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Some Properties of (p, q) - Lucas Number

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Abstract. In this paper, we consider the generalized Lucas sequence which is the (p, q) - Lucas sequence. Then we used the Binet's formula to show some properties of the (p, q) - Lucas number. We get some generalized identities of the (p, q) - Lucas number.

1. Introduction

Fibonacci number and Lucas number cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [4] and Vajda [5]. The Fibonacci number F_n is the term of the sequence where each term is the sum of the two previous terms beginning with the initial values $F_0 = 0$ and $F_1 = 1$. The well-known Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. And the Lucas sequence is defined as $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Falcon [3] studied the k-Lucas sequence $\{L_{k,n}\}$ which is defined as $L_{k,0} = 2, L_{k,1} = k$ and $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ for $n \geq 1,\ k \geq 1$. If $k = 1$, we get the classical Lucas sequence $\{2, 1, 3, 4, 7, 11, 18, ...\}$. If $k = 2$, we get the Pell-Lucas sequence $\{2, 2, 6, 14, 34, 82, 198, ...\}$.

The well-known Binet's formulas for k-Fibonacci number and k-Lucas number, see [1, 2, 3], are given by $F_{k,n} = \dfrac{r_1^n - r_2^n}{r_1 - r_2}$ and $L_{k,n} = r_1^n + r_2^n$ where $r_1 = \dfrac{k + \sqrt{k^2+4}}{2}$ and $r_2 = \dfrac{k - \sqrt{k^2+4}}{2}$ are roots of the characteristic equation $r^2 - kr - 1 = 0$. We note that $r_1 + r_2 = k$, $r_1 r_2 = -1$ and $r_1 - r_2 = \sqrt{k^2+4}$.

The generalized of Fibonacci sequence $\{F_{p,q,n}\}$ is defined as $F_{p,q,0} = 0, F_{p,q,1} = 1$ and $F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}$ for $n \geq 2$ which we call the (p, q) - Fibonacci sequence. So, each term of the (p, q) - Fibonacci sequence is called (p, q) - Fibonacci

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number. Moreover, the generalized of Lucas sequence $\{L_{p,q,n}\}$ is defined as $L_{p,q,0} = 2, L_{p,q,1} = p$ and $L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}$. So, it is called the (p, q) - Lucas sequence. Then each term of the (p, q) - Lucas sequence is called (p, q) - Lucas number. The Binet's formulas for the (p, q) - Fibonacci number and the (p, q) - Lucas number are given by $F_{p,q,n} = \dfrac{r_1^n - r_2^n}{r_1 - r_2}$ and $L_{p,q,n} = r_1^n + r_2^n$ where $r_1 = \dfrac{p + \sqrt{p^2 + 4q}}{2}$ and $r_2 = \dfrac{p - \sqrt{p^2 + 4q}}{2}$ are roots of the characteristic equation $r^2 - pr - q = 0$. We note that $r_1 + r_2 = p$, $r_1 r_2 = -q$ and $r_1 - r_2 = \sqrt{p^2 + 4q}$.

In 2015, Suvarnamani and Tatong [6] proved some results of the (p, q) - Fibonacci number. Moreover, Raina and Srivastava [7] showed a class of numbers associated with the Lucas number. Then Djordjevicand Srivastava [8] showed the example for the application of the Fibonacci number to the generalized function. In this paper, we find some properties of the (p, q) - Lucas numbers.

2. Main Results

Theorem 2.1. *For $n \geq 1$, we get $L_{p,q,n+1}L_{p,q,n-1} - L_{p,q,n}^2 = (-q)^{n-1}(p^2 + 4q)$.*

Proof. For $n \geq 1$, we have $L_{p,q,n+1}L_{p,q,n-1} - L_{p,q,n}^2$

$$= (r_1^{n+1} + r_2^{n+1})(r_1^{n-1} + r_2^{n-1}) - (r_1^n + r_2^n)^2$$
$$= (r_1^{2n} + r_2^{2n} + r_1^{n-1}r_2^{n+1} + r_1^{n+1}r_2^{n-1}) - (r_1^{2n} + 2r_1^n r_2^n + r_2^{2n})$$
$$= r_1^{n+1}r_2^{n-1} + r_1^{n-1}r_2^{n+1} - 2r_1^n r_2^n$$
$$= r_1^{n-1}r_2^{n-1}(r_1^2 - 2r_1r_2 + r_2^2)$$
$$= (-q)^{n-1}(r_1 - r_2)^2$$
$$= (-q)^{n-1}(p^2 + 4q).$$ □

Theorem 2.2. *For $n \geq 2$, we get*

$$L_{p,q,n-2}L_{p,q,n+1} - L_{p,q,n-1}L_{p,q,n} = (-q)^{n-2}(p^3 + 4pq).$$

Proof. For $n \geq 2$, we have $L_{p,q,n-2}L_{p,q,n+1} - L_{p,q,n-1}L_{p,q,n}$

$$= (r_1^{n-2} + r_2^{n-2})(r_1^{n+1} + r_2^{n+1}) - (r_1^{n-1} + r_2^{n-1})(r_1^n + r_2^n)$$
$$= (r_1^{2n-1} + r_2^{2n-1} + r_1^{n-2}r_2^{n+1} + r_1^{n+1}r_2^{n-2}) - (r_1^{2n-1} + r_2^{2n-1} + r_1^n r_2^{n-1} + r_1^{n-1}r_2^n)$$
$$= r_1^{n-2}r_2^{n+1} + r_1^{n+1}r_2^{n-2} - r_1^n r_2^{n-1} - r_1^{n-1}r_2^n.$$
$$= r_1^{n-2}r_2^{n-2}(r_1^3 + r_2^3 - r_1^2 r_2 - r_1 r_2^2)$$
$$= (-q)^{n-2}(r_1 - r_2)(r_1^2 - r_2^2)$$
$$= (-q)^{n-2}(r_1 + r_2)(r_1 - r_2)^2$$
$$= (-q)^{n-2}(p)(p^2 + 4q)$$
$$= (-q)^{n-2}(p^3 + 4pq).$$ □

Theorem 2.3. *For $n \geq 1$, we get $L_{p,q,n+1}L_{p,q,n-1} + (-q)^n(p^2 + 4q) = L_{p,q,n}^2$.*

Proof. For $n \geq 1$, by Theorem 2.1, we have

$$L_{p,q,n+1}L_{p,q,n-1} - L_{p,q,n}^2 = (-q)^{n-1}(p^2 + 4q).$$

We get $L_{p,q,n+1}L_{p,q,n-1} - (-q)^{n-1}(p^2+4q) = L^2_{p,q,n}$.
So, $L_{p,q,n+1}L_{p,q,n-1} + (-q)^{n}(p^2+4q) = L^2_{p,q,n}$. □

Theorem 2.4. *For $m, n \geq 1$, we get*

$$L_{p,q,m}L_{p,q,n+1} + qL_{p,q,m-1}L_{p,q,n} = (p^2+4q)F_{p,q,m+n}.$$

Proof. For $m, n \geq 1$, we have

$$\begin{aligned}
&L_{p,q,m}L_{p,q,n+1} + qL_{p,q,m-1}L_{p,q,n}\\
&= (r_1^m + r_2^m)(r_1^{n+1} + r_2^{n+1}) + (-r_1r_2)(r_1^{m-1} + r_2^{m-1})(r_1^n + r_2^n)\\
&= r_1^{m+n+1} + r_2^{m+n+1} + r_1^{n+1}r_2^m + r_1^m r_2^{n+1} - r_1^{m+n}r_2 - r_2^{m+n}r_1 - r_1^{n+1}r_2^m - r_1^m r_2^{n+1}\\
&= r_1^{m+n+1} + r_2^{m+n+1} - r_1^{m+n}r_2 - r_2^{m+n}r_1\\
&= r_1^{m+n}(r_1 - r_2) - r_2^{m+n}(r_1 - r_2)\\
&= (r_1^{m+n} - r_2^{m+n})(r_1 - r_2)\\
&= \frac{r_1^{m+n} - r_2^{m+n}}{r_1 - r_2}(r_1 - r_2)^2\\
&= F_{p,q,m+n}(p^2+4q).
\end{aligned}$$
□

Theorem 2.5. *For $m, n \geq 1$, we get*

$$L_{p,q,m-n}L_{p,q,m+n} - L^2_{p,q,m} = (-q)^{m-n}(p^2+4q).$$

Proof. For $m, n \geq 1$, we have

$$\begin{aligned}
&L_{p,q,m-n}L_{p,q,m+n} - L^2_{p,q,n}\\
&= (r_1^{m-n} + r_2^{m-n})(r_1^{m+n} + r_2^{m+n}) - (r_1^m + r_2^m)^2\\
&= r_1^{2m} + r_1^{m+n}r_2^{m-n} + r_1^{m-n}r_2^{m+n} + r_2^{2m} - r_1^{2m} - 2r_1^m r_2^m - r_2^{2m}\\
&= r_1^{m+n}r_2^{m-n} + r_1^{m-n}r_2^{m+n} - 2r_1^m r_2^m\\
&= r_1^{m-n}r_2^{m-n}(r_1^{2n} - 2r_1^n r_2^n + r_2^{2n})\\
&= (-q)^{m-n}(r_1^n - r_2^n)^2\\
&= (-q)^{m-n}(r_1^n - r_2^n)^2\frac{r_1 - r_2}{r_1 - r_2}\\
&= (-q)^{m-n}\sqrt{p^2+4q}\, F^2_{p,q,n}.
\end{aligned}$$
□

Theorem 2.6. *For $m, n, k \geq 1$, we get*

$$L_{p,q,m+n}L_{p,q,m+k} - L_{p,q,m}L_{p,q,m+n+k} = (-1)^{m+1}q^m(p^2+4q)F_{p,q,n}F_{p,q,k}.$$

Proof. For $m, n \geq 1$, we have

$$\begin{aligned}
&L_{p,q,m-n}L_{p,q,m+n} - L^2_{p,q,n}\\
&= (r_1^{m+n} + r_2^{m+n})(r_1^{m+k} + r_2^{m+k}) - (r_1^m + r_2^m)(r_1^{m+n+k} + r_2^{m+n+k})\\
&= r_1^{m+n}r_2^{m+k} + r_1^{m+k}r_2^{m+n} - r_1^m r_2^{m+n+k} - r_1^{m+n+k}r_2^m\\
&= r_1^m r_2^m(r_1^n r_2^k + r_1^k r_2^n - r_2^{n+k} - r_1^{n+k})\\
&= (-q)^m(r_2^k(r_1^n - r_2^n) - r_1^k(r_1^n - r_2^n))\\
&= (-1)(-q)^m(r_1^n - r_2^n)(r_1^k - r_2^k)\\
&= (-1)^{m+1}q^m(p^2+4q)F_{p,q,n}F_{p,q,k}.
\end{aligned}$$
□

Theorem 2.7.

$$\lim_{n\to\infty} \frac{L_{p,q,n}}{L_{p,q,n-1}} = r_1.$$

Proof.
$$\lim_{n\to\infty} \frac{L_{p,q,n}}{L_{p,q,n-1}} = \lim_{n\to\infty} \frac{r_1^n + r_2^n}{r_1^{n-1} + r_2^{n-1}} = \lim_{n\to\infty} \frac{r_1^n(1+(\frac{r_2}{r_1})^n)}{r_1^{n-1}(1+(\frac{r_2}{r_1})^{n-1})} = \lim_{n\to\infty} \frac{r_1(1+(\frac{r_2}{r_1})^n)}{1+(\frac{r_2}{r_1})^{n-1}}.$$

Using the ratio $\frac{r_2}{r_1}$, then $\lim_{n\to\infty}(\frac{r_2}{r_1})^n = 0$.

Next, we get $\lim_{n\to\infty} \frac{L_{p,q,n}}{L_{p,q,n-1}} = r_1$. □

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