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Derivations with Power Values on Lie Ideals in Rings and Banach Algebras

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Abstract. Let R be a 2-torsion free prime ring with center Z, U be the Utumi quotient ring, Q be the Martindale quotient ring of R, d be a derivation of R and L be a Lie ideal of R. If $d(uv)^n = d(u)^m d(v)^l$ or $d(uv)^n = d(v)^l d(u)^m$ for all $u, v \in L$, where m, n, l are fixed positive integers, then $L \subseteq Z$. We also examine the case when R is a semiprime ring. Finally, as an application we apply our result to the continuous derivations on non-commutative Banach algebras. This result simultaneously generalizes a number of results in the literature.

1. Introduction

In all that follows, unless specifically stated otherwise, R is a (semi) prime ring, Z is the center of R, Q is the Martindale quotient ring and U is the Utumi quotient ring of R. The center of U, denoted by C, is called the extended centroid of R (we

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refer the reader to [2], for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. A ring R is called 2-torsion free, if whenever $2x = 0$ with $x \in R$, then $x = 0$. An additive subgroup L of R is said to be a Lie ideal if $[l, r] \in L$ for all $l \in L$ and $r \in R$. A Lie ideal L is said to be non-commutative if $[L, L] \neq 0$. Let L be a non-commutative Lie ideal of R. It is well known that $[R[L, L]R, R] \subseteq L$ (see the proof of [9, Lemma 1.3]). Since $[L, L] \neq 0$, we have $0 \neq [I, R] \subseteq L$, for $I = R[L, L]R$ a nonzero ideal of L. Recall that a ring R is prime if $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + yd(x)$ holds for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$, if $d(x) = [q, x]$ holds for all $x \in R$. For any nonempty subset S of R. If $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in S$, then d is called a derivation which acts as a homomorphism or an anti-homomorphism on S, respectively. A derivation d of R can be extended uniquely to a derivation on Q (see [2, Proposition 2.5.1]) which will be also denoted by d. A derivation d is said to be Q-inner if there exists $q \in Q$ such that $d = ad(q)$, i.e., $d(x) = ad(q)(x) = [q, x]$ for all $x \in R$, otherwise d is Q-outer. Moreover, we remark that the main theory of differential identities initiated by Kharchenko [13].

Let us introduce the background of our investigation. Many results in literature indicate that global structure of a prime ring R is often tightly connected to the behaviour of additive mappings defined on R. A well-known result proved by Herstein [10], stated that if d is a nonzero derivation of a prime ring R such that $d(x)^n \in Z$ for all $x \in R$, then R satisfies s_4, the standard identity in four variables. The Herstein's result was extended to the case of Lie ideals of prime rings by Bergen and Carini [4] and they obtained the same conclussion by proving that if R is a prime ring of characteristic not 2 and d is a nonzero derivation of R such that $d(u)^n \in Z$ for all u in some noncentral Lie ideal of R. The number of authors extended this theorem in several ways.

In 1989, Bell and Kappe [3, Theorem 3] proved that if d is a derivation of a prime ring R such that $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in I$, a nonzero right ideal of R, then $d = 0$ on R. Further Ali et al. [1] extend this result to Lie ideal of a 2-torsion free prime rings. More precisely they prove that if L is a noncentral Lie ideal of R such that $u^2 \in L$ for all $u \in L$ and $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in L$, then either $d = 0$ or $L \subset Z$. In 2007, Wang and You [25] eliminate the hypothesis $u^2 \in L$ for all $u \in L$ and proved the same result as Ali et al. [1]. To be more specific, the statement of Wang and You theorem is the following.

Theorem 1.1 *Let R be a 2-torsion free prime ring and L be a nonzero Lie ideal of R. If d is a derivation of R such that $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in L$, then either $d = 0$ or $L \subseteq Z$.*

The present paper is motivated by the above mention results and here our aim is to generalize all the above results by studying the following theorem:

Theorem 1.2 *Let R be a 2-torsion free prime ring with center Z, d be a nonzero derivation of R and L be a Lie ideal of R. If $d(uv)^n = d(u)^m d(v)^l$ (or $d(uv)^n = d(v)^l d(u)^m$) for all $u, v \in L$, where m, n, l are fixed positive integers, then $L \subseteq Z$.*

Theorem 1.3 *Let R be a 2-torsion free semiprime ring and d be a nonzero derivation of R. If $d(rs)^n = d(r)^m d(s)^l$ (or $d(rs)^n = d(s)^l d(r)^m$) for all $r, s \in R$, where m, n, l are fixed positive integers, then there exists a central idempotent element e of U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

In the last section of this paper we will consider $\mathfrak{A}$ as a Banach algebra with Jacobson radical $rad(\mathfrak{A})$. The classical result of Singer and Wermer [23] stated that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Wermer also formulated the conjecture that the continuity assumption can be removed. In 1988, Thomas [24] verified this conjecture. It is clear that the same result of Singer-Wermer does not hold in non-commutative Banach algebras (because of inner derivations).

However, this situation raises a very interesting question as how to obtain the non-commutative version of Singer-Wermer theorem. A first answer to this problem was obtained by Sinclair [22] and he proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. Since then many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebras. In [18], Mathieu and Murphy proved that if d is a continuous derivation on an arbitrary Banach algebra such that $[d(x), x] \in Z(\mathfrak{A})$ for all $x \in \mathfrak{A}$, then d maps into the radical. Later in [19], Mathieu and Runde removed the continuity assumption using the classical result of Posner's on centralizing derivations of prime rings [21] and Thomas's theorem [24] in which they showed that if d is a derivation which satisfies $[d(x), x] \in Z(\mathfrak{A})$ for all $x \in \mathfrak{A}$, then d has its range in the radical of the algebra. More recently Park [20] proved that if d is a derivation of a non-commutative Banach algebra $\mathfrak{A}$ such that $[[d(x), x], d(x)] \in rad(\mathfrak{A})$ for all $x \in \mathfrak{A}$, then d maps into $rad(\mathfrak{A})$. In [7], De Filippis extended the Park's result to the generalized derivation. Here we will continue the investigation about the relationship between the structure of an algebra $\mathfrak{A}$ and the behaviour of derivations defined on $\mathfrak{A}$. After that we apply our first result on prime rings to the study of analogous conditions for continuous derivations on non-commutative Banach algebras.

More precisely, we will prove the following.

Theorem 1.4 *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and m, n, l be the fixed positive integers. Suppose that there exist a continuous derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $d(rs)^n - d(r)^m d(s)^l$ (or $d(rs)^n - d(s)^l d(r)^m) \in rad(\mathfrak{A})$ for all $r, s \in \mathfrak{A}$, then d maps into the radical of $\mathfrak{A}$.*

2. The Results in Prime Rings

For the proof of main results, we need the following facts, which might be of some independent interest.

Fact 2.1 ([5]) *If I is a two-sided ideal of R, then R, I and U satisfy the same generalized polynomial identities with coefficients in U.*

Fact 2.2 ([2, Proposition 2.5.1]) *Every derivation d of R can be uniquely extended to a derivation of U.*

Fact 2.3 ([15]) *If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities.*

For a complete and detailed description of the theory of generalized polynomial identities involving derivations, we refer the reader to [2, Chapter 7].

We denote by $Der(U)$ the set of all derivations on U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \dots d_m$ with each $d_i \in Der(U)$. Then a differential polynomial is a generalized polynomial with coefficients in U of the form $\Phi(\Delta_j x_i)$ involving non-commutative indeterminates x_i on which the derivation words Δ_j act as unary operations. The differential polynomial $\Phi(\Delta_j x_i)$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of $Der(U)$ consisting of all inner derivations on U and let d be a nonzero derivation on R. By [13, Theorem 2], we have the following result (see also [15, Theorem 1]):

If $\Phi(x_1, \dots, x_n, {}^d x_1, \dots, {}^d x_n)$ is a differential identity on R, then one of the following assertions holds:

(i) either $d \in D_{int}$;

(ii) or, R satisfies the generalized polynomial identity $\Phi(x_1, \cdots, x_n, y_1, \cdots, y_n)$.

Now, we are in a position to prove the main result of this section.

Theorem 1.2 *Let R be a 2-torsion free prime ring with center Z, d be a nonzero derivation of R and L be a Lie ideal of R. If $d(uv)^n = d(u)^m d(v)^l$ (or $d(uv)^n = d(v)^l d(u)^m$) for all $u, v \in L$, where m, n, l are fixed positive integers, then $L \subseteq Z$.*

Proof. Assume on contrary that $L \nsubseteq Z$. Since R is a prime ring and $char(R) \neq 2$, it follows from a result of Herstien [9, p. 4-5], there exists a nonzero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$. In particular, $[I, I] \subseteq L$. By the given hypothesis we divide the proof into two cases:

Case 1. When d satisfy $d(uv)^n = d(u)^m d(v)^l$ for all $u, v \in L$, then by the same argument presented above, we have

$$d([x, y][z, w])^n = d([x, y])^m d([z, w])^l \text{ for all } x, y, z, w \in I.$$

Thus, for all $x, y, z, w \in I$, I satisfies the differential identity

$$\big(([d(x), y] + [x, d(y)])[z, w] + [x, y]([d(z), w] + [z, d(w)])\big)^n = ([d(x), y] + [x, d(y)])^m([d(z), w] + [z, d(w)])^l. \tag{2.1}$$

In the light of Kharchenko's theory [13], we split the proof into two steps:

Step 1. If the derivation d is Q-outer, then by Kharchenko's theorem [13], I satisfies the polynomial identity

$$\big(([s, y] + [x, t])[z, w] + [x, y]([s_1, w] + [z, t_1])\big)^n = ([s, y] + [x, t])^m([s_1, w] + [z, t_1])^l \text{ for all } x, y, z, w, s, t, s_1, t_1 \in I.$$

In particular, for $y = s_1 = t_1 = 0$, I satisfies the blended component $([x, t][z, w])^n = 0$ for all $x, t, z, w \in I$. By Chuang [5, Theorem 2], this polynomial identity is also satisfied by Q and hence R does as well. Note that this is a polynomial identity and hence there exist a field $\mathbb{F}$ such that $R \subseteq \mathcal{M}_k(\mathbb{F})$, the ring of $k \times k$ matrices over a field $\mathbb{F}$, where $k \geq 1$. Moreover, R and $\mathcal{M}_k(\mathbb{F})$ satisfy the same polynomial identity [14, Lemma 1], i.e., $([x, t][z, w])^n = 0$ for all $x, t, z, w \in \mathcal{M}_k(\mathbb{F})$. Let e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. By choosing $x = e_{11}$, $t = e_{12}$, $z = e_{21}$, and $w = e_{22}$, we see that

$$([x, t][z, w])^n = (-1)^n e_{11} \neq 0, \text{ a contradiction.}$$

Step 2. We now assume that d be an inner derivation induced by an element $q \in Q$, i.e, $d(x) = [q, x]$ for all $x \in R$. Then, for any $x, y, z, w \in I$, we have

$$\big(([[q, x], y] + [x, [q, y]])[z, w] + [x, y]([[q, z], w] + [z, [q, w]])\big)^n = ([[q, x], y] + [x, [q, y]])^m([[q, z], w] + [z, [q, w]])^l.$$

By Chuang [5, Theorem 2], I and Q satisfy same generalized polynomial identities (GPIs), we have

$$\big(([[q, x], y] + [x, [q, y]])[z, w] + [x, y]([[q, z], w] + [z, [q, w]])\big)^n = ([[q, x], y] + [x, [q, y]])^m([[q, z], w] + [z, [q, w]])^l, \quad \text{for all } x, y, z, w \in I.$$

Let

$$\phi(x, y) = \big(([[q, x], y] + [x, [q, y]])[z, w] + [x, y]([[q, z], w] + [z, [q, w]])\big)^n - ([[q, x], y] + [x, [q, y]])^m([[q, z], w] + [z, [q, w]])^l.$$

Since $d \neq 0$, $q \notin C$. Moreover, since L is noncentral, R must be noncommutative. Hence $\phi(x, y) = 0$ is a nontrivial generalized polynomial identity for Q. When the center C of Q is infinite, we have $\phi(x,y)=0$ for all $x,y\in Q\otimes_C\overline{C}$, where $\overline{C}$ is algebraic closure of C (see [16, Proposition]). Since both Q and $Q\otimes_C\overline{C}$ are prime and centrally closed [8, Theorems 2.5 and 3.5], we may replace R by Q or $Q\otimes_C\overline{C}$ according as C is finite or infinite. Thus, we may assume that $C=Z$ and R is C-algebra centrally closed, which is either finite or algebraically closed and that $q\in R$, $q\notin Z$ such that R satisfies the generalized polynomial identity

$$\begin{aligned}\big(([[q,x],y]+[x,[q,y]])[z,w]+[x,y]([[q,z],w]+[z,[q,w]])\big)^n\\ =([[q,x],y]+[x,[q,y]])^m([[q,z],w]+[z,[q,w]])^l.\end{aligned}$$

By Martindale's theorem [17], R is a primitive ring having nonzero socle H and the commuting division ring $\mathcal{D}$, which is a finite dimensional central division algebra over Z. Since Z is either finite or algebraically closed, $\mathcal{D}$ must coincide with Z. Hence by Jacobson's theorem [11, p. 75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over Z, i.e., $R\cong End(\mathcal{V}_Z)$. If $\mathcal{V}$ is a finite dimensional over Z, then the density of R on $\mathcal{V}$ implies that $R\cong\mathcal{M}_k(Z)$, where $k=dim_Z\mathcal{V}$.

Assume that $dim(\mathcal{V}_Z)=1$, then $R=Z$ so $I\subseteq Z$, a contradiction. Therefore $dim(\mathcal{V}_Z)\geq 2$. In this case, our aim is to show that, for any $v\in\mathcal{V}$, v and qv are linearly Z-dependent. If $v=0$, then $\{v,qv\}$ is linearly Z-dependent, so we assume that $v\neq 0$. On contrary suppose that v and qv are linearly Z-independent. By the density of R in $End(\mathcal{V}_Z)$ there exists elements $x_0,y_0,z_0,w_0\in R$ such that

$$\begin{aligned}&x_0v=0,\quad x_0qv=qv,\quad y_0v=0,\quad y_0qv=v;\\ &z_0v=0,\quad z_0qv=v,\quad w_0v=0,\quad w_0qv=qv.\end{aligned}$$

With all these, we obtain from the assumption that

$$\begin{aligned}0&=\big(([[q,x_0],y_0]+[x_0,[q,y_0]])[z_0,w_0]+[x_0,y_0]([[q,z_0],w_0]+[z_0,[q,w_0]])\big)^n\\ &=([[q,x_0],y_0]+[x_0,[q,y_0]])^m([[q,z_0],w_0]+[z_0,[q,w_0]])^l\\ &=(-1)^lv,\ \text{a contradiction.}\end{aligned}$$

Thus we conclude that $\{v,qv\}$ is a linearly Z-dependent for any $v\in\mathcal{V}$. From above we have prove that $qv=v\mu(v)$ for all $v\in V$, where $\mu(v)\in Z$ depends on $v\in V$. We claim that $\mu(v)$ is independent of the choice of $v\in V$. Indeed for any $v,w\in V$, if v and w are Z-independent, by the above situation, there exist $\mu(v),\mu(w),\mu(v+w)\in Z$ such that

$$qv=v\mu(v),\ qw=w\mu(w),\ \text{and } q(v+w)=(v+w)\mu(v+w),$$

$$\text{which gives, } v\mu(v)+w\mu(w)=q(v+w)=(v+w)\mu(v+w).$$

Therefore

$$v(\mu(v)-\mu(v+w))+w(\mu(w)-\mu(v+w))=0.$$

Since v and w are Z-independent, we have $\mu(x) = \mu(v+w) = \mu(w)$. If v and w are Z-dependent, say $v = w\beta$, where $\beta \in Z$, then $v\mu(v) = qv = qw\beta = w\mu(w)\beta = v\mu(w)$ and so $\mu(v) = \mu(w)$ as claimed, i.e., $\mu(v)$ is independent of the choice of $v \in \mathcal{V}$. So, there exist $\gamma \in Z$ such that $qv = v\gamma$, for all $v \in V$. Therefore $q \in Z$ and $d = 0$, a contradiction. This completes the proof.

Case 2. Assume that d satisfy $d(uv)^n = d(v)^l d(u)^m$ for all $u, v \in L$. By hypothesis, we can see that

$$d([x,y][z,w])^n = d([z,w])^l d([x,y])^m \text{ for all } x,y,z,w \in I.$$

Equivalently we have

$$\begin{aligned}(2.2)\qquad &\big(([d(x),y]+[x,d(y)])[z,w]+[x,y]([d(z),w]+[z,d(w)])\big)^n \\ &= ([d(z),w]+[z,d(w)])^l([d(x),y]+[x,d(y)])^m.\end{aligned}$$

This condition is a differential identity satisfied by I. By using Kharchenko's theorem [13], either $d = ad(q)$ is the inner derivation induced by an element $q \in Q$ or I satisfies the polynomial identity for all $x, y, z, w, s, t, s_1, t_1 \in I$

$$\begin{aligned}&\big(([s,y]+[x,t])[z,w]+[x,y]([s_1,w]+[z,t_1])\big)^n \\ &= ([s_1,w]+[z,t_1])^l([s,y]+[x,t])^m.\end{aligned}$$

In the latter case set $z = t = s = 0$ to obtain the identity $[x,y][s_1,w] = 0$ for all $x, y, s_1, w \in I$. Thus we can get a contradiction by using the similar technique as presented in Case 1. Assume now that for $q \in Q$ such that $d = ad(q)$, i.e., $d(x) = ad(q)(x) = [q,x]$ for all $x \in R$. By [5, Theorem 2], I and Q satisfies the same generalized polynomial identities and hence by (2.2) we have

$$\begin{aligned}&\big(([[q,x],y]+[x,[q,y]])[z,w]+[x,y]([[q,z],w]+[z,[q,w]])\big)^n \\ &= ([[q,z],w]+[z,[q,w]])^l([[q,x],y]+[x,[q,y]])^m \\ &\qquad \text{for all} x,y,z,w \in Q.\end{aligned}$$

In view of the above situation in Case 1, we assume that R is centrally closed over Z which is either finite or algebraically closed and that $q \in R$, $q \notin Z$ such that R satisfies the nontrivial generalized polynomial identity

$$\begin{aligned}&\big(([[q,x],y]+[x,[q,y]])[z,w]+[x,y]([[q,z],w]+[z,[q,w]])\big)^n \\ &= ([[q,z],w]+[z,[q,w]])^l([[q,x],y]+[x,[q,y]])^m.\end{aligned}$$

Moreover, we know that R is isomorphic to a dense subring of $End(\mathcal{V}_Z)$ for some vector space $\mathcal{V}$ over Z. For any $v \in \mathcal{V}$, we claim that v and qv are Z-dependent. Suppose to the contrary that v and qv are Z-independent, then by the density of R in $End(\mathcal{V}_Z)$ there exist elements $x_0, y_0, z_0, w_0 \in R$ such that

$$\begin{aligned}&x_0v = 0, \quad x_0qv = qv, \quad y_0v = 0, \quad y_0qv = v; \\ &z_0v = 0, \quad z_0qv = v, \quad w_0v = 0, \quad w_0qv = qv.\end{aligned}$$

One can see that

$$\begin{aligned}0 &= \big(([[q,x_0],y_0]+[x_0,[q,y_0]])[z_0,w_0]+[x_0,y_0]([[q,z_0],w_0]+[z_0,[q,w_0]])\big)^n\\ &= ([[q,z_0],w_0]+[z_0,[q,w_0]])^l([[q,x_0],y_0]+[x_0,[q,y_0]])^m\\ &= (-1)^m v, \ \text{ a contradiction.}\end{aligned}$$

Thus, v and qv are Z-dependent as claimed. In view of Case 1, we know that $q \in Z$ and so $d = 0$, a contradiction. This completes the proof. □

We immediately get the following corollary from the above theorem:

Corollary 2.1 *Let R be a 2-torsion free prime ring and d be a nonzero derivation of R. If $d(rs)^n = d(r)^m d(s)^l$ (or $d(rs)^n = d(s)^l d(r)^m$) for all $r, s \in R$, where m, n, l are fixed positive integers, then R is commutative.*

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 2.1 Let S be any ring.

(i) Let $R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S\right\}$ and $L = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S\right\}$. We define a map $d : R \to R$ by $d(x) = e_{11}x - xe_{11}$. Then it is easy to see that d is a nonzero derivation and L is a Lie ideal of R such that for all fixed positive integers m, n, l, d satisfies the properties, $d(uv)^n = d(u)^m d(v)^l$ and $d(uv)^n = d(v)^l d(u)^m$ for $u, v \in L$, however $L \nsubseteq Z$.

(ii) Let $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in S\right\}$ and $L = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S\right\}$. Define a map $d : R \to R$ by $d(x) = [x, e_{11} + e_{12}]$. Then R is a ring under usual operations. It is easy to see that d is a nonzero derivation and L is a Lie ideal of R which satisfies the properties, $d(uv)^n = d(u)^m d(v)^l$ and $d(uv)^n = d(v)^l d(u)^m$ for $u, v \in L$, where m, n, l are fixed positive integers, but $L \nsubseteq Z$.

3. The Results in Semiprime Rings

From now on, R is the semiprime ring and U is the left Utumi quotient ring of R. In order to prove the main result of this section we will make use of the following facts:

Fact 3.1 ([2, Proposition 2.5.1]) *Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole U.*

Fact 3.2 ([6, p. 38]) *If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 3.3 ([6, p. 42]) *Let B be the set of all the idempotents in C, the extended centroid of R. Suppose that R is an orthogonally complete B-algebra. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any nonzero derivation of R.*

Now, we prove the following.

Theorem 1.3. *Let R be a 2-torsion free semiprime ring and d be a nonzero derivation of R. If $d(rs)^n = d(r)^m d(s)^l$ for all $r, s \in R$, where m, n, l are fixed positive integers, then there exists a central idempotent element e of U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Proof. Since R is semiprime and we have given that $d(rs)^n = d(r)^m d(s)^l$ for all $r, s \in I$. By Fact , $Z(U) = C$, the extended centroid of R, and by Fact , derivation d can be uniquely extended on U. In view of Lee [15], R and U satisfy the same differential identities, hence $d(rs)^n = d(r)^m d(s)^l$ for all $r, s \in U$.

Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Since U is a B-algebra which is orthogonally complete (see Chuang [6, p. 42], and Fact), by Fact , MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. Therefore $\overline{d}$ has in $\overline{U}$ the same property as d on U, i.e., for all $\overline{r}, \overline{s} \in \overline{U}$, $\overline{d}(\overline{rs})^n = \overline{d}(\overline{r})^m \overline{d}(\overline{s})^l$. It is obvious that $\overline{U}$ is prime. Therefore, by Corollary , either $\overline{U}$ is commutative or $\overline{d} = 0$. This implies that, for any maximal ideal M of B, either $d(U) \subseteq MU$ or $[U, U] \subseteq MU$. In any case $d(U)[U, U] \subseteq MU$ for all M, where MU runs over all prime ideals of U. Therefore $d(U)[U, U] \subseteq \bigcap_M MU = 0$, we obtain $d(U)[U, U] = 0$.

By using the theory of orthogonal completion for semiprime rings [2, Chapter 3], it is clear that there exists a central idempotent element e in U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative. This completes the proof of the theorem. □

Using arguments similar to those used in the proof of the Theorem , we may conclude with the following (we omit the proof brevity). We can prove

Theorem 3.1 *Let R be a 2-torsion free semiprime ring and d be a nonzero derivation of R. If $d(rs)^n = d(s)^l d(r)^m$ for all $r, s \in R$, where m, n, l are fixed positive integers, then there exists a central idempotent element e of U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

4. Applications on Banach Algebras

In this section we obtain some results on non-commutative Banach algebras by using the preceding algebraic results. Here $\mathfrak{A}$ will denote a complex Banach algebra. Let us state some well known and elementary definitions for the sake of completeness. By Banach algebra we shall mean a complex normed algebra $\mathfrak{A}$ whose underlying vector space is a Banach space. The Jacobson radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, $\mathfrak{A}$ is called semisimple. In fact any Banach algebra $\mathfrak{A}$ without a unity can be embedded into a unital Banach algebra $\mathfrak{A}_I = \mathfrak{A} \oplus \mathbb{C}$ as an ideal of codimension one. In particular we may identify $\mathfrak{A}$ with the ideal $\{(x, 0) : x \in \mathfrak{A}\}$ in $\mathfrak{A}_I$ via the isometric isomorphism $x \mapsto (x, 0)$.

In this section, we apply the purely algebraic results which is derived in section 2 and obtain the conditions that every continuous derivation on a Banach algebra maps into the radical. The proofs of the results rely on a Sinclair's theorem [22] which stated that every continuous derivation d of a Banach algebra $\mathfrak{A}$ leaves the primitive ideals of $\mathfrak{A}$ invariant. As we have mentioned before, Thomas [24] has generalized the Singer-Wermer theorem by proving that any derivation on a commutative Banach algebra maps the algebra into its radical, this result leads to the question whether the theorem can be proven without any commutativity assumption. On this manuscript there are many papers [17, 18, 22, 23] which shows that the theorem holds without any commutativity assumption. We also acquire that every derivation maps into its radical with some property without any commutativity assumption. Our first result in this section is about continuous derivations on Banach algebras:

Theorem 1.4. *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and m, n, l be the fixed positive integers. Suppose that there exist a continuous derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $d(rs)^n - d(r)^m d(s)^l \in rad(\mathfrak{A})$ for all $r, s \in \mathfrak{A}$, then d maps into the radical of $\mathfrak{A}$.*

Proof. We have given that $d(rs)^n - d(r)^m d(s)^l \in rad(\mathfrak{A})$ for all $r, s \in \mathfrak{A}$. Under the assumption that d is nonzero continuous derivation with Jacobson radical $rad(\mathfrak{A})$. In [22], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Since the Jacobson radical $rad(\mathfrak{A})$ is the intersection of all primitive ideals, we have $d(rad(\mathfrak{A})) \subseteq rad(\mathfrak{A})$, which means that there is no loss of generality in assuming that $\mathfrak{A}$ is semisimple. Since d leaves all primitive ideals invariant, one can introduce for any primitive ideal $P \subseteq \mathfrak{A}$, a nonzero derivation

$$d_P : \mathfrak{A}/P \to \mathfrak{A}/P$$

$$\text{where } \mathfrak{A}/P = \overline{A} \text{ is a factor Banach algebra}$$

$$d_P(r + P) = d(r) + P \text{ for all } r \in \mathfrak{A}.$$

Note that every derivation on a semisimple Banach algebra is continuous [12, Remark 4.3]. First, in case $\mathfrak{A}/P$ is commutative, combining this result with the Singer-Wermer theorem gives $d_P = 0$ since $\mathfrak{A}/P$ is semisimple. We intend to show that $d_P = 0$ in case when $\mathfrak{A}/P$ is non-commutative. Since the assumption of the theorem gives $d(rs)^n - d(r)^m d(s)^l \in rad(\mathfrak{A})$ for all $r, s \in \mathfrak{A}/P$. Thus by Corollary

2.1, it is immediate that either $\overline{\mathfrak{A}}$ is commutative or $d_P = 0$ on $\mathfrak{A}/P$. Consequently $d(\mathfrak{A}) \subseteq P$ for any primitive ideal P. Since the radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals in $\mathfrak{A}$, we get the required conclusion. □

Using arguments similar to those used in the proof of the above theorem, we can prove

Theorem 4.1 *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and m, n, l be the fixed positive integers. Suppose that there exist a continuous derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $d(rs)^n - d(s)^l d(r)^m \in rad(\mathfrak{A})$ for all $r, s \in \mathfrak{A}$, then d maps into the radical of $\mathfrak{A}$.*

In order to prove our last result, we will use the following well-known result concerning semisimple Banach algebra contained [12].

Lemma 4.1 *Every nonzero derivation on a semisimple Banach algebra is continuous.*

In view of the above Lemma 4.1, and Theorem 1.4, we may prove the following theorem in the special case when $\mathfrak{A}$ is a semisimple Banach algebra.

Corollary 4.1 *Let $\mathfrak{A}$ be a non-commutative semisimple Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and m, n, l be the fixed positive integers. Suppose that there exist a continuous derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $d(rs)^n - d(r)^m d(s)^l \in rad(\mathfrak{A})$ for all $r, s \in \mathfrak{A}$, then $d(\mathfrak{A}) = 0$.*

Proof. By the hypothesis d is continuous. In view of the above Lemma 4.1, every nonzero derivation on a semisimple Banach algebra is continuous. Thus every nonzero derivation on a semisimple Banach algebra leaves the primitive ideals of the algebra invariant. Now by using the same argument as used in the proof of the Theorem and the fact that $rad(\mathfrak{A}) = 0$, as $\mathfrak{A}$ is semisimple, we get the required result. □

We immediately get the following corollary from the above theorem.

Corollary 4.2 *Let $\mathfrak{A}$ be a non-commutative semisimple Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and m, n, l be the fixed positive integers. Suppose that there exist a continuous derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $d(rs)^n = d(s)^l d(r)^m \in rad(\mathfrak{A})$ for all $r, s \in \mathfrak{A}$, then $d(\mathfrak{A}) = 0$.*

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