KYUNGPOOK Math. J. 56(2016), 349-355
http://dx.doi.org/10.5666/KMJ.2016.56.2.349
pISSN 1225-6951 eISSN 0454-8124

On the Numbers of Palindromes

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Abstract. For any integer $n \geq 2$, each palindrome of n induces a circulant graph of order n. It is known that for each integer $n \geq 2$, there is a one-to-one correspondence between the set of (resp. aperiodic) palindromes of n and the set of (resp. connected) circulant graphs of order n (cf. [2]). This bijection gives a one-to-one correspondence of the palindromes σ with $\gcd(\sigma) = 1$ to the connected circulant graphs. It was also shown that the number of palindromes σ of n with $\gcd(\sigma) = 1$ is the same number of aperiodic palindromes of n. Let a_n (resp. b_n) be the number of aperiodic palindromes σ of n with $\gcd(\sigma) = 1$ (resp. $\gcd(\sigma) \neq 1$). Let c_n (resp. d_n) be the number of periodic palindromes σ of n with $\gcd(\sigma) = 1$ (resp. $\gcd(\sigma) \neq 1$). In this paper, we calculate the numbers a_n, b_n, c_n, d_n in two ways. In Theorem 2.3, we find recurrence relations for a_n, b_n, c_n, d_n in terms of a_d for $d|n$ and $d \neq n$. Afterwards, we find formulae for $a_n,\ b_n,\ c_n,\ d_n$ explicitly in Theorem 2.5.

1. Introduction

A *composition* of n is an ordered word $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$ of positive integers that sum to n. A composition is *aperiodic* if it is not the concatenation of a proper part of itself. A *numeral palindrome* (or simply, *palindrome*) is a composition which is

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Received July 27, 2015; revised December 9, 2015; accepted December 23, 2015.
2010 Mathematics Subject Classification: Primary 05C30; Secondary 05C25.
Key words and phrases: compositions of n, palindromes, circulant graphs.

unchanged by reversing the order of the summands (*i.e.*, $\sigma = \sigma^{-1}$, where $\sigma^{-1} = \sigma_m \ldots \sigma_2\sigma_1$ for $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$). For each composition $\sigma = \sigma_1\sigma_2 \cdots \sigma_\ell$, define $\gcd(\sigma) := \gcd\{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$. The gcd of the integers of a word σ is denoted by $\gcd(\sigma)$, and σ is *prime* if $\gcd(\sigma) = 1$. If an integer d divides $\gcd(\sigma)$ then we define a composition $\frac{1}{d}\sigma$ by $\frac{1}{d}\sigma := \frac{\sigma_1}{d}\frac{\sigma_2}{d}\cdots\frac{\sigma_\ell}{d}$. For each integer $n \geq 1$, we define the set of palindromes and aperiodic palindromes as follows:

$$\mathcal{P}(n) = \{\sigma \mid \sigma \text{ is a composition of } n, \sigma = \sigma^{-1}\},$$

and

$$\mathcal{P}_A(n) = \{\sigma \in \mathcal{P}(n) \mid \sigma \text{ is aperiodic}\}.$$

The cardinality of these two sets are considered in [1, 2, 3, 4] (see also Lemma 2.1). A *circulant graph* is a graph whose automorphism group includes a cyclic subgroup which acts transitively on the vertex set of the graph. It is shown in [2, Theorem 3.1] that for each integer $n \geq 2$, there is a one-to-one correspondence between the set of palindromes of n and the set of circulant graphs of order n. In particular, they showed in [2, Theorem 3.5] that for each integer $n \geq 2$, there is a one-to-one correspondence between the set of aperiodic palindromes of n and the set of connected circulant graphs of order n. This bijection gives a one-to-one correspondence of the palindromes having a gcd of 1 to the connected circulant graphs. It was also shown that the number of prime palindromes of n is the same number of aperiodic palindromes. In this paper, we divide the set $\mathcal{P}(n)$ into four parts as follows.

Definition 1.1. For each integer $n \geq 2$, we define $A_n, B_n, C_n, D_n, a_n, b_n, c_n, d_n$ by

$$\begin{aligned} A_n &:= \{\sigma \in \mathcal{P}_A(n) \mid \gcd(\sigma) = 1\}, \\ B_n &:= \{\sigma \in \mathcal{P}_A(n) \mid \gcd(\sigma) \neq 1\}, \\ C_n &:= \{\sigma \in \mathcal{P}(n) \setminus \mathcal{P}_A(n) \mid \gcd(\sigma) = 1\}, \\ D_n &:= \{\sigma \in \mathcal{P}(n) \setminus \mathcal{P}_A(n) \mid \gcd(\sigma) \neq 1\} \end{aligned}$$

and $a_n := |A_n|,\ b_n := |B_n|,\ c_n := |C_n|,\ d_n := |D_n|$. Set $a_1 = 1$, $b_1 = 0$, $c_1 = 0$ and $d_1 = 0$.

In this paper we calculate the numbers a_n, b_n, c_n, d_n ($n \geq 2$) in two ways. In Theorem 2.3, we find recurrence relations for a_n, b_n, c_n, d_n in terms of a_d for $d|n$ and $d \neq n$. These recurrence relations enable us to calculate the numbers a_n, b_n, c_n, d_n recursively. Afterwards, we find formulae for $a_n,\ b_n,\ c_n,\ d_n$ explicitly in Theorem 2.5. By using these formulae, we can compute a_n, b_n, c_n, d_n without using recursive relations. Moreover, we calculate a_n, b_n, c_n, d_n with n up to 20 in Table 1.

2. Main Results

In this section, we prove main results Theorem 2.3 and Theorem 2.5. To do this, we first need to review some basic results. The following lemma comes from Definition 1.1 and [2, Corollaries 3.2 and 3.6].

Lemma 2.1. *For each integer $n \geq 2$, the following hold:*

(i) $a_n + b_n + c_n + d_n = |\mathcal{P}(n)| = 2^{\lfloor \frac{n}{2} \rfloor}$,

(ii) $a_n + b_n = |\mathcal{P}_A(n)| = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^{\lfloor \frac{d}{2} \rfloor}$,

where μ is the Möbius function.

Note that the numbers $\mu(n)$ for $1 \leq n \leq 20$ are given in Table 1. For our convenience, we need to review some notation. We refer the reader to [2] for more details. For an aperiodic palindrome ω of k, denote by $\sigma = \omega^r$ the palindrome of $n = kr$ we get by concatenating r copies of ω.

Lemma 2.2. *For any integer $n \geq 2$, we have $d_n = \sum_{d|n \ d \neq n} c_d$.*

Proof. We will show that there is a bijection between the set D_n and the disjoint union of the sets C_d over all $d|n,\ d \neq n$. For each $\sigma \in D_n$, there exist an integer $2 \leq d < n$ with $d|n$ and an aperiodic palindrome $\omega \in \mathcal{P}_A\left(\frac{n}{d}\right)$ satisfying $\sigma = \omega^d$, where $\gcd(\sigma) = \gcd(\omega) \neq 1$. Put $\sigma' := \left(\frac{1}{\gcd(\omega)}\omega\right)^d$. Then $\frac{1}{\gcd(\omega)}\omega$ is an aperiodic palindrome in $\mathcal{P}_A\left(\frac{n}{\gcd(\omega)d}\right)$ and $\gcd\left(\frac{1}{\gcd(\omega)}\omega\right) = \gcd(\sigma') = 1$. Hence $\sigma' \in C_{d'}$ for $d' := \frac{n}{\gcd(\omega)}$ satisfying $d'|n$ and $d' \neq n$. Hence it is straightforward that map $f : D_n \to \cup_{d|n, d\neq n} C_d$ defined by $f(\sigma) = \sigma'$ is injective and thus $d_n \leq \sum_{d|n \ d \neq n} c_d$. On the other hand, let $\sigma \in C_d$ for some d satisfying $d|n$ and $d \neq n$. Then there exist an integer $\ell \geq 2$ and an aperiodic palindrome $\omega \in \mathcal{P}_A\left(\frac{d}{\ell}\right)$ such that $\sigma = \omega^\ell$ and $\gcd(\sigma) = \gcd(\omega) = 1$ all hold. Put $\sigma' := \left(\frac{n}{d}\omega\right)^\ell$. As $\gcd(\sigma') = \gcd\left(\frac{n}{d}\omega\right) = \frac{n}{d} \neq 1$, $\sigma' \in D_n$. This shows f is onto and thus f is a bijection. This completes the proof. □

In Theorem 2.3, we find recurrence relations for a_n, b_n, c_n, d_n in terms of a_d for $d|n$ and $d \neq n$. We first show $b_n = c_n$ and we find a recurrence relation for c_n in terms of a_d for $d|n$ and $d \neq n$. Using this recurrence relation for c_n, we will also find recurrence relations for a_n and d_n.

Theorem 2.3. *For each integer $n \geq 2$, the following hold:*

(i) $a_n = 2^{\lfloor \frac{n}{2} \rfloor} - \sum_{d|n \ d \neq n} \tau\left(\frac{n}{d}\right) a_d$,

(ii) $b_n = c_n = \sum_{d|n \ d \neq n} a_d$,

(iii) $d_n = \sum_{d|n \ d \neq n} \left(\tau\left(\frac{n}{d}\right) - 2\right) a_d$,

where $\tau(m)$ is the number of divisors of an integer m.

Proof. Let an integer $n \geq 2$ be given.
(ii): It is shown in [2, Theorem 3.5] that there is a one-to-one correspondence between the set of aperiodic palindromes of n and the set of connected circulant

graphs of order n. Hence we have $a_n + b_n = a_n + c_n$ and this shows $b_n = c_n$. Now we will prove $c_n = \sum_{d|n\ d\neq n} a_d$ by showing that there is a bijection between the set C_n and the disjoint union of the sets A_d over all $d|n,\ d \neq n$. Let $\sigma \in C_n$. Then there exist an integer $\ell \geq 2$ and an aperiodic palindrome $\omega \in A_{\frac{n}{\ell}}$ such that $\sigma = \omega^\ell$ and $\gcd(\sigma) = \gcd(\omega) = 1$ all hold with $\ell|n$. Since $\ell \geq 2$, $d := \frac{n}{\ell}$ satisfies $\omega \in A_d$, $d|n$ and $1 \leq d < n$. On the other hand, if $\sigma \in A_d$ for some $d \neq n$ satisfying $d|n$ then $\gcd(\sigma) = 1$. Hence palindrome $\omega := \sigma^{\frac{n}{d}}$ satisfies $\gcd(\omega) = \gcd(\sigma) = 1$ and thus $\omega \in C_n$.
(i) and (iii): It follows by Lemma 2.1 (i), Lemma 2.2 and Theorem 2.3 (ii) that

$$\begin{aligned}
2^{\lfloor \frac{n}{2} \rfloor} - a_n &= b_n + c_n + d_n = 2c_n + d_n = 2\sum_{d|n\ d\neq n} a_d + \sum_{d|n\ d\neq n} c_d \\
&= 2\sum_{d|n\ d\neq n} a_d + \left(\sum_{d|n\ d\neq n}\left(\sum_{\alpha|d\ \alpha\neq d} a_\alpha\right)\right) \\
&= 2\sum_{d|n\ d\neq n} a_d + \left(\sum_{d|n\ d\neq n}\left(\tau\left(\frac{n}{d}\right) - 2\right) a_d\right) = \sum_{d|n\ d\neq n} \tau\left(\frac{n}{d}\right) a_d
\end{aligned}$$

holds, where $\tau(m)$ is the number of divisors of m (Note that the numbers $\tau(n)$ for $1 \leq n \leq 20$ are given in Table 1.). This shows the result (i). By Lemma 2.1 (i) and Theorem 2.3 (i)-(ii), we have

$$2^{\lfloor \frac{n}{2} \rfloor} = a_n + b_n + c_n + d_n = \left(2^{\lfloor \frac{n}{2} \rfloor} - \sum_{d|n\ d\neq n} \tau\left(\frac{n}{d}\right) a_d\right) + 2\left(\sum_{d|n\ d\neq n} a_d\right) + d_n$$

which shows part (iii). This completes the proof. □

By Theorem 2.3, we can compute the numbers a_n, b_n, c_n, d_n recursively. The result for a_n, b_n, c_n, d_n with n up to 20 are given in the following example.

Example 2.4. The numbers $\tau(n)$ $(1 \leq n \leq 20)$ are listed in Table 1. Using these numbers and Theorem 2.3, the numbers a_n, b_n, c_n, d_n for n up to 20 are calculated in Table 1. For our convenience, let α_n and β_n be the number of connected and disconnected circulant graphs of order n, respectively. Then $\alpha_n = a_n + c_n = a_n + b_n = |\mathcal{P}_A(n)|$ and $\beta_n = b_n + d_n = c_n + d_n = |\mathcal{P}(n) \setminus \mathcal{P}_A(n)|$. The numbers α_n, β_n for n up to 20 are also calculated in Table 1. We note here that for any positive integer n the number γ_n of circulant graphs of order n is equal to $\alpha_n + \beta_n = |\mathcal{P}(n)| = 2^{\lfloor \frac{n}{2} \rfloor}$ (see Lemma 2.1 (i) and [2, Theorem 3.1]).

Now, we find formulae for $a_n,\ b_n,\ c_n,\ d_n$ in Theorem 2.5. These results enable us to compute the values a_n, b_n, c_n, d_n directly without using recursive relations (cf. Theorem 2.3). We first find a formula for a_n, and using the formula for a_n we find formulae for b_n, c_n, d_n.

Theorem 2.5. *For each integer $n \geq 2$, the following hold:*

	a_n	b_n	c_n	d_n	$\tau(n)$	$f(n)$	$\mu(n)$	α_n	β_n	γ_n
$n=1$	1	0	0	0	1	1	1	1	0	$2^0=1$
$n=2$	0	1	1	0	2	-2	-1	1	1	$2^1=2$
$n=3$	0	1	1	0	2	-2	-1	1	1	$2^1=2$
$n=4$	1	1	1	1	3	1	0	2	2	$2^2=4$
$n=5$	2	1	1	0	2	-2	-1	3	1	$2^2=4$
$n=6$	4	1	1	2	4	4	1	5	3	$2^3=8$
$n=7$	6	1	1	0	2	-2	-1	7	1	$2^3=8$
$n=8$	10	2	2	2	4	0	0	12	4	$2^4=16$
$n=9$	13	1	1	1	3	1	0	14	2	$2^4=16$
$n=10$	24	3	3	2	4	4	1	27	5	$2^5=32$
$n=11$	30	1	1	0	2	-2	-1	31	1	$2^5=32$
$n=12$	48	6	6	4	6	-2	0	54	10	$2^6=64$
$n=13$	62	1	1	0	2	-2	-1	63	1	$2^6=64$
$n=14$	112	7	7	2	4	4	1	119	9	$2^7=128$
$n=15$	120	3	3	2	4	4	1	123	5	$2^7=128$
$n=16$	228	12	12	4	5	0	0	240	16	$2^8=256$
$n=17$	254	1	1	0	2	-2	-1	255	1	$2^8=256$
$n=18$	472	18	18	4	6	-2	0	490	22	$2^9=512$
$n=19$	510	1	1	0	2	-2	-1	511	1	$2^9=512$
$n=20$	962	28	28	6	6	-2	0	990	34	$2^{10}=1024$

Table 1: $a_n, b_n, c_n, d_n, \alpha_n, \beta_n$ $(1 \le n \le 20)$

(i) $a_n = \sum_{d|n} f\left(\frac{n}{d}\right) 2^{\lfloor \frac{d}{2} \rfloor}$,

(ii) $b_n = c_n = \sum_{d|n} \left(\mu\left(\frac{n}{d}\right) - f\left(\frac{n}{d}\right)\right) 2^{\lfloor \frac{d}{2} \rfloor}$,

(iii) $d_n = \sum_{d|n\ d\neq n} \left(f\left(\frac{n}{d}\right) - 2\mu\left(\frac{n}{d}\right)\right) 2^{\lfloor \frac{d}{2} \rfloor}$,

where μ is the Möbius function and the function f is defined as follows: for each integer $m = p_1^{m_1} \cdots p_r^{m_r} \ge 1$ with r distinct primes,

$$f(m) = \begin{cases} (-2)^{|\{1\le i\le r|\ m_i=1\}|} & \text{if } \max\{m_i \mid 1 \le i \le r\} \le 2, \\ 0 & \text{if } \max\{m_i \mid 1 \le i \le r\} \ge 3. \end{cases}$$

Proof. (i): Let $n \ge 2$ be an integer. It follows by Lemma 2.1 (ii) and Theorem 2.3 (ii) that

$$|\mathcal{P}_A(n)| = a_n + b_n = a_n + c_n = a_n + \sum_{d|n,\ d\neq n} a_d = \sum_{d|n} a_d.$$

By applying the Möbius inversion formula, $a_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) |\mathcal{P}_A(d)|$ holds, where μ is the Möbius function. Now, by Lemma 2.1 (ii), we have

$$\begin{aligned} a_n &= \sum_{d|n} \mu\left(\frac{n}{d}\right)|\mathcal{P}_A(d)| = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left(\sum_{\alpha|d} \mu\left(\frac{d}{\alpha}\right) 2^{\lfloor\frac{\alpha}{2}\rfloor}\right) \\ &= \sum_{\alpha|n} \sum_{d'|\frac{n}{\alpha}} \mu\left(\frac{n}{\alpha d'}\right)\mu(d')2^{\lfloor\frac{\alpha}{2}\rfloor}. \end{aligned}$$

It is not hard to show that for each integer $m = p_1^{m_1}\cdots p_r^{m_r} \geq 1$ with r distinct primes, $\sum_{e|m} \mu\left(\frac{m}{e}\right)\mu(e) = f(m)$ holds. Note that the numbers $f(n)$ for $1 \leq n \leq 20$ are given in Table 1. Since $\sum_{d'|\frac{n}{\alpha}} \mu\left(\frac{n}{\alpha d'}\right)\mu(d') = f\left(\frac{n}{\alpha}\right)$, we have (i).

(ii): As $b_n = c_n = |\mathcal{P}_A(n)| - a_n = \sum_{d|n}\left(\mu\left(\frac{n}{d}\right) - f\left(\frac{n}{d}\right)\right)2^{\lfloor\frac{d}{2}\rfloor}$ follows by Theorem 2.3 (ii), Lemma 2.1 (ii) and Theorem 2.5 (i), the result (ii) follows immediately.

(iii): It follows by Lemma 2.1 (i), Theorem 2.3 (ii) and Theorem 2.5 (i)-(ii) that

$$\begin{aligned} d_n &= 2^{\lfloor\frac{n}{2}\rfloor} - (a_n + b_n + c_n) = 2^{\lfloor\frac{n}{2}\rfloor} - (a_n + 2b_n) \\ &= 2^{\lfloor\frac{n}{2}\rfloor} + \sum_{d|n}\left(f\left(\frac{n}{d}\right) - 2\mu\left(\frac{n}{d}\right)\right)2^{\lfloor\frac{d}{2}\rfloor} \\ &= \sum_{d|n\ d\neq n}\left(f\left(\frac{n}{d}\right) - 2\mu\left(\frac{n}{d}\right)\right)2^{\lfloor\frac{d}{2}\rfloor} \end{aligned}$$

holds. This completes the proof. □

Remark 2.6. The formulae in Theorem 2.5 give us a way to compute the values a_n, b_n, c_n, d_n directly. For example, we can also obtain the numbers a_n, b_n, c_n, d_n $(1 \leq n \leq 20)$ in Table 1 by using Theorem 2.5 and the numbers $f(n)$, $\mu(n)$ $(1 \leq n \leq 20)$ which are also listed in Table 1.

Remark 2.7. Circulant graphs can be described as follows. For a subset $S \subseteq \mathbb{Z}_n$ satisfying $S = -S \pmod{n}$, a circulant graph of order n denoted by $G(n, S)$ is a graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set E, where $\{i, j\} \in E$ if and only if $i \neq j$ and $j - i \in S \pmod{n}$. For each composition $\sigma = \sigma_1\sigma_2\cdots\sigma_\ell$ of n, we define a subset $\Omega_\sigma = \{s_1, s_2 \cdots s_\ell\} \subseteq \mathbb{Z}_n$, where $s_1 = 0$, $s_i = \sum_{j=1}^{i-1} \sigma_j$ for each $i = 2, \ldots, \ell$. For example, for compositions $\sigma = n$ and $\sigma' = \overbrace{11\cdots1}^{n \text{ ones}}$ we have $\Omega_\sigma = \{0\}$ and $\Omega_{\sigma'} = \{0, 1, 2, \ldots, n-1\}$. For each palindrome $\sigma \in \mathcal{P}(n)$, the corresponding set Ω_σ defines a circulant graph $G(n, \Omega_\sigma)$ of order n. Any aperiodic (resp. periodic) palindrome σ does not imply that $G(n, \Omega_\sigma)$ is connected (resp. disconnected). For example, $\sigma = 242 \in \mathcal{P}_A(8)$ is aperiodic with disconnected graph $G(8, \{0, 2, 6\})$, and $\sigma = (121)^2 \in \mathcal{P}(8) \setminus \mathcal{P}_A(8)$ is periodic with connected graph $G(8, \{0, 1, 3, 4, 5, 7\})$.

Now we can reformulate Definition 1.1 as the following:

$$\begin{aligned} A_n &= \{\sigma \in \mathcal{P}_A(n) \mid G(n, \Omega_\sigma) \text{ is a connected graph of order } n.\}, \\ B_n &= \{\sigma \in \mathcal{P}_A(n) \mid G(n, \Omega_\sigma) \text{ is a disconnected graph of order } n.\}, \\ C_n &= \{\sigma \in \mathcal{P}(n) \setminus \mathcal{P}_A(n) \mid G(n, \Omega_\sigma) \text{ is a connected graph of order } n.\}, \\ D_n &= \{\sigma \in \mathcal{P}(n) \setminus \mathcal{P}_A(n) \mid G(n, \Omega_\sigma) \text{ is a disconnected graph of order } n.\}. \end{aligned}$$

Hence the results of this paper give us the number of circulant graphs in the above four cases.

Acknowledgements. The authors would like to thank the referee for valuable comments on this work. Sejeong Bang was supported by NRF grant NRF-2014R1A1A1002094. Yan-Quan Feng was supported by the National Natural Science Foundation of China (11571035, 11231008). Jaeun Lee was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF)funded by the Ministry of Education, Science and Technology (NRF-2013K2A2A2000416).

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