

Korobov Polynomials of the Fifth Kind and of the Sixth Kind

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ABSTRACT. Recently, Korobov polynomials have been received a lot of attention, which are discrete analogs of Bernoulli polynomials. In particular, these polynomials are used to derive some interpolation formulas of many variables and a discrete analog of the Euler summation formula. In this paper, we extend these family of polynomials to consider the Korobov polynomials of the fifth kind and of the sixth kind. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. Introduction

Carlitz [3, 4] introduced the degenerate version of the Bernoulli polynomials called the *degenerate Bernoulli polynomials*. On the other hand, Korobov [14, 15] studied the first degenerate version of the Bernoulli polynomials of the second kind called *Korobov polynomials of the first kind*. It is noted here, in passing, that the degenerate Bernoulli polynomials were rediscovered by Ustinov [19] under the name

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of *Korobov polynomials of the second kind*. Recently, various kinds of degenerate versions of the familiar polynomials like Bernoulli polynomials, Euler polynomials and their variants were investigated by many researchers. In [10], introduced were two kinds of new degenerate versions of Bernoulli polynomials of the second kind. Here, we would like to present yet two other degenerate versions of Bernoulli polynomials of the second kind. We will give properties, explicit expression, recurrence relations, and connections with other familiar polynomials by using the technique of umbral calculus.

The *Bernoulli polynomials of the second kind* $b_n(x)$ are given by the generating function

$$(1.1) \quad \frac{t}{\log(1+t)}(1+t)^x = \sum_{n \geq 0} b_n(x) \frac{t^n}{n!}.$$

For $x = 0$, $b_n = b_n(0)$ are called the *Bernoulli numbers of the second kind*. The *Korobov polynomials* $K_n(\lambda, x)$ of the first kind are given by

$$\frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n \geq 0} K_n(\lambda, x) \frac{t^n}{n!}.$$

When $x = 0$, $K_n(\lambda) = K_n(\lambda, 0)$ are called the *Korobov numbers of the first kind*. Since 2002, Korobov polynomials have been received a lot of attention, which are discrete analogs of Bernoulli polynomials (see [15]). In particular, these polynomials are used to derive some interpolation formulas of many variables and a discrete analog of the Euler summation formula (see [19]). Also, these polynomials are used to study a class of two-player games on posets with a rank function, in which each move of the winning strategy is unique (see [7]). Here, we introduce *Korobov polynomials of the fifth kind* $K_{n,5}(\lambda, x)$ and of the sixth kind $K_{n,6}(\lambda, x)$ respectively given by

$$(1.2) \quad \frac{t}{\log(1+t)}(1+\lambda)^{\frac{x}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} = \sum_{n \geq 0} K_{n,5}(\lambda, x) \frac{t^n}{n!},$$

$$(1.3) \quad \frac{\lambda t}{(1+t)^\lambda - 1}(1+\lambda)^{\frac{x}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} = \sum_{n \geq 0} K_{n,6}(\lambda, x) \frac{t^n}{n!}.$$

For $x = 0$, $K_{n,5}(\lambda) = K_{n,5}(\lambda, 0)$ and $K_{n,6}(\lambda) = K_{n,6}(\lambda, 0)$ are called the Korobov numbers of the fifth kind and of the sixth kind, respectively. We observe that

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^x &= \lim_{\lambda \rightarrow 0} \frac{t}{\log(1+t)}(1+\lambda)^{\frac{x}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \\ &= \lim_{\lambda \rightarrow 0} \frac{\lambda t}{(1+t)^\lambda - 1}(1+\lambda)^{\frac{x}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} = \frac{t}{\log(1+t)}(1+t)^x, \end{aligned}$$

which implies that $\lim_{\lambda \rightarrow 0} K_n(\lambda, x) = \lim_{\lambda \rightarrow 0} K_{n,5}(\lambda, x) = \lim_{\lambda \rightarrow 0} K_{n,6}(\lambda, x) = b_n(x)$.

It is immediate to see that $K_{n,5}(\lambda, x)$ and $K_{n,6}(\lambda, x)$ are Sheffer sequences (see [17, 18]) for the respective pairs $\left(\frac{\log(1+f(t))}{f(t)}, f(t)\right)$ and $\left(\frac{\lambda t}{\log(1+\lambda)f(t)}, f(t)\right)$, where

$$f(t) = \left(1 + \frac{\lambda^2 t}{\log(1 + \lambda)}\right)^{1/\lambda} - 1.$$

Note that $\bar{f}(t) = \frac{\log(1+\lambda)}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}$. According the notation of [17, 18], we have

$$(1.4) \quad K_{n,5}(\lambda, x) \sim \left(\frac{\log(1 + f(t))}{f(t)}, f(t)\right), K_{n,6}(\lambda, x) \sim \left(\frac{\lambda t}{\log(1 + \lambda)f(t)}, f(t)\right).$$

In this paper, we will use umbral calculus in order to study some properties, explicit formulas, recurrence relations and identities about the Korobov polynomials of the fifth kind and of the sixth kind. Also, we present connections between our polynomials and several known families of polynomials.

2. Explicit Expressions

In this section we present several explicit formulas for the Korobov polynomials of the fifth kind and of the sixth kind, namely $K_{n,5}(\lambda, x)$ and $K_{n,6}(\lambda, x)$. To do so, we recall that the *degenerate Stirling numbers of the first kind* $S_1(\ell, k|\lambda)$ via the generating function (see [9])

$$(2.1) \quad \frac{((1 + t)^\lambda - 1)^k}{k! \lambda^k} = \sum_{\ell \geq k} S_1(\ell, k|\lambda) \frac{t^\ell}{\ell!}.$$

Theorem 2.1. *For all $n \geq 0$,*

$$K_{n,5}(\lambda, x) = \sum_{k=0}^n \sum_{\ell=k}^n \binom{n}{\ell} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k|\lambda) b_{n-\ell} x^k,$$

$$K_{n,6}(\lambda, x) = \sum_{k=0}^n \sum_{\ell=k}^n \binom{n}{\ell} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k|\lambda) K_{n-\ell}(\lambda) x^k.$$

Proof. We proceed the proof by using the conjugation representation for Sheffer sequences (see [17, 18]): $s_n(x) = \sum_{k=0}^n \frac{1}{k!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^k | x^n \rangle x^k$, for any $s_n(x) \sim (g(t), f(t))$. Thus, by (1.4), we have

$$K_{n,5}(\lambda, x) = \sum_{k=0}^n \frac{1}{k!} \left\langle \frac{t}{\log(1 + t)} \frac{\log^k(1 + \lambda) ((1 + t)^\lambda - 1)^k}{\lambda^{2k}} | x^n \right\rangle x^k$$

$$= \sum_{k=0}^n \frac{\log^k(1 + \lambda)}{\lambda^k} \left\langle \frac{t}{\log(1 + t)} \frac{((1 + t)^\lambda - 1)^k}{k! \lambda^k} | x^n \right\rangle x^k,$$

which, by (1.1) and (2.1), implies

$$\begin{aligned} K_{n,5}(\lambda, x) &= \sum_{k=0}^n \frac{\log^k(1 + \lambda)}{\lambda^k} \left\langle \frac{t}{\log(1 + t)} \middle| \sum_{\ell \geq k} S_1(\ell, k|\lambda) \frac{t^\ell}{\ell!} x^n \right\rangle x^k \\ &= \sum_{k=0}^n \sum_{\ell=k}^n \binom{n}{\ell} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k|\lambda) \left\langle \frac{t}{\log(1 + t)} \middle| x^{n-\ell} \right\rangle x^k \\ &= \sum_{k=0}^n \sum_{\ell=k}^n \binom{n}{\ell} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k|\lambda) b_{n-\ell} x^k. \end{aligned}$$

By similar techniques, we obtain the explicit expression for $K_{n,6}(\lambda, x)$. □

In the next theorem, we express our polynomials in terms of degenerate falling factorial polynomials $(x)_{n,\lambda}$, which are defined by the generating function

$$(2.2) \quad (1 + \lambda)^{\frac{x}{\lambda}} \frac{(1+t)^\lambda - 1}{\lambda} = \sum_{n \geq 0} (x)_{n,\lambda} \frac{t^n}{n!}.$$

Theorem 2.2. For all $n \geq 0$,

$$\begin{aligned} K_{n,5}(\lambda, x) &= \sum_{m=0}^n \binom{n}{m} b_{n-m}(x)_{m,\lambda}, \\ K_{n,6}(\lambda, x) &= \sum_{m=0}^n \binom{n}{m} K_{n-m}(\lambda)(x)_{m,\lambda}. \end{aligned}$$

Proof. Let us prove only the first expression (the second expression can be obtained by using very similar techniques). By (1.2), we have

$$K_{n,5}(\lambda, y) = \left\langle \frac{t}{\log(1 + t)} (1 + \lambda)^{\frac{y((1+t)^\lambda - 1)}{\lambda^2}} \middle| x^n \right\rangle,$$

which, by (2.2), implies

$$K_{n,5}(\lambda, y) = \left\langle \frac{t}{\log(1 + t)} \middle| \sum_{m \geq 0} (y)_{m,\lambda} \frac{t^m}{m!} x^n \right\rangle.$$

Thus, by (1.1), we obtain

$$K_{n,5}(\lambda, y) = \sum_{m=0}^n \binom{n}{m} (y)_{m,\lambda} \left\langle \frac{t}{\log(1 + t)} \middle| x^{n-m} \right\rangle = \sum_{m=0}^n \binom{n}{m} (y)_{m,\lambda} b_{n-m},$$

which completes the proof. □

Theorem 2.3. For all $n \geq 0$,

$$\begin{aligned}
 &K_{n,5}(\lambda, x) \\
 &= \sum_{j=0}^n \left(\sum_{k=j}^n \sum_{\ell=0}^k \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m}}{\ell!} \binom{\ell}{m} \binom{k}{j} (m|\lambda)_{k-j} \frac{\log^j(1+\lambda)}{\lambda^j} S_1(n, k|\lambda) b_{\ell} \right) x^j, \\
 &K_{n,6}(\lambda, x) \\
 &= \sum_{j=0}^n \left(\sum_{k=j}^n \frac{1}{k+1} \binom{k+1}{j} (1|\lambda)_{k+1-j} \frac{\log^j(1+\lambda)}{\lambda^j} S_1(n, k|\lambda) \right) x^j.
 \end{aligned}$$

Proof. By (1.4) and (2.1), we have that $(x)_{n,\lambda} = \sum_{k=0}^n \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) x^k \sim (1, f(t))$, see [9, 16]. Thus, by (1.2), we have

$$\frac{\log(1+f(t))}{f(t)} K_{n,5}(\lambda, x) = \sum_{k=0}^n \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) x^k.$$

Therefore, by (1.1), we obtain

$$\begin{aligned}
 K_{n,5}(\lambda, x) &= \sum_{k=0}^n \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) \frac{f(t)}{\log(1+f(t))} x^k \\
 &= \sum_{k=0}^n \sum_{\ell=0}^k \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) b_{\ell} \frac{f^{\ell}(t)}{\ell!} x^k \\
 &= \sum_{k=0}^n \sum_{\ell=0}^k \sum_{m=0}^{\ell} \binom{\ell}{m} (-1)^{\ell-m} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) \frac{b_{\ell}}{\ell!} \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{m/\lambda} x^k \\
 &= \sum_{k=0}^n \sum_{\ell=0}^k \sum_{m=0}^{\ell} \sum_{j=0}^k \binom{\ell}{m} \frac{(-1)^{\ell-m} \log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) \frac{b_{\ell}}{\ell!} \frac{(m/\lambda)_j}{j!} \left(\frac{\lambda^2 t}{\log(1+\lambda)}\right)^j x^k \\
 &= \sum_{k=0}^n \sum_{\ell=0}^k \sum_{m=0}^{\ell} \sum_{j=0}^k \binom{\ell}{m} \binom{k}{j} \frac{(-1)^{\ell-m} \log^{k-j}(1+\lambda)}{\lambda^{k-j}} S_1(n, k|\lambda) \frac{b_{\ell}}{\ell!} (m|\lambda)_j x^{k-j} \\
 (2.3) \quad &= \sum_{k=0}^n \sum_{\ell=0}^k \sum_{m=0}^{\ell} \sum_{j=0}^k \frac{(-1)^{\ell-m}}{\ell!} \binom{\ell}{m} \binom{k}{j} (m|\lambda)_{k-j} \frac{\log^j(1+\lambda)}{\lambda^j} S_1(n, k|\lambda) b_{\ell} x^j,
 \end{aligned}$$

which gives the explicit expression of $k_{n,5}(\lambda, x)$. Similarly, by (1.4) and (2.1), we have

$$\frac{\lambda t}{\log(1+\lambda)f(t)} K_{n,6}(\lambda, x) = \sum_{k=0}^n \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) x^k.$$

Therefore we obtain

$$\begin{aligned}
K_{n,6}(\lambda, x) &= \sum_{k=0}^n \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) \frac{\log(1+\lambda)f(t)}{\lambda t} x^k \\
&= \sum_{k=0}^n \frac{\log^{k+1}(1+\lambda)}{(k+1)\lambda^{k+1}} S_1(n, k|\lambda) f(t) x^{k+1} \\
&= \sum_{k=0}^n \sum_{j=1}^{k+1} \frac{\log^{k+1}(1+\lambda)}{(k+1)\lambda^{k+1}} S_1(n, k|\lambda) (1/\lambda)_j \frac{1}{j!} \left(\frac{\lambda^2 t}{\log(1+\lambda)} \right)^j x^{k+1} \\
&= \sum_{k=0}^n \sum_{j=1}^{k+1} \binom{k+1}{j} \frac{\log^{k-j+1}(1+\lambda)}{(k+1)\lambda^{k-j+1}} S_1(n, k|\lambda) (1|\lambda)_j x^{k-j+1} \\
&= \sum_{j=0}^n \left(\sum_{k=j}^n \frac{1}{k+1} \binom{k+1}{j} (1|\lambda)_{k+1-j} \frac{\log^j(1+\lambda)}{\lambda^j} S_1(n, k|\lambda) \right) x^j,
\end{aligned}$$

as claimed. \square

In next theorem, we express our polynomials $K_{n,5}(\lambda, x)$ and $K_{n,6}(\lambda, x)$ in terms of degenerate Bernoulli numbers $\beta_\ell^{(n)}(\lambda)$ of order n , which are given by the generating function $\frac{t^n}{((1+\lambda t)^{1/\lambda}-1)^n} = \sum_{\ell \geq 0} \beta_\ell^{(n)}(\lambda) \frac{t^\ell}{\ell!}$.

Theorem 2.4. For all $n \geq 1$, the polynomial $K_{n,5}(\lambda, x)$ is given by

$$\sum_{j=0}^n \left(\sum_{\ell=j}^n \sum_{k=0}^{\ell} \sum_{m=0}^k \frac{(-1)^{k-m}}{k!} \binom{n-1}{\ell-1} \binom{k}{m} \binom{\ell}{j} (m|\lambda)_{\ell-j} \frac{\log^j(1+\lambda)}{\lambda^j} b_k \beta_{n-\ell}^{(n)}(\lambda) \right) x^j.$$

Proof. Note that $(\log(1+\lambda)x/\lambda)^n \sim (1, \lambda t/\log(1\lambda))$. Thus, by (1.2), we have

$$\begin{aligned}
&\frac{\log(1+f(t))}{f(t)} K_{n,5}(\lambda, x) \\
&= x \left(\frac{\lambda t/\log(1+\lambda)}{(1+\lambda^2 t/\log(1+\lambda))^{1/\lambda}-1} \right)^n x^{-1} (\log(1+\lambda)x/\lambda)^n \\
&= \frac{\log^n(1+\lambda)}{\lambda^n} x \left(\frac{s}{(1+\lambda s)^{1/\lambda}-1} \right)^n \Big|_{s=\frac{\lambda t}{\log(1+\lambda)}} x^{n-1} \\
&= \frac{\log^n(1+\lambda)}{\lambda^n} x \sum_{\ell \geq 0} \beta_\ell^{(n)}(\lambda) \frac{1}{\ell!} \left(\frac{\lambda t}{\log(1+\lambda)} \right)^\ell x^{n-1} \\
&= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \frac{\log^{n-\ell}(1+\lambda)}{\lambda^{n-\ell}} \beta_\ell^{(n)}(\lambda) x^{n-\ell} = \sum_{\ell=1}^n \binom{n-1}{\ell-1} \frac{\log^\ell(1+\lambda)}{\lambda^\ell} \beta_{n-\ell}^{(n)}(\lambda) x^\ell.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned} K_{n,5}(\lambda, x) &= \sum_{\ell=1}^n \binom{n-1}{\ell-1} \frac{\log^\ell(1+\lambda)}{\lambda^\ell} \beta_{n-\ell}^{(n)}(\lambda) \frac{f(t)}{\log(1+f(t))} x^\ell \\ &= \sum_{\ell=0}^n \sum_{k=0}^{\ell} \sum_{m=0}^k \sum_{j=0}^{\ell} (-1)^{k-m} \binom{n-1}{\ell-1} \binom{k}{m} \binom{\ell}{j} (m|\lambda)_{\ell-j} \frac{\log^j(1+\lambda)}{\lambda^j} \beta_{n-\ell}^{(n)}(\lambda) \frac{b_k}{k!} x^j \\ &= \sum_{j=0}^n \left(\sum_{\ell=j}^n \sum_{k=0}^{\ell} \sum_{m=0}^k \frac{(-1)^{k-m} \binom{n-1}{\ell-1} \binom{k}{m} \binom{\ell}{j}}{k!} (m|\lambda)_{\ell-j} \frac{\log^j(1+\lambda)}{\lambda^j} b_k \beta_{n-\ell}^{(n)}(\lambda) \right) x^j, \end{aligned}$$

which completes the proof of the expression of $K_{n,5}(\lambda, x)$. □

Very similar techniques, as in the proof of the pervious theorem, lead to the expression of $K_{n,6}(\lambda, x)$, where we leave the details for the interested reader.

Theorem 2.5. *For all $n \geq 0$, the polynomial $K_{n,6}(\lambda, x)$ is given by*

$$\sum_{j=0}^n \left(\sum_{\ell=j}^n \frac{1}{\ell+1} \binom{n-1}{\ell-1} \binom{\ell+1}{j} (1|\lambda)_{\ell+1-j} \frac{\log^j(1+\lambda)}{\lambda^j} \beta_{n-\ell}^{(n)}(\lambda) \right) x^j.$$

3. Recurrences

In this section, we present several recurrences for the Korobov polynomials of the fifth kind and of the sixth kind. Note that, by (1.4) and the fact that $(x)_{n,\lambda} \sim (1, f(t))$, we obtain $K_{n,d}(\lambda, x+y) = \sum_{j=0}^n \binom{n}{j} K_{j,d}(\lambda, x)(y)_{n-j,\lambda}$, for $d = 5, 6$.

Proposition 3.1.

For all $n \geq 1$,

$$\begin{aligned} &K_{n,5}(\lambda, x) + nK_{n-1,5}(\lambda, x) \\ &= \sum_{m=0}^n \left(\sum_{k=m}^n \sum_{\ell=k}^n \binom{n}{\ell} \binom{k}{m} (1|\lambda)_{k-m} \frac{\log^m(1+\lambda)}{\lambda^m} S_1(\ell, k|\lambda) b_{n-\ell} \right) x^m, \\ &K_{n,6}(\lambda, x) + nK_{n-1,6}(\lambda, x) \\ &= \sum_{m=0}^n \left(\sum_{k=m}^n \sum_{\ell=k}^n \binom{n}{\ell} \binom{k}{m} (1|\lambda)_{k-m} \frac{\log^m(1+\lambda)}{\lambda^m} S_1(\ell, k|\lambda) K_{n-\ell}(\lambda) \right) x^m. \end{aligned}$$

Proof. It is well-known that if $s_n(x) \sim (g(t), f(t))$, then we have $f(t)s_n(x) = ns_{n-1}(x)$ (see [17, 18]). Thus, by (1.4), we obtain

$$\left(\left(1 + \frac{\lambda^2 t}{\log(1+\lambda)} \right)^{\frac{1}{\lambda}} - 1 \right) K_{n,5}(\lambda, x) = nK_{n-1,5}(\lambda, x),$$

which implies $K_{n,5}(\lambda, x) + nK_{n-1,5}(\lambda, x) = \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} K_{n,5}(\lambda, x)$. Therefore, by Theorem 2.1 we have

$$\begin{aligned} & K_{n,5}(\lambda, x) + nK_{n-1,5}(\lambda, x) \\ &= \sum_{k=0}^n \sum_{\ell=k}^n \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) b_{n-\ell} \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} x^k \\ &= \sum_{k=0}^n \sum_{\ell=k}^n \sum_{m=0}^k \binom{n}{\ell} \frac{\log^{k-m}(1+\lambda)}{\lambda^{k-m}} S_1(\ell, k|\lambda) b_{n-\ell} (1|\lambda)_m \frac{t^m}{m!} x^k \\ &= \sum_{k=0}^n \sum_{\ell=k}^n \sum_{m=0}^k \binom{n}{\ell} \binom{k}{m} \frac{\log^{k-m}(1+\lambda)}{\lambda^{k-m}} S_1(\ell, k|\lambda) b_{n-\ell} (1|\lambda)_m x^{k-m} \\ &= \sum_{k=0}^n \sum_{\ell=k}^n \sum_{m=0}^k \binom{n}{\ell} \binom{k}{m} \frac{\log^m(1+\lambda)}{\lambda^m} S_1(\ell, k|\lambda) b_{n-\ell} (1|\lambda)_{k-m} x^m \\ &= \sum_{m=0}^n \left(\sum_{k=m}^n \sum_{\ell=k}^n \binom{n}{\ell} \binom{k}{m} (1|\lambda)_{k-m} \frac{\log^m(1+\lambda)}{\lambda^m} S_1(\ell, k|\lambda) b_{n-\ell} \right) x^m. \end{aligned}$$

By using similar techniques to the above, with replacing $b_{n-\ell}$ by $K_{n-\ell}(\lambda)$, we obtain the recurrence relation for $K_{n,6}(\lambda, x)$. □

In the next result we express $\frac{d}{dx} K_{n,5}(\lambda, x)$ and $\frac{d}{dx} K_{n,6}(\lambda, x)$ in terms of $K_{n,5}(\lambda, x)$ and $K_{n,6}(\lambda, x)$, respectively.

Proposition 3.2. For all $n \geq 0$,

$$\frac{d}{dx} K_{n,d}(\lambda, x) = \frac{\log(1+\lambda)}{\lambda^2} \sum_{\ell=0}^{n-1} \binom{n}{\ell} (\lambda)_{n-\ell} K_{\ell,d}(\lambda, x),$$

where $d = 5, 6$.

Proof. Note that $\frac{d}{dx} s_n(x) = \sum_{\ell=0}^{n-1} \binom{n}{\ell} \langle \bar{f}(t) | x^{n-\ell} \rangle s_\ell(x)$ for all $s_n(x) \sim (g(t), f(t))$, see [17, 18]. So, for $s_n(x) = K_{n,d}(\lambda, x)$, it remains to compute $A = \langle \bar{f}(t) | x^{n-\ell} \rangle$. By (1.4), we have

$$A = \frac{\log(1+\lambda)}{\lambda^2} \left\langle \sum_{j \geq 1} (\lambda)_j \frac{t^j}{j!} | x^{n-\ell} \right\rangle = \frac{\log(1+\lambda)}{\lambda^2} (\lambda)_{n-\ell},$$

which completes the proof. □

Theorem 3.3. For all $n \geq 1$,

$$\begin{aligned}
 K_{n,5}(\lambda, x) &= \frac{x \log(1 + \lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda - 1)_{n-1-\ell} K_{\ell,5}(\lambda, x) \\
 &\quad - \frac{1}{n} \sum_{\ell=0}^{n-1} \sum_{k=0}^{n-\ell} (-1)^k \binom{n}{k} b_{n-k-\ell} \binom{n-k}{\ell} K_{\ell,5}(\lambda, x) \\
 K_{n,6}(\lambda, x) &= \frac{x \log(1 + \lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda - 1)_{n-1-\ell} K_{\ell,6}(\lambda, x) \\
 &\quad - \frac{1}{n} \sum_{\ell=0}^{n-1} \sum_{k=0}^{n-\ell} \binom{n}{k} \binom{n-k}{\ell} (\lambda - 1)_k K_{n-k-\ell}(\lambda) K_{\ell,6}(\lambda, x).
 \end{aligned}$$

Proof. Since the similarity between $K_{n,5}(\lambda, x)$ and $K_{n,6}(\lambda, x)$ (see (1.2) and (1.2)), we omit the proof of the case $K_{n,6}(\lambda, x)$ and give only the details of the case $K_{n,5}(\lambda, x)$. By (1.2), we have

$$K_{n,5}(\lambda, y) = \left\langle \frac{d}{dt} \left(\frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda}} \frac{(1+t)^{\lambda-1}}{\lambda} \right) |x^{n-1} \right\rangle = A + B,$$

where

$$A = \left\langle \frac{d}{dt} \frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda}} \frac{(1+t)^{\lambda-1}}{\lambda} |x^{n-1} \right\rangle$$

and

$$B = \left\langle \frac{t}{\log(1+t)} \frac{d}{dt} (1+\lambda)^{\frac{y}{\lambda}} \frac{(1+t)^{\lambda-1}}{\lambda} |x^{n-1} \right\rangle.$$

First, we compute the term B .

$$\begin{aligned}
 B &= \left\langle \frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda}} \frac{(1+t)^{\lambda-1}}{\lambda} \frac{\log(1+\lambda)}{\lambda} y (1+t)^{\lambda-1} |x^{n-1} \right\rangle \\
 &= \frac{y \log(1+\lambda)}{\lambda} \left\langle \frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda}} \frac{(1+t)^{\lambda-1}}{\lambda} |(1+t)^{\lambda-1} x^{n-1} \right\rangle \\
 &= \frac{y \log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda - 1)_{\ell} \left\langle \frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda}} \frac{(1+t)^{\lambda-1}}{\lambda} |x^{n-1-\ell} \right\rangle \\
 &= \frac{y \log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda - 1)_{\ell} K_{n-1-\ell,5}(\lambda, y) \\
 &= \frac{y \log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda - 1)_{n-1-\ell} K_{\ell,5}(\lambda, y).
 \end{aligned}$$

Now, we compute the first term A ,

$$\begin{aligned} A &= \left\langle \frac{t}{\log(1+t)} \frac{1+t-t/\log(1+t)}{t(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \middle| x^{n-1} \right\rangle \\ &= \left\langle \frac{t}{(1+t)\log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \middle| \frac{1+t-t/\log(1+t)}{t} x^{n-1} \right\rangle \\ &= \frac{1}{n} \left\langle \frac{t}{(1+t)\log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \middle| (1+t-t/\log(1+t)) x^n \right\rangle \\ &= \frac{1}{n} \left\langle \frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \middle| \left(1 - \frac{t}{(1+t)\log(1+t)}\right) x^n \right\rangle. \end{aligned}$$

Note that $1+t-t/\log(1+t)$ has order at least one. Thus, $A = \frac{1}{n} K_{n,5}(\lambda, y) - \frac{1}{n} C$, where $C = \left\langle \frac{t^2}{\log^2(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \middle| \frac{1}{1+t} x^n \right\rangle$. By the definitions, we have

$$\begin{aligned} C &= \sum_{k=0}^n (-1)^k (n)_k \left\langle \frac{t^2}{\log^2(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \middle| x^{n-k} \right\rangle \\ &= \sum_{k=0}^n \sum_{m=0}^{n-k} (-1)^k (n)_k b_m \binom{n-k}{m} \left\langle \frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \middle| x^{n-k-m} \right\rangle \\ &= \sum_{k=0}^n \sum_{m=0}^{n-k} (-1)^k (n)_k b_m \binom{n-k}{m} K_{n-k-m,5}(\lambda, y) \\ &= \sum_{k=0}^n \sum_{m=0}^{n-k} (-1)^k (n)_k b_{n-k-m} \binom{n-k}{m} K_{m,5}(\lambda, y). \end{aligned}$$

Hence, for all $n \geq 1$,

$$\begin{aligned} K_{n,5}(\lambda, x) &= \frac{x \log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda-1)_{n-1-\ell} K_{\ell,5}(\lambda, x) \\ &\quad + \frac{1}{n} \left(K_{n,5}(\lambda, x) - \sum_{\ell=0}^n \sum_{k=0}^{n-\ell} (-1)^k (n)_k b_{n-k-\ell} \binom{n-k}{\ell} K_{\ell,5}(\lambda, x) \right), \end{aligned}$$

which completes the proof. \square

4. Connections with Families of Polynomials

In this section, we present some examples on the connections with families of polynomials. To do that, we recall for any two Sheffer sequences $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, we have that $s_n(x) = \sum_{m=0}^n C_{n,m} r_m(x)$, where (see [17,

18, 11])

$$(4.1) \quad C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (\ell(\bar{f}(t)))^m |x^n \right\rangle.$$

We start with the connection to *Bernoulli polynomials* $B_n^{(s)}(x)$ of order s . Recall that the *Bernoulli polynomials* $B_n^{(s)}(x)$ of order s are defined by the generating function $\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n \geq 0} B_n^{(s)}(x) \frac{t^n}{n!}$, equivalently,

$$(4.2) \quad B_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s, t \right)$$

(see[2, 6, 18]). In the next result, we express our polynomials in terms of Bernoulli polynomials of order s .

Theorem 4.1. For all $n \geq 0$, $K_{n,5}(\lambda, x) = \sum_{k=0}^n C_{n,m} B_m^{(s)}(x)$, where

$$C_{n,m} = \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \frac{\binom{n}{\ell} \binom{k+m}{m}}{\binom{k+s}{s}} b_\ell S_2(k+s, s) \frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}} S_1(n-\ell, k+m|\lambda).$$

Proof. Let $K_{n,5}(\lambda, x) = \sum_{k=0}^n C_{n,m} B_m^{(s)}(x)$. So, by (1.2) and (4.2), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{(e^{\bar{f}(t)} - 1)^s}{\bar{f}^s(t)} \frac{t}{\log(1+t)} \bar{f}^m(t) |x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \frac{(e^{\bar{f}(t)} - 1)^s}{\bar{f}^s(t)} \bar{f}^m(t) \left| \frac{t}{\log(1+t)} x^n \right. \right\rangle \\ &= \frac{1}{m!} \left\langle \frac{(e^{\bar{f}(t)} - 1)^s}{\bar{f}^s(t)} \bar{f}^m(t) \left| \sum_{\ell \geq 0} b_\ell \frac{t^\ell}{\ell!} x^n \right. \right\rangle \\ &= \frac{1}{m!} \sum_{\ell=0}^n \binom{n}{\ell} b_\ell \left\langle s! \sum_{k \geq 0} S_2(k+s, s) \frac{\bar{f}^{k+m}(t)}{(k+s)!} |x^{n-\ell} \right\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} C_{n,m} &= \frac{s!}{m!} \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \binom{n}{\ell} b_\ell S_2(k+s, s) \left\langle \frac{\bar{f}^{k+m}(t)}{(k+s)!} |x^{n-\ell} \right\rangle \\ &= \frac{s!}{m!} \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \frac{\binom{n}{\ell} b_\ell}{(k+s)!} S_2(k+s, s) \frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}} \left\langle \frac{((1+t)^\lambda - 1)^{k+m}}{\lambda^{k+m}} |x^{n-\ell} \right\rangle \\ &= \frac{s!}{m!} \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \frac{\binom{n}{\ell} b_\ell}{(k+s)!} S_2(k+s, s) \frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}} (k+m)! S_1(n-\ell, k+m|\lambda), \end{aligned}$$

where $S_1(n, k|\lambda)$ is given in [9] as $S_1(n, k|\lambda) = \frac{1}{k!} \langle \frac{(1+t)^\lambda - 1}{\lambda} \rangle^k |x^n\rangle$. Therefore,

$$C_{n,m} = \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \frac{\binom{n}{\ell} \binom{k+m}{m}}{\binom{k+s}{s}} b_\ell S_2(k+s, s) \frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}} S_1(n-\ell, k+m|\lambda),$$

as required. □

Similar techniques as in the proof of the previous theorem, we can express our polynomials $K_{n,6}(\lambda, x)$ in terms of Bernoulli polynomials of order s (we leave the proof for the interested reader).

Theorem 4.2. For all $n \geq 0$, $K_{n,6}(\lambda, x) = \sum_{k=0}^n C_{n,m} B_m^{(s)}(x)$, where

$$C_{n,m} = \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \frac{\binom{n}{\ell} \binom{k+m}{m}}{\binom{k+s}{s}} K_\ell(\lambda) S_2(k+s, s) \frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}} S_1(n-\ell, k+m|\lambda).$$

Similar techniques as in the proof of the previous theorem, we can express our polynomials $K_{n,5}(\lambda, x), K_{n,6}(\lambda, x)$ in terms of other families. Below we present two examples, where we leave the proofs to the interested reader. In the first example, we express our polynomials in terms of Frobenius-Euler polynomials. Note that the *Frobenius-Euler polynomials* $H_n^{(s)}(x|\mu)$ of order s are defined by the generating function

$$\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n \geq 0} H_n^{(s)}(x|\mu) \frac{t^n}{n!}, (\mu \neq 1),$$

or equivalently, $H_n^{(s)}(x|\mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu}\right)^s, t\right)$ (see [1, 12, 13]).

Theorem 4.3. For all $n \geq 0$, $K_{n,5}(\lambda, x) = \sum_{m=0}^n C_{n,m} H_m^{(s)}(x|\mu)$ and $K_{n,6}(\lambda, x) = \sum_{m=0}^n D_{n,m} H_m^{(s)}(x|\mu)$, where

$$\begin{aligned} C_{n,m} &= \sum_{\ell=0}^{n-m} \sum_{k=0}^s \sum_{j=k}^{n-\ell-m} \frac{k! \binom{j+m}{m} \binom{n}{\ell} \binom{s}{k}}{(1-\mu)^k} \frac{\log^{j+m}(1+\lambda)}{\lambda^{j+m}} S_1(n-\ell, j+m|\lambda) S_2(j, k) b_\ell \\ D_{n,m} &= \sum_{\ell=0}^{n-m} \sum_{k=0}^s \sum_{j=k}^{n-\ell-m} \frac{k! \binom{j+m}{m} \binom{n}{\ell} \binom{s}{k}}{(1-\mu)^k} \frac{\log^{j+m}(1+\lambda)}{\lambda^{j+m}} S_1(n-\ell, j+m|\lambda) S_2(j, k) K_\ell(\lambda). \end{aligned}$$

For what follows, we define the associated sequence for

$$1 - (1 + \lambda^2 t / \log(1 + \lambda))^{-1/\lambda},$$

namely $(x)^{(n,\lambda)}$. Thus,

$$(x)^{(n,\lambda)} \sim (1, 1 - (1 + \lambda^2 t / \log(1 + \lambda))^{-1/\lambda}).$$

Recall here that $(x)_n \sim (1, e^t - 1)$, $(x)^{(n)} \sim (1, 1 - e^{-t})$,

$$(x)_{n,\lambda} \sim (1, (1 + \lambda^2 t / \log(1 + \lambda))^{1/\lambda} - 1)$$

and $(1 + \lambda^2 t / \log(1 + \lambda))^{1/\lambda} - 1 \rightarrow e^t - 1$, as $\lambda \rightarrow 0$. Now, we ready to present our second example.

Theorem 4.4. For all $n \geq 0$, $K_{n,5}(\lambda, x) = \sum_{m=0}^n C_{n,m}(x)^{(m,\lambda)}$ and $K_{n,6}(\lambda, x) = \sum_{m=0}^n D_{n,m}(x)^{(m,\lambda)}$, where

$$C_{n,m} = \sum_{\ell=0}^n (-1)^{n-\ell-m} \binom{n}{\ell} \binom{n-\ell}{m} (n-1-\ell)_{n-\ell-m} b_\ell,$$

$$D_{n,m} = \sum_{\ell=0}^n (-1)^{n-\ell-m} \binom{n}{\ell} \binom{n-\ell}{m} (n-1-\ell)_{n-\ell-m} K_\ell(\lambda).$$

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