha-1}) \Gamma_q(\alpha + 1) - M}.$$

So, we have $\mathcal{T}(B_R) \subset B_R$. Then, combining with Lemma 2.5, the Schauder fixed point theorem assures that operator \mathcal{T} has at least one fixed point in B_R and then the problem (1.1) has at least one positive solution. The proof is complete. \square

Theorem 3.2. Suppose that $f : [0, 1] \times \mathbb{R} \rightarrow [0, +\infty)$ is a jointly continuous function satisfying the condition

$$|f(t, u) - f(t, v)| \leq L|u - v|, \quad \text{for } t \in [0, 1], \quad u, v \in [0, +\infty).$$

Then the problem (1.1) has a unique positive solution if

$$L \leq \frac{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)}{2(2 - \beta\xi^{\alpha-1} + \beta\xi^\alpha)}.$$

Proof. Defining $\sup_{t \in [0, 1]} |f(t, 0)| = K < \infty$ and selecting

$$r \geq \frac{2K(2 - \beta\xi^{\alpha-1} + \beta\xi^\alpha)}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)},$$

we show that $TB_r \subset B_r$, where $B_r = \{u \in \mathbb{C} : \|u\| \leq r\}$. For $u \in B_r$, we have

$$\begin{aligned} |\mathcal{T}u(t)| &\leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s))| d_qs + \int_0^1 \frac{t^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} |f(s, u(s))| d_qs \\ &\quad + \int_0^\xi \frac{\beta t^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} |f(s, u(s))| d_qs \\ &\leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) d_qs \\ &\quad + \int_0^1 \frac{t^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) d_qs \\ &\quad + \int_0^\xi \frac{\beta t^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) d_qs \\ &\leq (Lr + K) \left(\int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs + \int_0^1 \frac{t^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} d_qs \right. \\ &\quad \left. + \int_0^\xi \frac{\beta t^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} d_qs \right) \leq (Lr + K) \frac{2 - \beta\xi^{\alpha-1} + \beta\xi^\alpha}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)} \leq r. \end{aligned}$$

Taking the maximum over the interval $[0, 1]$, we get $\|\mathcal{T}u(t)\| \leq r$. Now, for $u, v \in \mathbb{C}$ and for each $t \in [0, 1]$, we obtain

$$\begin{aligned} \|\mathcal{T}u(t) - \mathcal{T}v(t)\| &\leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_qs \\ &\quad + \int_0^1 \frac{t^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_qs \\ &\quad + \int_0^\xi \frac{\beta t^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_qs \end{aligned}$$

$$\begin{aligned}
&\leq L\|u - v\| \left(\int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs \right. \\
&\quad \left. + \int_0^1 \frac{t^{\alpha-1}(1 - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} d_qs + \int_0^\xi \frac{\beta t^{\alpha-1}(\xi - qs)^{(\alpha-1)}}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha)} d_qs \right) \\
&\leq \frac{L(2 - \beta\xi^{\alpha-1} + \beta\xi^\alpha)}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)} \|u - v\| = \Lambda_{L,\alpha,\beta,\xi} \|u - v\|,
\end{aligned}$$

where

$$\Lambda_{L,\alpha,\beta,\xi} = \frac{L(2 - \beta\xi^{\alpha-1} + \beta\xi^\alpha)}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)},$$

which depends only on the parameters involved in the problem. As $\Lambda_{L,\alpha,\beta,\xi} < 1$, Then, combining with Lemma 2.5, the Banach fixed point theorem assures that operator \mathcal{S} has a unique fixed point in \mathbb{C} and then the problem (1.1) has a unique positive solution. The proof is complete. \square

4. Two Examples

In this section, we will present some examples to illustrate the main results.

Example 4.1. Consider the following q -fractional three-point boundary value problem

$$(4.1) \quad \begin{cases} (D_q^\alpha u)(t) = \frac{(2u^2 + u)(2 + \sin u)}{9u + 1}, & t \in [0, 1], \\ u(0) = 0, & u(1) = \beta u(\xi). \end{cases}$$

where $\alpha = 1.5$ and $\beta = \xi = q = 0.5$. By simple computation, we can easily have

$$0 \leq \limsup_{u \rightarrow +\infty} \max_{t \in [0,1]} \frac{f(t, u)}{u} \approx 0.666667 < (1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1) \approx 0.769655.$$

Thus, all the assumptions of Theorem 3.1 holds. Consequently, the conclusion of Theorem 3.1 implies that the problem (4.1) has at least one positive solution.

Example 4.2. Consider the following q -fractional three-point boundary value problem

$$(4.2) \quad \begin{cases} (D_q^\alpha u)(t) = \frac{e^{-\pi t}|u(t)|}{(6 + e^{-\pi t})(1 + |u(t)|)}, & t \in [0, 1], \\ u(0) = 0, & u(1) = \beta u(\xi), \end{cases}$$

where $\alpha = 1.5$ and $\beta = \xi = q = 0.5$. Let

$$f(t, u) = \frac{e^{-\pi t}|u|}{(6 + e^{-\pi t})(1 + |u|)}.$$

Clearly, $L = 1/6$ as $|f(t, u) - f(t, v)| \leq 1/6|u - v|$. Further,

$$\frac{2L(2 - \beta\xi^{\alpha-1} + \beta\xi^\alpha)}{(1 - \beta\xi^{\alpha-1})\Gamma_q(\alpha + 1)} = \frac{2(2 - 0.5 * 0.5^{0.5} + 0.5 * 0.5^{1.5})}{6(1 - 0.5 * 0.5^{0.5})\Gamma_{0.5}(2.5)} \approx 0.789628 < 1.$$

Thus, all the assumptions of Theorem 3.2 are satisfied. Therefore, the conclusion of Theorem 3.2 implies that the problem (4.2) has a unique positive solution.

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