

On Deferred Statistical Convergence of Sequences

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ABSTRACT. In this paper, deferred statistical convergence is defined by using deferred Cesàro mean instead of Cesàro mean in the definition of statistical convergence. The obtained method is compared with strong deferred Cesàro mean and statistical convergence under some certain assumptions. Also, some inclusion theorems and examples are given.

1. Introduction and Definitions

The concept of statistical convergence was introduced by I.J. Steinhaus in [17] and H. Fast in [6] independently in the same year. Nowadays, this subject has become one of the most active research area in the theory of summability. It was applied in different areas of mathematics such as number theory by P. Erdős-G.Tenenbaum [5] and summability theory by A. R. Freedman-J. J. Sember-M. Raphael [7].

Furthermore, this subject was studied in [3], [4], [8], [9], [10], [15], [16] etc.

Statistical convergence is also closely related to the subject of asymptotic density (or natural density) of the subset of natural numbers (see, [2]) and its root goes back to A. Zygmund [19].

In 1932, R.P. Agnew [1] defined the deferred Cesaro mean as a generalization of Cesàro mean of real (or complex) valued sequence $x = (x_k)$ by

$$(1.1) \quad (D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \quad n = 1, 2, 3, \dots,$$

where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of nonnegative

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integers satisfying

$$(1.2) \quad p(n) < q(n) \text{ and } \lim_{n \rightarrow \infty} q(n) = \infty.$$

R.P. Agnew also showed that the method in (1.1) has some important properties besides regularity (see for regularity [11, Theorem 3]).

A sequence $x = (x_k)$ is said to be strong $D_{p,q}$ -convergent to l if

$$\lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - l| = 0,$$

holds and it is denoted by

$$\lim_{n \rightarrow \infty} x_n = l (D [p, q]).$$

Recall that a sequence $x = (x_k)$ is said to be statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k : k \leq n, |x_k - l| \geq \varepsilon\}| = 0,$$

satisfied where the vertical bars indicate the numbers of elements inside the set and it is denoted by $\lim_{n \rightarrow \infty} x_n = l(S)$.

There is a natural relationship between statistical convergence and strong summability of sequences. This relation has been investigated in [3], [4], [12], [13], [14] and etc.

Definition 1.1 (Deferred Statistical Convergence (DS)) A sequence $x = (x_k)$ is said to be deferred statistically convergent to $l \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| = 0,$$

holds and it is denoted by

$$\lim_{n \rightarrow \infty} x_n = l (DS [p, q]).$$

It is clear that;

(i) If $q(n) = n$ and $p(n) = 0$, then Definition 1.1. is coincide with the definition of statistical convergence,

(ii) If we consider $q(n) = k_n$ and $p(n) = k_{n-1}$ (for any lacunary sequence of nonnegative integers with $k_n - k_{n-1} \rightarrow \infty$ as $n \rightarrow \infty$), then Definition 1.1. is turned to Lacunary Statistical convergence [9],

(iii) If $q(n) = \lambda_n$ and $p(n) = 0$ (where λ_n is a strictly increasing sequence of natural numbers such that $\lim_{n \rightarrow \infty} \lambda_n = \infty$), then Definition 1.1. is coincide λ -statistical convergence of sequences which is given by Osikievich [18] and Mursaleen [13].

2. Inclusion Theorems

Throughout the paper, we consider the sequence of nonnegative natural numbers $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ satisfying (1.1). Any other restrictions on (if needed) $p(n)$ and $q(n)$ will be given in related theorems.

2.1 Comparison of D with DS

In this section, strong deferred Cesàro mean $D[p, q]$ and deferred statistical convergence $DS[p, q]$ will be compared. It is going to show that these two methods are equivalent only for bounded sequences.

Theorem 2.1.1. *If $x_n \rightarrow l (D[p, q])$, then $x_n \rightarrow l (DS[p, q])$.*

Proof. Assume $x_n \rightarrow l (D[p, q])$. For an arbitrary $\varepsilon > 0$, following inequality

$$\begin{aligned} & \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - l| \\ = & \frac{1}{q(n) - p(n)} \left(\sum_{\substack{k=p(n)+1 \\ |x_k - l| \geq \varepsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ |x_k - l| < \varepsilon}}^{q(n)} \right) |x_k - l| \geq \frac{1}{q(n) - p(n)} \sum_{\substack{k=p(n)+1 \\ |x_k - l| \geq \varepsilon}}^{q(n)} |x_k - l| \\ \geq & \varepsilon \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| \end{aligned}$$

holds. After taking limit when $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| = 0.$$

Therefore, desired result is obtained. □

Corollary 2.1.2. *If $x_n \rightarrow l (n \rightarrow \infty)$, then $x_n \rightarrow l (DS[p, q])$.*

Remark 2.1.3. The converse of Theorem 2.1.1 and Corollary 2.1.2 are not true, in general.

For this, consider a sequence $x = (x_k)$ as

$$x_k := \begin{cases} k^2, & \left[\left[\sqrt{q(n)} \right] \right] - m_0 < k \leq \left[\left[\sqrt{q(n)} \right] \right], \quad n = 1, 2, 3, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

where $q(n)$ is a monotone increasing sequence and $m_0 \neq 0$ is an arbitrary fixed natural number.

If we consider $D[p, q]$ for the sequence $p(n)$ satisfying

$$0 < p(n) \leq \left[\left[\sqrt{q(n)} \right] \right] - m_0,$$

then for an arbitrary $\varepsilon > 0$ we have

$$\frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - 0| \geq \varepsilon\}| = \frac{m_0}{q(n) - p(n)} \rightarrow 0,$$

when $n \rightarrow \infty$, i.e., $x_k \rightarrow 0(DS[p, q])$.

On the other hand,

$$\frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} |x_k - 0| \geq \frac{m_0 \left(\left[\left[\sqrt{q(n)} \right] \right] - m_0 \right)^2}{q(n) - p(n)} \rightarrow m_0,$$

when $n \rightarrow \infty$, i.e., (x_k) is not $D[p, q]$ convergent to zero. It is also clear that the sequence does not convergent to zero in usual case.

Let us recall that l_∞ is the set of all bounded sequences.

Theorem 2.1.4. *If $x = (x_n) \in l_\infty$ and $x_n \rightarrow l(DS[p, q])$ then $x_n \rightarrow l(D[p, q])$.*

Proof. Suppose that $x = (x_n) \in l_\infty$ and $x_n \rightarrow l(DS[p, q])$. Under the assumption on (x_n) there exists positive reel number M such that $|x_n - l| \leq M$ holds for all n .

So, the inequality

$$\begin{aligned} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - l| &= \frac{1}{q(n) - p(n)} \left(\sum_{\substack{k=p(n)+1 \\ |x_k - l| \geq \varepsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ |x_k - l| < \varepsilon}}^{q(n)} \right) |x_k - l| \\ &\leq \frac{1}{q(n) - p(n)} \left(M \sum_{\substack{k=p(n)+1 \\ |x_k - l| \geq \varepsilon}}^{q(n)} 1 + \varepsilon \sum_{\substack{k=p(n)+1 \\ |x_k - l| < \varepsilon}}^{q(n)} 1 \right) \\ &\leq M \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| \\ &\quad + \varepsilon \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| < \varepsilon\}|, \end{aligned}$$

is hold. From the limit relation we have

$$\lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - l| = 0.$$

So, the proof is completed. \square

2.2. Comparison of S with $DS[p, q]$

In this section, statistical convergence and deferred statistical convergence will be compared under some restrictions on $p(n)$ or $q(n)$.

Theorem 2.2.1. *If the sequence $\left\{ \frac{p(n)}{q(n)-p(n)} \right\}_{n \in \mathbb{N}}$ is bounded, then $x_n \rightarrow l(S)$ implies $x_n \rightarrow l(DS[p, q])$.*

Proof. Let's give a note about the sequences of positive natural numbers $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ without proof: if $\lim_{n \rightarrow \infty} a_n = a$, $a \in \mathbb{R}$ and $\lim_{n \rightarrow \infty} b_n = +\infty$, then $\lim_{n \rightarrow \infty} a_{b_n} = a$.

From the assumption on (x_n) , the limit relation

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} |\{k : k \leq n, |x_k - l| \geq \varepsilon\}| = 0,$$

holds for every $\varepsilon > 0$. Since the sequence $q(n)$ satisfies (1.2), then the sequence

$$\left\{ \frac{|\{k : k \leq q(n), |x_k - l| \geq \varepsilon\}|}{q(n)} \right\}_{n \in \mathbb{N}}$$

is convergent to zero.

Therefore, the inclusion

$$\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\} \subset \{k : k \leq q(n), |x_k - l| \geq \varepsilon\},$$

and the inequality

$$|\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| \leq |\{k : k \leq q(n), |x_k - l| \geq \varepsilon\}|,$$

are hold. From the last inequality we have

$$\begin{aligned} & \frac{1}{q(n)-p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| \\ & \leq \left(1 + \frac{p(n)}{q(n)-p(n)} \right) \cdot \frac{1}{q(n)} |\{k : k \leq q(n), |x_k - l| \geq \varepsilon\}|, \end{aligned}$$

and from the limit relation we get

$$x_k \rightarrow l(DS[p, q]).$$

So, desired result is obtained. □

Corollary 2.2.2. *Let $\{q(n)\}_{n \in \mathbb{N}}$ be an arbitrary sequence with $q(n) < n$ for all $n \in \mathbb{N}$ and $\left\{ \frac{n}{q(n)-p(n)} \right\}_{n \in \mathbb{N}}$ be a bounded sequence. Then, $x_n \rightarrow l(S)$ implies $x_n \rightarrow l(DS[p, q])$.*

Remark 2.2.3. The converse of Theorem 2.2.1 is not true even if $\left\{ \frac{p(n)}{q(n)-p(n)} \right\}_{n \in \mathbb{N}}$ is bounded.

Example 2.2.4. Let us consider $p(n) = 2n$, $q(n) = 4n$ and a sequence $x = (x_n)$ as

$$x_n = \begin{cases} \frac{n+1}{2}, & n \text{ is odd,} \\ -\frac{n}{2}, & n \text{ is even.} \end{cases}$$

It is clear that the assumption of Theorem 2.2.1 is fulfilled and $x_n \rightarrow 0$ ($D[2n, 4n]$).

From Theorem 2.1.1 we get $x_n \rightarrow 0$ ($DS[2n, 4n]$). But, for an arbitrary small $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k : k \leq n, |x_n - 0| \geq \varepsilon\}| \neq 0.$$

Definition 2.2.5. A method $DS[p, q]$ is called properly deferred when $\{p(n)\}$ and $\{q(n)\}$ satisfy in addition to (1.2) the condition $\left\{\frac{p(n)}{q(n)-p(n)}\right\}$ is bounded for all n .

Remark 2.2.6. Two properly deferred statistically convergence method must not be include each other. Let

$$x_n := \begin{cases} k + 1, & n = 2k + 1, \\ -k, & n = 2k. \end{cases}$$

It is clear that $x_n \rightarrow 0$ ($DS[2n, 4n]$) and $x_n \rightarrow \frac{1}{2}$ ($DS[2n - 1, 4n - 1]$).

Theorem 2.2.7. Let $q(n) = n$ for all $n \in \mathbb{N}$. Then, $x_n \rightarrow l$ ($DS[p, n]$) if and only if $x_n \rightarrow l$ (S).

Proof. (\implies) Let us assume that $x_k \rightarrow l$ ($DS[p, n]$). We shall apply the technique which was used by Agnew in [1]. Then, for any $n \in \mathbb{N}$,

$$p(n) = n^{(1)} > p(n^{(1)}) = n^{(2)} > p(n^{(2)}) = n^{(3)} > \dots,$$

and we may write the set $\{k \leq n : |x_k - l| \geq \varepsilon\}$ as

$$\{k \leq n : |x_k - l| \geq \varepsilon\} = \{k \leq n^{(1)} : |x_k - l| \geq \varepsilon\} \cup \{n^{(1)} < k \leq n : |x_k - l| \geq \varepsilon\},$$

and the set $\{1 < k \leq n^{(1)} : |x_k - l| \geq \varepsilon\}$ as

$$\begin{aligned} & \{1 < k \leq n^{(1)} : |x_k - l| \geq \varepsilon\} \\ &= \{k \leq n^{(2)} : |x_k - l| \geq \varepsilon\} \cup \{n^{(2)} < k \leq n^{(1)} : |x_k - l| \geq \varepsilon\}, \end{aligned}$$

and the set $\{k \leq n^{(2)} : |x_k - l| \geq \varepsilon\}$ as

$$\{k \leq n^{(2)} : |x_k - l| \geq \varepsilon\} = \{k \leq n^{(3)} : |x_k - l| \geq \varepsilon\} \cup \{n^{(3)} < k \leq n^{(2)} : |x_k - l| \geq \varepsilon\},$$

and if this process is continued we obtain

$$\{k \leq n^{(h-1)} : |x_k - l| \geq \varepsilon\}$$

$$= \left\{ k \leq n^{(h)} : |x_k - l| \geq \varepsilon \right\} \cup \left\{ n^{(h)} < k \leq n^{(h-1)} : |x_k - l| \geq \varepsilon \right\}$$

for a certain positive integer $h > 0$ depending on n such that $n^{(h)} \geq 1$ and $n^{(h+1)} = 0$. From the above discussion, the relation

$$\begin{aligned} & \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| \\ &= \sum_{m=0}^h \frac{n^{(m)} - n^{(m+1)}}{n} \frac{1}{n^{(m)} - n^{(m+1)}} \left| \left\{ n^{(m+1)} < k \leq n^{(m)} : |x_k - l| \geq \varepsilon \right\} \right| \end{aligned}$$

holds for every n . This relation gives that statistical convergency of the sequence (x_n) to l is a linear combination of following sequence.

$$\left\{ \frac{1}{n^{(m)} - n^{(m+1)}} \left| \left\{ n^{(m+1)} < k \leq n^{(m)} : |x_k - l| \geq \varepsilon \right\} \right| \right\}_{m \in \mathbb{N}}.$$

Let us consider the matrix

$$b_{n,m} := \begin{cases} \frac{n^{(m)} - n^{(m+1)}}{n}, & m = 0, 1, 2, \dots, h, \\ 0, & \text{otherwise.} \end{cases}$$

where $n^{(0)} := n$.

The matrix $(b_{n,m})$ is satisfied the Silverman Toeplitz theorem (see in [11]). So, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0$$

since

$$\frac{1}{n^{(m)} - n^{(m+1)}} \left| \left\{ n^{(m+1)} < k \leq n^{(m)} : |x_k - l| \geq \varepsilon \right\} \right| \rightarrow 0,$$

when $n \rightarrow \infty$.

(\Leftarrow) Since $q(n) = n$ is satisfied (1.2), then the inverse of the theorem is a simple consequence of Theorem 2.2.1. \square

Corollary 2.2.8. *Assume that $\{q(n)\}_{n \in \mathbb{N}}$ contains almost all positive integers. Then, $x_n \rightarrow l(DS [p, q])$ implies $x_n \rightarrow l(S)$.*

Proof. Let $x_n \rightarrow l(DS [p, q])$ for an arbitrary $\{p(n)\}$ and choose sufficiently large positive integer m such that the set $\{q(n)\}$ contains all positive integers which is greater than m . Then, it can be constructed a sequence (k_n) as follows:

$$k_1 = k_2 = \dots = k_m = 1$$

and for each $n > m$ an index k_n such that $q_{k_n} = n$.

It is clear from the construction that (k_n) is a monotone increasing sequence. So, from the assumption $x_n \rightarrow l(DS [p_{k_n}, q_{k_n}])$. Hence, the proof of Corollary follows from Theorem 2.2.5. \square

Corollary 2.2.9. *Let us assume $\{q(n)\}$ contains almost all positive integers. If $x = (x_n)$ is a sequence such that $x_n \rightarrow l(DS[p, q])$ for an arbitrary $\{p(n)\}_{n \in \mathbb{N}}$ and $\Delta x_n = O(\frac{1}{n})$ then $x_n \rightarrow l(n \rightarrow \infty)$.*

Proof. It is clear from Corollary 2.2.6 and the assumption on $q(n)$ that $x_n \rightarrow l(S)$. Therefore, if we use [9, Theorem 3] we obtained the proof. \square

2.3. Comparison of $DS[p', q']$ with $DS[p, q]$

In this section, the methods $DS[p, q]$ and $DS[p', q']$ will be compared under the following restriction

$$(3.1) \quad p(n) \leq p'(n) < q'(n) \leq q(n)$$

for all $n \in \mathbb{N}$.

Theorem 2.3.1. *Let $p' = \{p'(n)\}_{n \in \mathbb{N}}$ and $q' = \{q'(n)\}_{n \in \mathbb{N}}$ be sequences of positive natural numbers satisfying (3.1) such that the sets*

$$\{k : p(n) < k \leq p'(n)\} \text{ and } \{k : q'(n) < k \leq q(n)\}$$

are finite sets for all $n \in \mathbb{N}$. Then, $x_k \rightarrow l(DS[p', q'])$ implies $x_k \rightarrow l(DS[p, q])$.

Proof. Let us consider the sequence $x = (x_k)$ such that $x_k \rightarrow l(DS[p', q'])$. For an arbitrary $\varepsilon > 0$, the equality

$$\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\} = \{k : p(n) < k \leq p'(n), |x_k - l| \geq \varepsilon\}$$

$$\cup \{k : p'(n) < k \leq q'(n), |x_k - l| \geq \varepsilon\} \cup \{k : q'(n) < k \leq q(n), |x_k - l| \geq \varepsilon\},$$

and

$$\begin{aligned} & \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| \\ & \leq \frac{1}{q'(n) - p'(n)} |\{k : p(n) < k \leq p'(n), |x_k - l| \geq \varepsilon\}| \\ & + \frac{1}{q'(n) - p'(n)} |\{k : p'(n) < k \leq q'(n), |x_k - l| \geq \varepsilon\}| \\ & + \frac{1}{q'(n) - p'(n)} |\{k : q'(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}|, \end{aligned}$$

are hold.

On taking limits when $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), |x_k - l| \geq \varepsilon\}| = 0.$$

This proves our assertion. \square

Theorem 2.3.2. Let $\{p(n)\}_{n \in \mathbb{N}}$, $\{q(n)\}_{n \in \mathbb{N}}$ and $\{p'(n)\}_{n \in \mathbb{N}}$, $\{q'(n)\}_{n \in \mathbb{N}}$ be sequences of positive natural numbers satisfying (3.1) such that

$$\lim_{n \rightarrow \infty} \frac{q(n) - p(n)}{q'(n) - p'(n)} = d > 0.$$

Then, $x_k \rightarrow l(DS[p, q])$ implies $x_k \rightarrow l(DS[p', q'])$.

Proof. It is easy to see that the inclusion

$$\{k : p'(n) + 1 \leq k \leq q'(n), |x_k - l| \geq \varepsilon\} \subset \{k : p(n) + 1 \leq k \leq q(n), |x_k - l| \geq \varepsilon\}$$

and the inequality

$$|\{k : p'(n) + 1 \leq k \leq q'(n), |x_k - l| \geq \varepsilon\}| \leq |\{k : p(n) + 1 \leq k \leq q(n), |x_k - l| \geq \varepsilon\}|$$

are true. So, we have

$$\begin{aligned} & \frac{1}{q'(n) - p'(n)} |\{k : p'(n) + 1 \leq k \leq q'(n), |x_k - l| \geq \varepsilon\}| \\ & \leq \frac{q(n) - p(n)}{q'(n) - p'(n)} \frac{1}{q(n) - p(n)} |\{k : p(n) + 1 \leq k \leq q(n), |x_k - l| \geq \varepsilon\}|. \end{aligned}$$

Taking limits as $n \rightarrow \infty$ the desired results is obtained. \square

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