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Run related probability function and their application to start-up demonstration tests

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Abstract

A start-up demonstration test is a mechanism that is usually used to determine the reliability of equipment, for example water pumps, car batteries and power generators. The simplest and oldest start-up demonstration tests are called CS (consecutive successes) which have been studied by Hahn and Gage (1983), Viveros and Balakrishnan (1993).At first Hahn and Gage (1983) discussed the start-up demonstration test. It was based on i.i.d (independently and identically distributed) binary outcomes with the specified number of consecutive successful start-ups. Oh (2016) studied $CSNCF$ (consecutive successful, but not consecutive failures). In this paper, we investigated the CS and $CSNCF$ models, also their applications to start-up demonstration tests. The numerical results showed that the expectations and variances of the total number of attempted start-ups until the acceptance of the unit are gradually increasing in all of the specified number of successes as the p (probability of a successful start-up in any single trial) decreases from 0.99 to 0.90. The difference between means of the CS model and $CSNCF$ model is small, but variances of the CS and $CSNCF$ are big.

Keywords: CS, CSNCF, geometric distribution of order k, runs, start-updemonstration tests

1. Introduction

A start-up demonstration test is mechanism that is usually used to determine the acceptability of equipment. For example, all kinds of electronic communications, emergency light systems and other engineering systems in our life. By this method, we should judge whether the equipment has high start-up reliability before using it by observing the outcomes of a set of tests on the equipment.

A lot of research has been carried out on the start-up demonstration test in the past few decades. In the start-up demonstration test, the judging criterion of the reliability level is related to the specified number of consecutive successes and failures in test. While the success and failures have formed the acceptance and rejection criteria. Therefore, the simplest and

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oldest start-up demonstration tests are called CS (consecutive successes), which has been presented by Hahn and Gage (1983), Viveros and Balakrishnan (1993).

In early research on the demonstration test, Hahn and Gage (1983) were the first to discuss the start-up demonstration test. In a unit under test there was a specified number of consecutive successes start-up before the equipment was accepted. Also the failure number in this test was ignored. They considered the result of the start-ups and i.i.d binary random variables, and a recursion formula is derived for the probability of the waiting time until the specified numbers of consecutive successes are accepted.

Viveros and Balakrishnan (1993) also studied the CS start-up demonstration test for i.i.d start-up with probability of consecutive successes. In addition they derived the mean and variance of start-up demonstration tests by the recurrence relationships of the test length or the probability generating function given by Feller (1968).

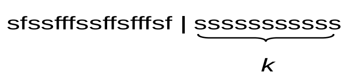
Balakrishnan, Balasubramanian and Viveros (1995) derived the joint probability generating function (p.g.f) of start-up demonstration test, and studied Markov dependent binary outcomes. In the same paper, the corrective actions had been studied in the case of independent outcomes. Subsequently, Balakrishnan, Mohanty and Aki (1997) investigated CS demonstration test, but in corrective action and the setting of Markov dependence structure. By this method, they studied the joint probability generating function (p.g.f), and they used it to derive characteristics with other p.g.f and waiting times. In particular, not only did they observe the joint probability generating function of the specified number of the consecutive successes until acceptance, but also the specified number of the failures until acceptance. As well as this they made note of the tests terminate the experiment. Also they studied the specified number of consecutive successes until acceptance and the number of the failures until acceptance, and the waiting time of the test length.

Kolev and Minkova (1997) analyzed the specified k of consecutive successes for a multistate Markov chain which has a successful state and unlimited fault state. They derived the joint probability generating function for the consecutive successes k, and the number of transitions among states until consecutive successes k was accepted. In addition, they also derived the exact distribution of the total number of consecutive successes, total number of failures, and the total number of tests for a consecutive success of length k. Oh (2016)studied $CSNCF$.

2. CS model and $CSNCF$ model

CS

A start-up demonstration experiment in which an equipment under test is accepted if a specified number of consecutive successful start-up.



CSNCF

A start-up demonstration test in which a unit under test is accepted if a consecutive successesfirst time no sequence of consecutive failures start-up.

2.1. CS model

According to the previous study of success run distribution, Todhunter (1865) studied the probability of the event u_n.Todhunter (1865)found the generating function of u_n,and obtained the formula (2.1). He interpreted that the run of k is completed at the n-th trial in a sequence of Bernoulli trials with success p. In addition, Todhunter (1865)argued that:

$$u_{n+1} = u_n + (1 - u_{n-k})qp^k, \text{ where } q = 1 - p. \tag{2.1}$$

In Feller's research (1957), an application of the theory of recurrent event was treated and hence showed that the distribution of the trial number n at which the first run of k occurs has probability generating function (p.g.f):

$$G(z) = \frac{p^k z^k(1 - pz)}{1 - z + qp^k z^{k+1}} \tag{2.2}$$

where $n = k, k + 1, k + 2, \cdots$, $k = 1, 2, \cdots$, and $0 < p < 1$, respectively.

The mean and variance are

$$\mu = \frac{1 - p^k}{qp^k},\ \mu_2 = \frac{1}{(qp^k)^2} - \frac{2k + 1}{qp^k} - \frac{p}{q^2} \tag{2.3}$$

The other related geometric distributions of order k are debated in the following content.

Consider $X_1, X_2, \cdots$ be a sequence of binary trial each resulting in the probability of success $p = Pr[X_i = 1]$ or probability of failure $q = 1 - p = Pr[X_i = 0]$. Let T_k be the waiting time until a k consecutive success occurs for the firsttime, that is:

$$\begin{aligned} T_k =& \min\ \{n : X_{n-k+1} = \cdots = X_n = 1\} \\ =& \min\ \{n : \prod_{i=n-k+1}^{n} X_i = 1\} \\ =& \min\ \{\sum_{i=n-k+1}^{n} X_i = k\} \end{aligned} \tag{2.4}$$

Philippou and Muwafi's (1982) obtained a exact non-recursive formula employing a simple combinatorial argument.

$$Pr[T_k = x] = p^x \sum_{x_1,\cdots,x_k} \binom{x_1 + \cdots + x_k}{x_1, \cdots, x_k} \left(\frac{q}{p}\right)^{x_1+\cdots+x_k} \tag{2.5}$$

where $x = k, k + 1, \cdots$

Uppuluri and Patil (1983) studied the simpler formula which involves binomial coefficients and single summations.

$$f(x) = p^k \sum_{j=0}^{\infty}(-1)^j \binom{x-k-jk}{j}(qp^k)^j - p^{k+1} \sum_{j=0}^{\infty}(-1)^j \binom{x-k-jk-1}{j}(qp^k)^j,\ x \geq k. \tag{2.6}$$

Muselli (1996) established the effective single summation formula.

$$f(x) = \sum_{j=1}^{[\frac{x+1}{k+1}]} (-1)^{j-1} p^{jk} q^{j-1} \left\{ \binom{x-jk-1}{j-2} + q \binom{x-jk-1}{j-1} \right\} \tag{2.7}$$

In the next sections, due to Hahn and Gage (1983), simple formulas have been derived and very efficient for computations.

2.1.1. CS model for the general solution

Hahn and Gage (1983) studied CS model and asked some questions in the following content. Assume that the start-up demonstration test is an independent event whose probability of the consecutive successes is p

1. What is the probability that for a specified unit 2 continuous successful start-ups can be obtained in exactly 25 attempted start-ups, 35 or less attempted start-ups, 45 or less attempted start-ups, 55 or less attempted start-upsand 65 or more attempted start-ups?
2. What is the number of attempted start-ups that results in the probability with 55%, 65%, 75%, 85% and 95%?

More detail will be done specifically as below:

$p=$ probability of a successful start-up in any single trial.

q =probability of a failure in any single trial, i.e. 1-p.

k = the number of consecutive successes required for acceptance.

$X=$ the total number of attempted start-ups until the acceptance of the unit.

$p(x) = Pr[X = x]$ = the probability of requiring exactly x attempted start-ups for acceptance of the unit.

Hahn and Gage (1983) show that

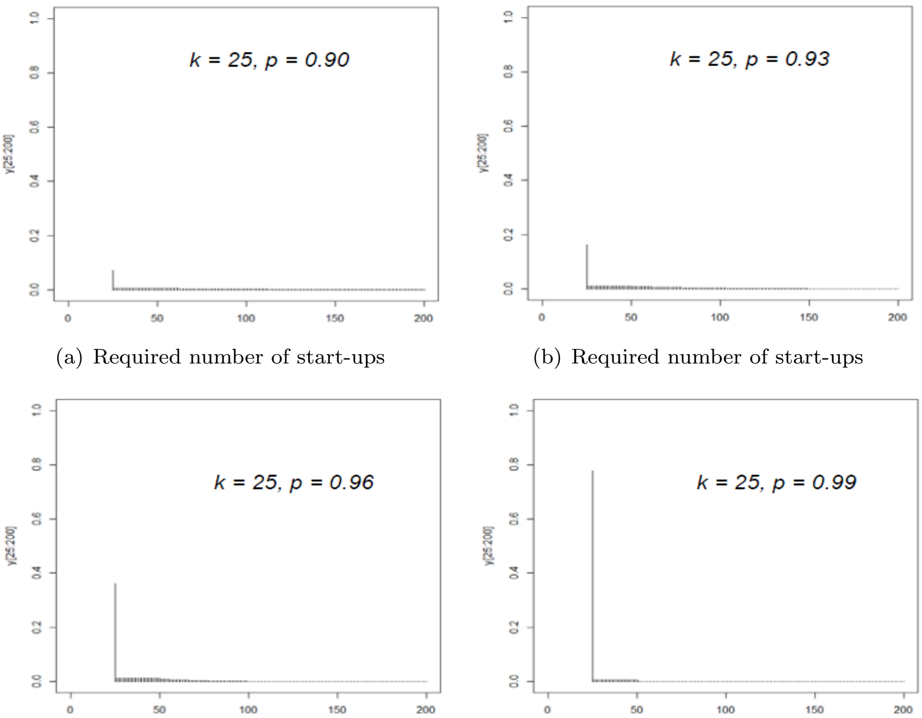
$$f(x) = \begin{cases} 0, & \text{when } x < k \\ p^k, & \text{when } x = k \\ (1-p)p^k, & \text{when } k+1 \le x \le 2k \\ (1-p)p^k p^*, & \text{when } x \ge 2k+1 \end{cases} \tag{2.8}$$

where $p^* = 1 - \sum_{i=1}^{x-2k} p(k+i-1)$.

2.1.2. Applications to start-up demonstration tests

In the following content, now we can use this new formula (2.8) to solve the questions 1and 2.

The Figure 2.1 is the graph of the geometric distribution of order $k = 25$, and $p = 0.90$, 0.93, 0.96, 0.99.



(a) Required number of start-ups (b) Required number of start-ups

(c) Required number of start-ups (d) Required number of start-ups

Figure 2.1 Geometric distribution of order $k = 25$

The specific answers to questions 1 and 2 can be obtained by using (2.8). The results are Table 2.1 and Table 2.2 when $k = 25$, $p = 0.90$, 0.93, 0.96 and 0.99.

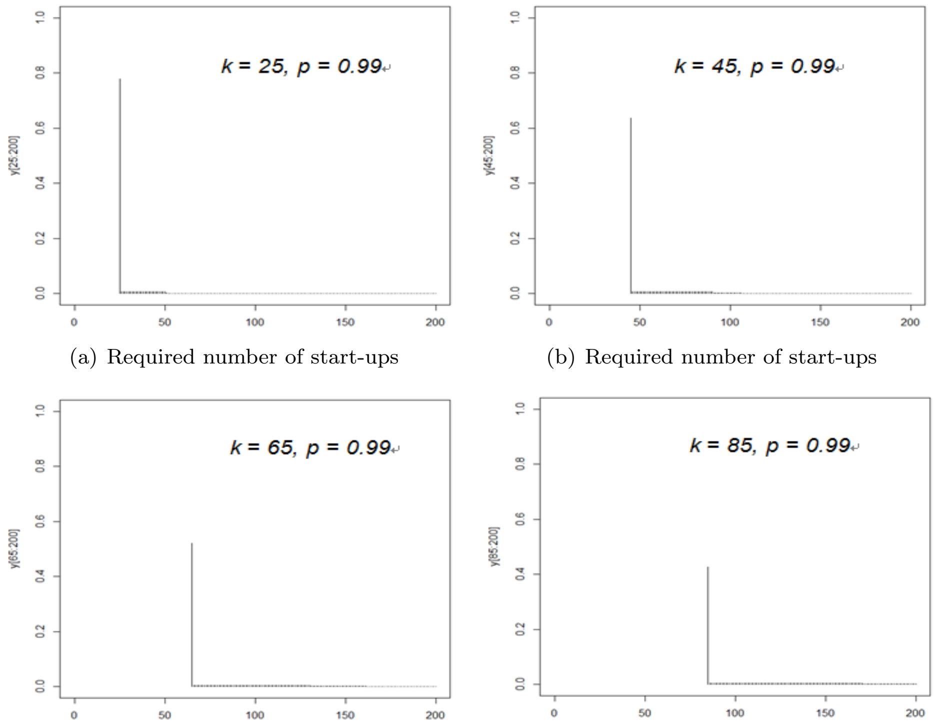
Table 2.1 The probability of a specified unit to obtain 25 continuous successful start-ups inattempted start-ups.

	$p = 0.90$	$p = 0.93$	$p = 0.96$	$p = 0.99$
Exactly 25 attempted start-ups [$p(25)$]	0.0717898	0.1629573	0.3603967	0.7778214
35 or less attempted start-ups [$p(35)$]	0.1435796	0.2770273	0.5045554	0.8556035
45 or less attempted start-ups [$p(45)$]	0.2153694	0.3910974	0.6487141	0.9333856
55 or less attempted start-ups [$p(55)$]	0.2840669	0.494572	0.7648174	0.9803125
65 or more attempted start-ups [1-$p(65)$]	0.6541931	0.4223078	0.1627208	0.0081536

Table 2.2 Required number of attempted start-ups withpercent for probability of single successful start-up at $p = 0.90$, $p = 0.93$, $p = 0.96$ and $p = 0.99$.

	$p = 0.90$	$p = 0.93$	$p = 0.96$	$p = 0.99$
55%	106	61	38	25
65%	133	75	45	25
75%	170	93	53	25
85%	-	121	67	34
95%	-	178	94	47

The Figure 2.2 is the graphs of geometric distribution of order $k = 25, 45, 65, 85$ and $p = 0.99$.



(a) Required number of start-ups (b) Required number of start-ups

(c) Required number of start-ups (d) Required number of start-ups

Figure 2.2 Geometric distribution of order $k = 25$, 45, 65 and 85, $p = 0.99$

2.2. $CSNCF$ model

Oh (2016) studied $CSNCF$ model, and also obtained the mean and variance for $CSNCF$ model. The mean and variance are

$$E(X) = \frac{1}{p^{k-2}}\left\{\frac{1}{1-p} + \frac{p^{k-2}}{p-1} - \frac{p^3(3-p+p^2)}{(1-p+p^2)^2} + \frac{p^2(5-p+p^2)}{(1-p+p^2)^2}\right\} \tag{2.9}$$

and

$$\begin{aligned} V(X) = &\frac{1}{p^{k-2}}\left\{\frac{8p^2}{(1-pq)^3} - \frac{4p^2}{(1-pq)^2} + \frac{p^2q}{(1-pq)} + \frac{8p^2q}{(1-pq)^3} - \left(\frac{2p^2}{(1-pq)^2} + \frac{p^2q}{(1-pq)} + \frac{2p^2q}{(1-pq)^2}\right)^2\right\} \\ &+ (1-p)\sum_{i=0}^{k-3}\frac{1}{p_i}\left\{\frac{1}{p^{k-i-2}}\left(\frac{2p^2}{(1-pq)^2} + \frac{p^2q}{(1-pq)} + \frac{2p^2q}{(1-pq)^2}\right) + \frac{1-p^{k-i-2}}{(1-p)}\right\}^2 \end{aligned} \tag{2.10}$$

3. Numerical results

The specified success probability for each attempted is p and the demonstration test will run until k consecutive successful start-ups are achieved.

Feller (1968) showed that the mean and variance of X for the CS model are

$$E(X) = \frac{1 - p^k}{(1-p)p^k} \tag{3.1}$$

and

$$V(X) = \frac{1 - (2k+1)qp^k - p^{2k+1}}{(qp^k)^2} \tag{3.2}$$

Oh (2016) show that the mean and variance of X for the $CSNC$ model as follows (2.9) and (2.10), respectively.In this study we also apply (2.9), (2.10), (3.1) and (3.2) to calculate the values in Table 3.1.

Table 3.1 The numerical results of the proposed CS model and CSNCF model for k and p = 0.90, 0.95 and 0.99.

k	p	CS		$CSNCF$	
		E (X)	V(X)	E(X)	V(X)
	0.99	10.57274	4.302397	10.57174	3.943335
10	0.95	13.40365	34.32725	13.36703	22.87593
	0.90	18.67972	130.2522	18.43637	61.9985
	0.99	12.81781	7.412993	12.8168	6.610665
12	0.95	17.01236	64.6056	16.97177	38.18922
	0.90	25.40706	278.4835	25.10663	109.0470
	0.99	15.10847	11.81393	15.10744	10.24607
14	0.95	21.01092	112.5789	20.96595	59.02275
	0.90	33.71242	553.1156	33.34152	179.2444
	0.99	17.44564	17.77156	17.44458	14.98624
16	0.95	25.44146	185.3583	25.39164	86.25834
	0.90	43.96595	1041.448	43.50805	281.3699
	0.99	19.83026	25.57135	19.82919	20.96344
18	0.95	30.35065	292.2140	30.29544	120.8534
	0.90	56.62463	1883.730	56.05932	427.7136
	0.99	22.2633	35.51891	22.26220	28.30576
20	0.95	35.79020	445.148	35.72902	163.8682
	0.90	72.25263	3303.138	71.55471	635.59
	0.99	24.74574	47.94107	24.74462	37.13737
22	0.95	41.81739	659.6073	41.74961	216.4941
	0.90	91.54646	5652.093	90.68483	929.526
	0.99	27.27858	63.187	27.27744	47.57877
24	0.95	48.49573	955.3738	48.42062	280.0835
	0.90	115.366	9483.7	114.3023	1344.452
	0.99	29.86285	81.62942	29.86169	59.74698
26	0.95	55.89554	1357.669	55.81232	356.183
	0.90	144.7728	15661.67	143.4596	1930.384
	0.99	32.49960	103.6658	32.49841	73.75578
28	0.95	64.09478	1898.530	64.00257	446.5695
	0.90	181.0776	25529.22	179.4563	2759.343
	0.99	35.18987	129.7196	35.18866	89.71594
30	0.95	73.17981	2618.509	73.07764	553.2918
	0.90	225.8982	41168.19	223.8966	3935.608
	0.99	37.93478	160.2419	37.93354	107.7354
32	0.95	83.24633	3568.793	83.13312	678.7178
	0.90	281.2324	65796.21	278.7613	5610.98
	0.99	40.73541	195.7124	40.73415	127.9195
34	0.95	94.40037	4813.819	94.27493	825.5893
	0.90	349.5462	104374.8	346.4954	8007.59
	0.99	43.59291	236.6412	43.59162	150.371
36	0.95	106.7594	6434.511	106.6204	997.0853
	0.90	433.8842	164539.6	430.1178	11452.06
	0.99	46.50843	283.5703	46.50711	175.1906
38	0.95	120.4536	8532.294	120.2996	1196.897
	0.90	538.0052	258023.2	533.3553	16426.84
	0.99	49.48314	337.0751	49.4818	202.4770
40	0.95	125.6273	11234.05	135.4567	1429.314
	0.90	666.5496	402828.8	660.809	23647.46
	0.99	52.51825	397.7664	52.51689	232.3266
42	0.95	152.4402	14698.21	152.2511	1699.328
	0.90	825.2464	626550.6	818.1593	34179.07
	0.99	55.61499	466.2917	55.6136	264.8344
44	0.95	171.0695	19122.37	170.86	2012.757
	0.90	1021.168	971444.2	1012.419	49612.56
	0.99	58.77461	543.3375	58.77319	300.0939
46	0.95	191.7114	24752.54	191.4792	2376.386
	0.90	1263.047	1502166	1252.245	72330.97
	0.99	61.99838	629.6312	61.99692	338.1968
48	0.95	214.5832	31894.71	214.326	2798.14
	0.90	1561.663	2317585	1548.328	105913.0
	0.99	65.2876	725.9428	65.28612	379.2338
50	0.95	239.926	40929	239.641	3287.291
	0.90	1930.325	3568799	1913.861	155745.2

4. Summary and concluding remarks

In this paper, we investigated the CS model and the $CSNCF$ model, also their application to start-up demonstration tests.In the start-up demonstration tests, we consider the run related probability function for k(10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50), and $p = 0.90$, 0.95 and 0.99. We get numerical results of the proposed CS model and $CSNCF$ model.

The numerical results showed that the $E(X)$ and $V(X)$ are gradually increasing in all of the specified number of successes as the p decreases from 0.99 to 0.90. That means the value of p had a positive correlation with CS and $CSNCF$. In addition, the difference between means of the CS and $CSNCF$ is small, but variances of the CS and $CSNCF$ are big.

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