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Noninformative priors for linear function of parameters in the lognormal distribution

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Abstract

This paper considers the noninformative priors for the linear function of parameters in the lognormal distribution. The lognormal distribution is applied in the various areas, such as occupational health research, environmental science, monetary units, etc. The linear function of parameters in the lognormal distribution includes the expectation, median and mode of the lognormal distribution. Thus we derive the probability matching priors and the reference priors for the linear function of parameters. Then we reveal that the derived reference priors do not satisfy a ?rst order matching criterion. Under the general priors including the derived noninformative priors, we check the proper condition of the posterior distribution. Some numerical study under the developed priors is performed and real examples are illustrated.

Keywords: Linear function of parameters, lognormal distribution, matching prior, reference prior.

1. Introduction

The lognormal distribution is applied in the various areas, such as occupational health research (Rappaport and Selvin, 1987), environmental science (Parkhurst, 1998), monetary units (Longford and Pittau, 2006; Zabel, 1999), etc. The reason is that the log-transformation provides symmetry for the distribution of data and eliminates the skew (Limpert *et al.*, 2001). In many contexts, the choice of normal distribution is the favored assumption of distribution, and logarithm is the usually used transformation for an analyst. However there are instances of interest on the original scale. For example, the population mean of lognormal distribution may be more interested in statistical inference than the population mean of the log-transformation.

This paper deals with the development of noninformative priors for the linear function of parameters of the lognormal distribution. We consider the priors with which the credible

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intervals for the linear function of parameters have approximately the frequentist validity in Bayesian inference. Although this matching is achieved asymptotically, the matching have a good performance for small and moderate sample sizes in many works.

The idea of the matching is starting from the study of Welch and Peers (1963). This matching idea had revived interest in the study of Stein (1985) and Tibshirani (1989). Among many works, we may refer to the studies of Datta and Ghosh (1995, 1996), Mukerjee and Dey (1993) and Mukerjee and Ghosh (1997). Meanwhile, the reference prior approach (Bernardo, 1979) extended by Berger and Bernardo (1989, 1992) and Ghosh and Mukerjee (1992) is an another idea developing a noninformative prior. Berger and Bernardo (1992) developed a general procedure for the derivation of a reference prior by several groups of parameters according to the inferential importance. This approach gives a good result in many practical applications (Kang, 2013; Kang *et al.,* 2013, 2014). Sometimes reference priors become to the matching priors described previously.

In the lognormal distribution, Zhou and Gao (1997) studies four methods for constructing confidence intervals of the lognormal mean. The four methods are the naive method, Cox's method, Angus's conservative method and Angus's parametric bootstrap method. They showed that Cox's method and the parametric bootstrap method are better than the other methods by numerical studies. Wu *et al.* (2003) revealed that Cox's method does not appropriate when the sample size is small, and the parametric bootstrap has good coverage in small variance cases, but has not good performance for large variance cases. However the modified signed log-likelihood ratio method of Wu *et al.* (2003) gives a good coverage even if the variance increases. Krishnamoorthy and Mathew (2003) derived the exact confidence interval and tests using the generalized p method, and showed that the coverage probabilities of confidence interval based on the generalized limits are close almost to the nominal coverage level.

The remaining works are as follows. In Section 2, we compute Fisher information matrix. Then we derive the first order probability matching priors for the linear function of parameters. Next we develop the reference priors for the linear function of parameters, and show that Jeffreys' prior, the reference prior and the first order matching prior have the different forms. In Section 3, we check the proper condition for the posterior distribution under a general prior including the developed noninformative priors. Under the developed priors, the frequentist coverage probabilities are investigated in Section 4. Also the confidence intervals using on Cox's method, the parametric bootstrap method, the modified signed log-likelihood ratio statistic method, and Bayesian credible intervals for the lognormal mean are illustrated using two examples.

2. The noninformative priors

2.1. The probability matching prior

Suppose that $\mathbf{X} = (X_1, \cdots, X_n)$ is a random sample from a lognormal population with parameters μ and σ^2. Thus $\log X_i$ has a normal distribution with mean μ and variance σ^2. Our interest is the parameter $\theta = a\mu + b\sigma^2$ with a and b are constants. Note that the expectation, median and mode of the lognormal distribution are $\exp\{\mu + \frac{1}{2}\sigma^2\}$, $\exp\{\mu\}$ and $\exp\{\mu - \sigma^2\}$, respectively. This motivates the general problem for the quantity $\exp\{a\mu + b\sigma^2\}$ with a and b are constants. Thus the problem for this general quantity is equivalent to θ in development of noninformative priors.

Let $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_t)^T$ and let θ_1 be the parameter of interest. Also $\theta_1^{1-\alpha}(\pi; \mathbf{X})$ is the $(1-\alpha)$th quantile of the posterior distribution of θ_1. We consider the priors π for which

$$P[\theta_1 \leq \theta_1^{1-\alpha}(\pi; \mathbf{X})|\boldsymbol{\theta}] = 1 - \alpha + o(n^{-r}), r > 0, \tag{2.1}$$

as n goes to infinity. Then priors π satisfying (2.1) are the matching priors. The prior π with $r = 1/2$ is called a first order matching prior, and for $r = 1$, π is called a second order matching prior.

Now we develop the matching priors π. Let

$$\theta_1 = a\mu + b\sigma^2 \text{ and } \theta_2 = \mu - \frac{a}{2b}\log\sigma^2.$$

Then the Jacobian matrix of the above transformation is given by

$$\frac{\partial(\theta_1, \theta_2)}{\partial(\mu, \sigma^2)} = \begin{pmatrix} a & b \\ 1 & -\dfrac{a}{2b}\sigma^{-2} \end{pmatrix}. \tag{2.2}$$

Thus the inverse of the expected Fisher information matrix is computed as follow:

$$\begin{aligned} I^{-1}(\theta_1, \theta_2) &= \left(\frac{\partial(\theta_1, \theta_2)}{\partial(\mu, \sigma^2)}\right) I^{-1}(\mu, \sigma^2) \left(\frac{\partial(\theta_1, \theta_2)}{\partial(\mu, \sigma^2)}\right)^t \\ &= \begin{pmatrix} a^2\sigma^2 + b^2\sigma^4 & 0 \\ 0 & \dfrac{a^2 + 2b^2\sigma^2}{2b} \end{pmatrix}. \end{aligned} \tag{2.3}$$

Therefore the Fisher information matrix from (2.3) is

$$I(\theta_1, \theta_2) = \begin{pmatrix} \dfrac{1}{a^2\sigma^2 + b^2\sigma^4} & 0 \\ 0 & \dfrac{2b}{a^2 + 2b^2\sigma^2} \end{pmatrix}. \tag{2.4}$$

Thus from (2.4), we know that θ_1 is orthogonal to θ_2 (Cox and Reid, 1987). By the results of Tibshirani(1989), the class of first order probability matching prior is given by

$$\pi_m(\theta_1, \theta_2) \propto \sigma^{-1}(a^2 + 2b^2\sigma^2)^{-\frac{1}{2}} g(\theta_2), \tag{2.5}$$

where $g(\theta_2) > 0$ is an arbitrary differentiable function. Note that in the original parametrization (μ, σ^2), the matching prior prior is characterized by

$$\pi_m(\mu, \sigma^2) \propto \sigma^{-3}(a^2 + 2b^2\sigma^2)^{\frac{1}{2}} g\left(\mu - \frac{a}{2b}\log\sigma^2\right). \tag{2.6}$$

2.2. The reference priors

Let

$$\theta_1 = a\mu + b\sigma^2, \text{ and } \theta_2 = \sigma^2. \tag{2.7}$$

Then the likelihood function of parameters θ_1 and θ_2 under this parametrization is given as

$$L(\theta_1, \theta_2) \quad \propto \quad \theta_2^{-1/2} \exp\left\{ -\frac{1}{2\theta_2} \left(\log x - \frac{\theta_1}{a} + \frac{b}{a}\theta_2 \right)^2 \right\}. \tag{2.8}$$

From the likelihood function (2.8), the Fisher information matrix is computed as

$$I(\theta_1, \theta_2) = \begin{pmatrix} \frac{1}{a^2\theta_2} & \frac{b}{a^2\theta_2} \\ \frac{b}{a^2\theta_2} & \frac{1}{2\theta_2^2} + \frac{b^2}{a^2\theta_2} \end{pmatrix}. \tag{2.9}$$

We develop the reference prior for the parameter grouping $\{\theta_1, \theta_2\}$ where θ_1 is the parameter of interest and θ_2 is the nuisance parameter. The algorithm of derivation for the reference prior is given in Berger and Bernardo (1992).

The compact subsets as a Cartesian products of sets is the form

$$\theta_1 \in [a_1, b_1], \theta_2 \in [a_2, b_2],$$

where a_1 will go to $-\infty$, a_2 will go to 0, and $b_i, i = 1, 2$, will go to ∞ in the limit. For the development of the reference prior, we require to compute the next elements using the Fisher information (2.4):

$$h_1 = \frac{1}{\theta_2(a^2 + b^2\theta_2)} \text{ and } h_2 = \frac{a^2 + 2b^2\theta_2}{2a^2\theta_2^2}.$$

Hereafter a subscripted C represents a constant that does not depend on any parameters, but does depend on the ranges of the parameters.

We note that

$$\int_{a_2}^{b_2} h_2^{1/2} d\theta_2 = \int_{a_2}^{b_2} \left(\frac{a^2 + 2b^2\theta_2}{2a^2\theta_2^2} \right)^{1/2} d\theta_2 = C_1.$$

Thus we can obtain the following conditional prior.

$$\pi_2^l(\theta_2|\theta_1) = C_1^{-1}\theta_2^{-1}(a^2 + 2b^2\theta_2)^{\frac{1}{2}}.$$

Also the expectation of $\log h_1$ is given by

$$\begin{aligned} E^l\{\log h_1|\theta_2\} &= \int_{a_2}^{b_2} C_1^{-1}\theta_2^{-1}(a^2 + 2b^2\theta_1)^{\frac{1}{2}} \log[\theta_2(a^2 + b^2\theta_2)] d\theta_2 \\ &= C_2. \end{aligned}$$

Then we obtain the prior as follow:

$$\pi_1^l(\theta_1) \propto \exp[E^l\{\log h_1|\theta_2\}/2] = \exp\{C_2/2\}.$$

Thus the reference prior is given by

$$\pi_1(\theta_1,\theta_2) = \lim_{l\to\infty} \frac{\pi_2^l(\theta_2|\theta_1)\pi_1^l(\theta_1)}{\pi_2^l(\theta_{20}|\theta_{10})\pi_1^l(\theta_{10})} \propto \theta_2^{-1}(a^2+2b^2\theta_2)^{\frac{1}{2}}, \tag{2.10}$$

where θ_{10} is some point in the interval $(-\infty,\infty)$, and θ_{20} is some point in the interval $(0,\infty)$. Also we notice that the reference prior in the original parametrization (μ,σ^2) is

$$\pi_r(\mu,\sigma^2) \propto \sigma^{-2}(a^2+b^2\sigma^2)^{\frac{1}{2}}. \tag{2.11}$$

Notice that the various matching priors exist in the matching priors (2.5) according to the choice of the function g. But there does not seem to improve in terms of the coverage probabilities for some functions. Thus we select a first order matching prior with the constant function. Then the matching prior is given as follow:

$$\pi_m(\mu,\sigma^2) \propto \sigma^{-3}(a^2+2b^2\sigma^2)^{\frac{1}{2}}. \tag{2.12}$$

Also from the Fisher information (2.9), Jeffreys' prior for the (θ_1,θ_2) can be

$$\pi_J(\theta_1,\theta_2) \propto \theta_2^{-3/2}. \tag{2.13}$$

Thus Jeffreys' prior in the original parametrization is given as

$$\pi_J(\mu,\sigma^2) \propto \sigma^{-3}. \tag{2.14}$$

Remark 2.1 Note that Jeffrey's prior, the first order matching prior and the reference prior have the different forms.

3. Propriety of the posterior distribution

For a general class of priors including Jeffreys' prior (2.13), the reference prior (2.10) and the first order matching prior (2.12), we check the proper condition of posteriors. We consider the following general priors

$$\pi(\theta_1,\theta_2) \propto \theta_2^{-c}(a^2+2b^2\theta_2)^d, \tag{3.1}$$

where $c>0$ and $d\geq 0$. Therefore we can obtain the following theorem.

Theorem 3.1 Under the general prior (3.1), the posterior distribution is proper if $n+2c-2d-3>0$.

Proof: Under the general prior (3.1), the joint posterior for θ_1 and θ_2 given $\mathbf{x}$ is given by

$$\pi(\theta_1,\theta_2|\mathbf{x}) \propto \theta_2^{-\frac{n}{2}-c}(a^2+2b^2\theta_2)^d \exp\left\{-\frac{1}{2\theta_2}\sum_{i=1}^n\left(\log x_i - \frac{\theta_1}{a}+\frac{b}{a}\theta_2\right)^2\right\}. \tag{3.2}$$

Firstly, we integrate with respect to θ_1 from (3.2). Then we can obtain the following marginal posterior

$$\pi(\theta_2|\mathbf{x}) \propto \theta_2^{-\frac{n-1}{2}-c}(a^2+2b^2\theta_2)^d \exp\left\{-\frac{S^2}{2\theta_2}\right\}, \tag{3.3}$$

where $S^2 = \sum_{i=1}^{n}(\log x_i - \bar{x})^2$ and $\bar{x} = \sum_{i=1}^{n} \log x_i/n$. Thus

$$\pi(\theta_2|\mathbf{x}) \leq k_1\theta_2^{-\frac{n-1}{2}-c} \exp\left\{-\frac{S^2}{2\theta_2}\right\} + k_2\theta_2^{-\frac{n-1}{2}-c+d} \exp\left\{-\frac{S^2}{2\theta_2}\right\}, \tag{3.4}$$

where k_1 and k_2 are a constant. Therefore the integration of the right side of (3.4) is proper if $n + 2c - 2d - 3 > 0$. □

Theorem 3.2 The marginal posterior density of θ_1 under the general prior (3.1) is given as follow:

$$\pi(\theta_1|\mathbf{x}) \propto \int_0^\infty \theta_2^{-\frac{n}{2}-c}(a^2 + 2b^2\theta_2)^d \exp\left\{-\frac{1}{2\theta_2}\sum_{i=1}^{n}\left(\log x_i - \frac{\theta_1}{a} + \frac{b}{a}\theta_2\right)^2\right\} d\theta_2. \tag{3.5}$$

Actually, we can know that the computation of the marginal distribution of θ_1 needs to one dimensional integration. So we can obtain the marginal posterior distribution of θ_1, and also the marginal moment of θ_1 can be compute. In the next section, we evaluate the frequentist coverage probabilities for the matching prior, reference prior and Jeffreys' prior, respectively.

4. Numerical studies

4.1. Simulation study

We investigate the frequentist coverage probabilities by computing the credible interval of the posterior density of θ_1 under the noninformative prior π given in (3.1) for some choice of parameters. That is, we investigate that the frequentist coverage for a $(1-\alpha)$th posterior quantile can be equal to $1-\alpha$. For the developed noninformative priors, Tables 4.1, 4.2 and 4.3 represent the numerical results for the frequentist coverage probabilities of 0.05 and 0.95 posterior quantiles, respectively. These numerical results are computed by the following equation for the fixed values of μ and σ^2, and the specified probability value α with 0.05 and 0.95. For the one sided credible interval of θ_1 the frequentist coverage probability is

$$P_{(\theta_1,\theta_2)}(\alpha;\theta_1) = P_{(\theta_1,\theta_2)}(0 < \theta_1 \leq \theta_1^{\alpha}(\pi;\mathbf{X})). \tag{4.1}$$

When $\alpha = 0.05(0.95)$, the computed values of $P_{(\theta_1,\theta_2)}(\alpha;\theta_1)$ are given in Tables 4.1, 4.2 and 4.3. For the given values of n, μ and σ^2, we generate $10,000$ independent random samples of $\mathbf{X}$ from the lognormal distribution.

From the results of Tables 4.1, 4.2 and 4.3, we can know that the matching prior π_m coincides the target coverage probabilities well than Jeffreys' prior π_J and the reference prior π_r. We notice that for the case of $(a,b) = (1,1)$, Jeffreys' prior overestimates the target coverage probability when values of σ and n are increasing, and for the cases of $(a,b) = (1,0.5)$ and $(a,b) = (1,1.282)$, Jeffreys' prior underestimates the target coverage probability when values of σ and n are increasing. Also we can observe that the results of tables are somewhat insensitive for the different values of (μ,σ^2). Therefore we can recommend the matching prior for Bayesian inference.

Table 4.1 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

(a,b)	μ	σ	n	π_J	π_r	π_m
(1,-1)	0.1	0.1	5	0.080 (0.930)	0.035 (0.927)	0.073 (0.930)
			10	0.069 (0.942)	0.047 (0.937)	0.066 (0.941)
			20	0.057 (0.950)	0.043 (0.946)	0.054 (0.950)
		0.5	5	0.092 (0.948)	0.028 (0.912)	0.070 (0.938)
			10	0.068 (0.954)	0.035 (0.932)	0.057 (0.947)
			20	0.063 (0.949)	0.042 (0.933)	0.055 (0.943)
		2.5	5	0.107 (0.969)	0.020 (0.907)	0.062 (0.949)
			10	0.080 (0.963)	0.030 (0.926)	0.056 (0.950)
			20	0.074 (0.959)	0.037 (0.934)	0.057 (0.948)
		10.0	5	0.109 (0.970)	0.015 (0.899)	0.054 (0.946)
			10	0.085 (0.965)	0.028 (0.920)	0.054 (0.947)
			20	0.068 (0.961)	0.033 (0.931)	0.050 (0.948)
	1.0	0.1	5	0.078 (0.929)	0.036 (0.927)	0.070 (0.928)
			10	0.060 (0.947)	0.037 (0.941)	0.057 (0.946)
			20	0.057 (0.948)	0.045 (0.944)	0.055 (0.947)
		0.5	5	0.089 (0.945)	0.028 (0.906)	0.067 (0.936)
			10	0.069 (0.950)	0.035 (0.923)	0.057 (0.941)
			20	0.062 (0.954)	0.042 (0.937)	0.055 (0.949)
		2.5	5	0.102 (0.969)	0.019 (0.900)	0.060 (0.948)
			10	0.076 (0.963)	0.030 (0.920)	0.053 (0.948)
			20	0.068 (0.960)	0.036 (0.933)	0.052 (0.947)
		10.0	5	0.109 (0.974)	0.016 (0.899)	0.055 (0.949)
			10	0.086 (0.968)	0.028 (0.921)	0.051 (0.951)
			20	0.075 (0.961)	0.037 (0.931)	0.054 (0.949)
	10.0	0.1	5	0.075 (0.931)	0.033 (0.927)	0.069 (0.930)
			10	0.061 (0.944)	0.040 (0.939)	0.058 (0.943)
			20	0.052 (0.948)	0.040 (0.943)	0.050 (0.947)
		0.5	5	0.090 (0.944)	0.030 (0.905)	0.070 (0.933)
			10	0.071 (0.951)	0.039 (0.928)	0.060 (0.944)
			20	0.063 (0.954)	0.043 (0.938)	0.056 (0.948)
		2.5	5	0.109 (0.967)	0.022 (0.903)	0.066 (0.949)
			10	0.087 (0.964)	0.032 (0.922)	0.060 (0.949)
			20	0.070 (0.959)	0.038 (0.931)	0.052 (0.948)
		10.0	5	0.108 (0.970)	0.017 (0.895)	0.053 (0.945)
			10	0.086 (0.965)	0.029 (0.921)	0.051 (0.947)
			20	0.071 (0.961)	0.034 (0.931)	0.052 (0.949)

Table 4.2 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

(a,b)	μ	σ	n	π_J	π_r	π_m
(1,0.5)	0.1	0.1	5	0.071 (0.922)	0.060 (0.957)	0.070 (0.924)
			10	0.060 (0.941)	0.056 (0.958)	0.059 (0.942)
			20	0.058 (0.945)	0.057 (0.953)	0.058 (0.945)
		0.5	5	0.070 (0.921)	0.078 (0.965)	0.072 (0.931)
			10	0.060 (0.936)	0.068 (0.960)	0.062 (0.939)
			20	0.053 (0.941)	0.058 (0.956)	0.054 (0.944)
		2.5	5	0.049 (0.910)	0.092 (0.974)	0.060 (0.936)
			10	0.044 (0.931)	0.074 (0.966)	0.055 (0.943)
			20	0.050 (0.942)	0.070 (0.962)	0.058 (0.949)
		10.0	5	0.033 (0.900)	0.096 (0.981)	0.051 (0.945)
			10	0.038 (0.921)	0.081 (0.972)	0.055 (0.945)
			20	0.043 (0.935)	0.071 (0.965)	0.054 (0.950)
	1.0	0.1	5	0.071 (0.927)	0.060 (0.962)	0.071 (0.928)
			10	0.061 (0.945)	0.058 (0.962)	0.061 (0.946)
			20	0.056 (0.945)	0.055 (0.954)	0.056 (0.945)
		0.5	5	0.069 (0.922)	0.077 (0.965)	0.071 (0.929)
			10	0.053 (0.941)	0.059 (0.962)	0.055 (0.945)
			20	0.054 (0.944)	0.060 (0.959)	0.055 (0.947)
		2.5	5	0.050 (0.912)	0.097 (0.972)	0.062 (0.934)
			10	0.047 (0.932)	0.076 (0.967)	0.056 (0.944)
			20	0.048 (0.937)	0.066 (0.960)	0.054 (0.945)
		10.0	5	0.033 (0.899)	0.096 (0.978)	0.053 (0.940)
			10	0.035 (0.923)	0.074 (0.969)	0.051 (0.945)
			20	0.040 (0.931)	0.068 (0.962)	0.051 (0.947)
	10.0	0.1	5	0.071 (0.924)	0.058 (0.959)	0.070 (0.926)
			10	0.059 (0.938)	0.056 (0.954)	0.059 (0.939)
			20	0.054 (0.947)	0.054 (0.954)	0.054 (0.947)
		0.5	5	0.071 (0.925)	0.081 (0.967)	0.073 (0.932)
			10	0.063 (0.939)	0.070 (0.962)	0.064 (0.944)
			20	0.056 (0.944)	0.061 (0.958)	0.057 (0.947)
		2.5	5	0.049 (0.908)	0.091 (0.974)	0.063 (0.932)
			10	0.046 (0.927)	0.074 (0.964)	0.055 (0.942)
			20	0.046 (0.940)	0.064 (0.963)	0.052 (0.948)
		10.0	5	0.036 (0.898)	0.102 (0.978)	0.056 (0.941)
			10	0.038 (0.918)	0.076 (0.967)	0.053 (0.941)
			20	0.041 (0.927)	0.068 (0.960)	0.052 (0.945)

Table 4.3 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

(a,b)	μ	σ	n	π_J	π_r	π_m
(1,1.282)	0.1	0.1	20	0.067 (0.925)	0.081 (0.967)	0.069 (0.933)
			10	0.057 (0.938)	0.067 (0.963)	0.059 (0.943)
			20	0.050 (0.945)	0.056 (0.959)	0.051 (0.947)
		0.5	5	0.048 (0.903)	0.090 (0.974)	0.060 (0.930)
			10	0.043 (0.930)	0.076 (0.968)	0.053 (0.945)
			20	0.043 (0.938)	0.064 (0.963)	0.051 (0.949)
		2.5	5	0.031 (0.892)	0.095 (0.982)	0.050 (0.941)
			10	0.034 (0.919)	0.075 (0.972)	0.051 (0.948)
			20	0.035 (0.924)	0.064 (0.961)	0.047 (0.942)
		10.0	5	0.030 (0.893)	0.103 (0.986)	0.053 (0.949)
			10	0.035 (0.914)	0.077 (0.972)	0.050 (0.949)
			20	0.037 (0.928)	0.066 (0.967)	0.049 (0.947)
	1.0	0.1	5	0.071 (0.918)	0.085 (0.965)	0.073 (0.927)
			10	0.057 (0.938)	0.066 (0.961)	0.058 (0.942)
			20	0.054 (0.943)	0.061 (0.959)	0.055 (0.947)
		0.5	5	0.049 (0.908)	0.096 (0.976)	0.062 (0.934)
			10	0.049 (0.933)	0.081 (0.968)	0.058 (0.946)
			20	0.044 (0.942)	0.065 (0.963)	0.051 (0.950)
		2.5	5	0.030 (0.897)	0.095 (0.982)	0.050 (0.945)
			10	0.033 (0.917)	0.074 (0.971)	0.049 (0.947)
			20	0.039 (0.933)	0.068 (0.965)	0.052 (0.950)
		10.0	5	0.028 (0.888)	0.102 (0.985)	0.051 (0.947)
			10	0.030 (0.916)	0.073 (0.975)	0.048 (0.949)
			20	0.038 (0.929)	0.070 (0.964)	0.054 (0.949)
	10.0	0.1	5	0.064 (0.918)	0.076 (0.967)	0.066 (0.929)
			10	0.057 (0.937)	0.067 (0.960)	0.059 (0.942)
			20	0.050 (0.940)	0.058 (0.955)	0.052 (0.944)
		0.5	5	0.044 (0.910)	0.094 (0.975)	0.058 (0.935)
			10	0.043 (0.929)	0.071 (0.969)	0.052 (0.944)
			20	0.048 (0.938)	0.068 (0.960)	0.054 (0.945)
		2.5	5	0.030 (0.890)	0.096 (0.980)	0.051 (0.936)
			10	0.032 (0.914)	0.076 (0.971)	0.048 (0.946)
			20	0.041 (0.931)	0.072 (0.968)	0.056 (0.949)
		10.0	5	0.029 (0.889)	0.105 (0.985)	0.056 (0.946)
			10	0.033 (0.914)	0.081 (0.972)	0.052 (0.948)
			20	0.037 (0.930)	0.068 (0.967)	0.050 (0.951)

4.2. Real examples

We compare the confidence interval based on Cox' method (Land, 1972), the parametric bootstrap method (Angus, 1994), the modified signed log-likelihood ratio method (Wu *et al.*, 2003) and Bayesian credible intervals for the lognormal mean. Two real examples are illustrated, and these examples analyzed by Wu *et al.* (2003).

Example 4.1 The first example is a study of the remission times of melanoma patients (Lee, 1992). Five melanoma (resected) patients receiving immunotherapy BCG (Bacillus Calmette-Guérin) are studied. The remission durations in weeks (in order of magnitude) are 8, 16, 23, 27 and 28. The Shapiro-Wilk test for normality on the logarithmically transformed data give a p-value of 0.1848 that supports the assumption that the remission times follow a lognormal distribution. The 90% confidence intervals for the mean remission time are (14.07, 32.24), (13.76, 41.92) and (14.12, 45.39) for Cox' method, the parametric bootstrap method and the modified signed log-likelihood ratio method, respectively. The 90% Bayesian credible intervals for the mean remission time are (14.79, 38.38), (14.75, 61.48) and (14.78, 40.48) for Jeffreys' prior, the reference prior and the matching prior, respectively. All methods yield similar lower bounds, but quite different upper bounds. That is, the parametric bootstrap method, the modified signed log-likelihood ratio method and Bayesian credible interval based on the matching prior are somewhat similar results, whereas Cox's method and Bayesian credible interval based on the reference prior have different results.

Example 4.2 The second example is a study of the survival times of ovarian cancer patients. Cameron and Pauling (1978) report the survival times of six women with terminal ovarian cancer who were treated with supplement vitamin C, and the data are 1234, 89, 201, 356, 2970 and 456. The Shapiro-Wilk test for normality on the logarithmically transformed data give a p-value of 0.9655 that presents strong evidence that the data follow a lognormal distribution. The 90% confidence intervals for the log mean survival time are (5.76, 8.12), (6.09, 9.16) and (6.02, 9.65) for Cox' method, the parametric bootstrap method and the modified signed log-likelihood ratio method, respectively. The 90% Bayesian credible intervals for the log mean time are (6.06, 8.89), (6.15, 11.17) and (6.08, 9.44) for Jeffreys' prior, the reference prior and the matching prior, respectively. Among the confidence intervals, the parametric bootstrap interval, the modified signed log-likelihood ratio interval, the Bayesian intervals based on Jeffreys' and the matching priors have the similar results. Furthermore the modified signed log-likelihood ratio interval and the Bayesian interval based on the matching prior give almost same confidence intervals.

5. Concluding remarks

The reference priors and a first order matching priors for the linear function of parameters in lognormal distribution have been derived. The linear function of parameters in the lognormal distribution includes the expectation, median and mode of the lognormal distribution. Therefore for the general cases including these parameters, the noninformative priors are developed. We revealed that Jeffreys' prior, the first order matching prior and the reference prior have the different forms, and also Jeffreys' prior and the reference prior do not satisfy the first order matching criterion. The propriety of posterior distributions are studied. From the results of our numerical study, the matching prior gives the best appropriate results of them all in terms of the asymptotic frequentist coverage probabilities and the confidence interval.

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