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REEB FLOW SYMMETRY ON ALMOST COSYMPLECTIC THREE-MANIFOLDS

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Abstract. We prove that the Ricci operator S of an almost cosymplectic three-manifold M is invariant along the Reeb flow, that is, M satisfies $\pounds_\xi S = 0$ if and only if M is either cosymplectic or locally isometric to the group $E(1,1)$ of rigid motions of Minkowski 2-space with a left invariant almost cosymplectic structure.

1. Introduction

Almost contact three-manifolds M are phrased as 3-dimensional manifolds whose structural group is reducible to $U(1) \times \{1\}$. Then there exist a vector field ξ, a 1-form η and a $(1,1)$-tensor field ϕ satisfying

$$\eta(\xi) = 1 \text{ and } \phi^2 = -\mathrm{id} + \eta \otimes \xi.$$

We call ξ the *Reeb vector field* of an almost contact manifold M. We can always find a compatible Riemannian metric g:

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M. Such $(M; \phi, \xi, \eta, g)$ is said to be an *almost contact metric manifold*. The *fundamental 2-form* Φ is defined by $\Phi(X,Y) = g(X, \phi Y)$. If M satisfies in addition $d\eta = \Phi$, then M is called a *contact metric manifold*, where d is the exterior differential operator.

An almost contact metric manifold $(M; \phi, \xi, \eta, g)$ is said to be almost cosympectic if $d\eta = 0$ and $d\Phi = 0$. Such a class was introduced by S. I. Goldberg and K. Yano [7]. The products of an almost Kähler manifold and a real line or a circle are the simplest examples of such manifolds. Besides such product manifolds, the class of almost cosymplectic manifolds includes plenty more examples (cf. [10], [12]). Recently, D. Perrone [14] classified all homogeneous almost cosymplectic three-manifolds. In the present paper, we prove a local classification theorem of almost cosymplectic three-manifolds whose Ricci operator S and the structure tensor ϕ commute, that is, $S\phi = \phi S$ (Theorem 4 in

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Section 3). Using this result, in Section 4, we prove that for a 3-dimensional almost cosymplectic manifold M the Ricci operator is invariant along the Reeb flow, that is, M satisfies $\pounds_\xi S = 0$ if and only if M is either cosymplectic or locally isometric to the group $E(1,1)$ of rigid motions of Minkowski 2-space with a left invariant almost cosymplectic structure (Theorem 9). It is remarkable that a Lie group $E(1,1)$ admits also a left invariant contact metric structure. But, such a contact metric structure does not enjoy this symmetry any more (see Remark 1 in Section 4).

In [14] it was shown that a locally symmetric almost cosymplectic three-manifold is cosymplectic and it is locally a Riemannian product of a curve and a Kähler surface of constant curvature c. In Section 5, we study pseudo-symmetry of unimodular Lie groups with left-invariant almost cosymplectic structure (Proposition 11).

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. First, we remind a fundamental equation on an almost contact metric three-manifold ([11]):

$$d\Phi = (\text{div } \xi)\eta \wedge \Phi.$$

Then we have (cf. [14]):

Proposition 1. *An almost contact metric three-manifold is almost cosymplectic if and only if $\nabla\xi$ is symmetric and div $\xi = 0$.*

Let $(M; \phi, \xi.\eta, g)$ be a 3-dimensional almost cosymplectic manifold. We prepare some fundamental formulas which are mainly established in [13]. Then we have

$$\nabla_\xi \phi = 0, \quad \nabla_X \xi = -\phi h X, \tag{1}$$

where $h = \frac{1}{2}\pounds_\xi \phi$. Moreover, we have the following properties:

$$h\phi + \phi h = 0, \quad h\xi = 0, \quad \text{tr } h = 0, \quad \text{div } \xi = 0. \tag{2}$$

For an almost contact structure (ϕ, ξ, η), one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be *normal*. It is known that M is normal if and only if M satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A normal almost cosymplectic manifold is called a cosymplectic manifold. An almost cosymplectic manifold is normal if and only if the canonical foliation $\mathcal{F}$, which is generated by the contact distribution $D = \ker \eta$, is tangentially Kähler and $h = 0$ (cf. [8]). From this, we observe that an almost cosymplectic three-manifold is cosymplectic if and only if $h = 0$. We refer [1] for more details on almost contact geometry.

3. Almost cosymplectic three-manifolds with $S\phi = \phi S$

For a Riemannian manifold (M, g), denote by R its Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for any vector fields X, Y, Z on M. Remind that the curvature tensor R of a 3-dimensional Riemannian manifold is expressed by

$$\begin{aligned} R(X,Y)Z = {} & \rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)SX - g(X,Z)SY \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{3}$$

for all vector fields X, Y, Z, where $\rho(Y, X) = g(SY, X)$ and r denotes the scalar curvature.

For a cosymplectic manifold $(M; \phi, \xi, \eta, g)$, from the 2nd equation of (1) we easily get $R(X, Y)\xi = 0$ and $S\xi = 0$. Then we have:

Proposition 2. *For a cosymplectic three-manifold we have the Ricci operator*:

$$S = \frac{r}{2}(I - \eta \otimes \xi), \tag{4}$$

where I denotes the identity transformation.

Moreover, using the well-known formula: $dr(\cdot) = 2\sum_i(\nabla_{e_i}\rho)(e_i, \cdot)$ for any local orthonormal frame field $\{e_i\}$, we obtain $dr(\xi) = 0$.

Corollary 3. *A cosymplectic three-manifold satisfies $S\phi = \phi S$.*

Now, we prove

Theorem 4. *Let M be a 3-dimensional almost cosymplectic manifold. Then the Ricci operator S and the structure tensor ϕ commute, that is, M satisfies $S\phi = \phi S$ if and only if M is either cosymplectic or locally isometric to the group $E(1, 1)$ of rigid motions of Minkowski 2-space with a left invariant almost cosymplectic structure.*

Proof. Let $M = (M^3; \phi, \xi, \eta, g)$ be an almost cosymplectic three-manifold. If $h = 0$ on M, then we see that M is cosymplectic. And M satisfies $S\phi = \phi S$ (Corollary 3). So, we suppose that $h \neq 0$ identically. We consider on M the maximal open subset U_1 on which $h \neq 0$ and the maximal open subset U_2 on which h is identically zero. $U_1 \cup U_2$ is open and dense in M. Then U_1 is non-empty and there is a local orthonormal frame field $\{e_1 = e, e_2 = \phi e, e_3 = \xi\}$ on U_1 such that $h(e_1) = \mu e_1$, $h(e_2) = -\mu e_2$ for some positive function μ. Then, using the 2nd equation of (1) we have

Lemma 5. *In U_1,*

$$\begin{aligned} &\nabla_\xi \xi = 0, && \nabla_\xi e_1 = ae_2, && \nabla_\xi e_2 = -ae_1, \\ &\nabla_{e_1} \xi = -\mu e_2, && \nabla_{e_1} e_1 = -be_2, && \nabla_{e_1} e_2 = be_1 + \mu\xi, \\ &\nabla_{e_2} \xi = -\mu e_1, && \nabla_{e_2} e_1 = ce_2 + \mu\xi, && \nabla_{e_2} e_2 = -ce_1, \end{aligned} \tag{5}$$

where a, b, c are smooth functions.

From (5) we compute the Poisson brackets:

$$[e_3, e_1] = (a + \mu)e_2, \quad [e_1, e_2] = be_1 - ce_2, \quad [e_2, e_3] = (a - \mu)e_1. \tag{6}$$

From the Jacobi identity

$$[e_3, [e_1, e_2]] + [e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] = 0,$$

we get the following:

$$\begin{cases} -e_1(\mu) + e_3(b) + e_1(a) + c(a - \mu) = 0, \\ e_2(\mu) - e_3(c) + e_2(a) + b(a + \mu) = 0. \end{cases} \tag{7}$$

By using (5) and (6), we compute the Riemannian curvature tensor R, and then we have the Ricci curvature tensor ρ:

$$\begin{cases} \rho(e_3, e_3) = -2\mu^2, \quad \rho(e_3, e_1) = \rho(e_1, e_3) = -e_2(\mu) - 2b\mu, \\ \rho(e_3, e_2) = \rho(e_2, e_3) = -e_1(\mu) - 2c\mu, \\ \rho(e_1, e_1) = -e_1(c) - e_2(b) - b^2 - c^2 - 2a\mu, \\ \rho(e_1, e_2) = \rho(e_2, e_1) = \xi(\mu), \\ \rho(e_2, e_2) = -e_1(c) - e_2(b) - b^2 - c^2 + 2a\mu, \end{cases} \tag{8}$$

where we have used (7). Now we prove

Lemma 6. *In U_1, $S\phi = \phi S \iff [a = 0,\ \xi(\mu) = 0,\ e_1(\mu) + 2b\mu = 0,\ e_2(\mu) + 2c\mu = 0]$.*

Proof. In U_1 we find that $S\phi = \phi S$ if and only if $\rho_{11} = \rho_{22}$ and $\rho_{ij} = 0$, $i \neq j$ for $i, j = 1, 2, 3$, where $\rho_{ij} = \rho(e_i, e_j)$. From this fact and (8) we obtain the above relations. □

Lemma 7. *μ is constant.*

Proof. From (3) with the help of Lemma 6 we have in U_1:

$$\begin{aligned} &R(e_1, e_2)e_2 = (\rho_{11} + \mu^2)e_1, \\ &R(e_1, e_2)e_1 = -(\rho_{22} + \mu^2)e_2, \\ &R(e_2, e_3)e_2 = R(e_1, e_3)e_1 = \mu^2 e_3, \\ &R(e_1, e_3)e_3 = -\mu^2 e_1, \\ &R(e_2, e_3)e_3 = -\mu^2 e_2, \\ &R(e_i, e_j)e_k = 0 \text{ for } i \neq j \neq k \neq i. \end{aligned} \tag{9}$$

Use (5) and (9) to calculate

$$(10)\qquad \begin{aligned} (\nabla_{e_1}R)(e_2,e_3)e_2 &= e_1(\mu^2)e_3,\\ (\nabla_{e_2}R)(e_3,e_1)e_2 &= 0,\\ (\nabla_{e_3}R)(e_1,e_2)e_2 &= e_3(\rho_{11}+\mu^2)e_1, \end{aligned}$$

and

$$(11)\qquad \begin{aligned} (\nabla_{e_2}R)(e_1,e_3)e_1 &= e_2(\mu^2)e_3,\\ (\nabla_{e_1}R)(e_3,e_2)e_1 &= 0,\\ (\nabla_{e_3}R)(e_2,e_1)e_1 &= e_3(\rho_{22}+\mu^2)e_2. \end{aligned}$$

By the second Bianchi identity, (10) and (11) yield respectively that $e_1(\mu)=0$ and $e_2(\mu)=0$. Hence, together with $\xi(\mu)=0$, we see that μ is constant on M, where we have used the continuity argument of μ. □

From $e_1(\mu)+2b\mu=0$ and $e_2(\mu)+2c\mu=0$ (see Lemma 6), we get $b=c=0$. Since $a=0$, from (6) we have

$$(12)\qquad [e_1,e_2]=\mu e_3,\ \ [e_2,e_3]=0,\ \ [e_3,e_1]=-\mu e_2,\ \ \mu\in\mathbb{R}.$$

Hence, due to J. Milnor's result ([9]) we see that M is locally isometric to the group $E(1,1)$ of rigid motions of Mikowski 2-space with a left invariant almost cosymplectic structure. Actually, from the data (12) we compute the Ricci tensor S:

$$(13)\qquad \begin{aligned} Se_1 &= 0,\\ Se_2 &= 0,\\ Se_3 &= \left(-2\mu^2\right)e_3. \end{aligned}$$

Therefore we have proved Theorem 4. □

4. The Reeb flow symmetry

In this section, we study almost cosymplectic three-manifolds whose Ricci operator is Reeb flow invariant: $\pounds_\xi S=0$. Since a cosymplectic three-manifold satisfies $\pounds_\xi\xi=\pounds_\xi\eta=0$, from Proposition 2 and $dr(\xi)=0$ we have:

Proposition 8. *A cosymplectic three-manifold satisfies* $\pounds_\xi S=0$.

We suppose that an almost cosymplectic three-manifold $(M;\phi,\xi,\eta,g)$ satisfies $\pounds_\xi S=0$ and suppose that $h\neq 0$ identically. Then, as in Section 3 we proceed our arguments in the maximal open set U_1 on which $h\neq 0$. We compute

$$\begin{aligned} 0 &= \pounds_\xi(SX)-S(\pounds_\xi X)\\ &= [\xi,SX]-S[\xi,X]. \end{aligned}$$

From this using the 2nd equation of (1) we get an equivalent equation to $\pounds_\xi S = 0$:

$$(\nabla_\xi S)X = (S\phi h - \phi h S)X. \tag{14}$$

Since $\nabla_\xi S$ is a self-adjoint operator, from (14) we find that $\nabla_\xi S = 0$ and $S\phi h = \phi h S$. Applying ξ to $S\phi h = \phi h S$, then we find that $S\xi = \sigma\xi$, where $\sigma = -2\mu^2$ in U_1. Developing $S\phi h = \phi h S$ with e_1 and e_2, then we also find that $\rho_{11} = \rho_{22}$ in U_1. We compute

$$0 = (\nabla_\xi S)\xi = -2(\xi\mu^2)\xi,$$

which yields that $\xi(\mu) = 0$. Then from (8) we obtain $\rho_{12} = \rho_{21} = 0$ in U_1. After all, we have that $S\phi = \phi S$ in U_1. By the same arguments in the proof of Theorem 4, we have that M is locally isometric to the group $E(1,1)$ with a left invariant almost cosymplectic structure. In addition, using (12) and (13) we can easily check that such $E(1,1)$ satisfies $\pounds_\xi S = 0$. Thus, we have

Theorem 9. *The Ricci operator S of 3-dimensional almost cosymplectic manifold is invariant along the Reeb flow, that is, M satisfies $\pounds_\xi S = 0$ if and only if M is either cosymplectic or locally isometric to the group $E(1,1)$ of rigid motions of Minkowski 2-space with a left invariant almost cosymplectic structure.*

The following remark gives interesting properties of a Lie group $E(1,1)$.

Remark 1. Besides an almost cosymplectic structure treated in this paper, a Lie group $E(1,1)$ admits also a left invariant contact metric structure. We refer to [4], [5] for the explicit description of different two structures of $E(1,1)$. For such a contact metric structure, $S\phi \neq \phi S$ (cf. [2]) and $\pounds_\xi S \neq 0$. Indeed, the present author [3] proved that a contact three-manifold satisfies $\pounds_\xi S = 0$ if and only if M is Sasakian or locally isometric to SU(2) (or SO(3)), $\mathrm{SL}(2,\mathbb{R})$ (or $O(1,2)$), $E(2)$ (the group of rigid motions of Euclidean 2-plane) with a left invariant contact Riemannian metric.

Remark 2. Almost Kenmotsu manifold is defined by the conditions: $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. In [5] we proved that the Ricci operator S of almost Kenmotsu three-manifolds is invariant along the Reeb flow, that is, M satisfies $\pounds_\xi S = 0$ if and only if M is locally isometric to either a hyperbolic space $\mathbb{H}^3(-1)$ or a non-unimodular Lie group with left-invariant almost Kenmotsu structure.

5. Pseudo-symmetry

Let (M, g) be a Riemannian manifold. We define a curvature-like tensor field $(X, Y, Z) \longmapsto (X \wedge Y)Z$ on M by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y$$

for all vector fields X, Y, Z on M. Then (M, g) is of constant curvature c if and only if its curvature tensor R satisfies $R(X, Y) = c(X \wedge Y)$ for all vector fields

X, Y on M. A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if there exists a function L such that

$$R(X,Y)\cdot R = L(X\wedge Y)\cdot R$$

holds for all vector fields X, Y on M ([6]). In particular, a pseudo-symmetric Riemannian manifold is called a *pseudo-symmetric space of constant type* if L is constant. A *semi-symmetric space* is a trivial example with $L = 0$.

On a Riemannian 3-manifold (M, g), the Riemannian curvature R is described by the Ricci curvature tensor field ρ and corresponding Ricci operator S by (3). From this, the following characterization of pseudo-symmetry is deduced.

Proposition 10. *A Riemannian* 3*-manifold is a pseudo-symmetric space of constant type if and only if the principal Ricci curvatures* (*eigenvalues of the Ricci operator*) *locally satisfy the following relations* (*up to numeration*):

$$\rho_1 = \rho_2, \quad \rho_3 = 2L.$$

Then, from (4) and (13) we easily find that both a cosymplectic three-manifold and the group $E(1,1)$ of rigid motions of Minkowski 2-space with a left invariant almost cosymplectic structure are pseudo-symmetric of constant type. In this section, we study pseudo-symmetry of unimodular Lie groups with left invariant almost cosymplectic structure.

By Perrone's classification of homogeneous almost cosymplectic three-manifolds ([14]), the simply connected unimodular Lie groups which admit a left invariant almost cosymplectic structure are the following: the universal covering $\tilde{E}(2)$ of the group of rigid motion of Euclidean 2-space, the group $E(1,1)$ of rigid motion of Minkowski 2-space, the Heisenberg group Nil_3 and the commutative group $\mathbb{R}^3$. Among the above model spaces, only a special $\tilde{E}(2)$ and $\mathbb{R}^3$ are normal.

Let G be a 3-dimensional unimodular Lie group with left invariant metric g. Then there exists an orthonomal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1,e_2] = c_3e_3,\ [e_2,e_3] = c_1e_1,\ [e_3,e_1] = c_2e_2, \quad c_i \in \mathbb{R}\ (i = 1,2,3).$$

Take $\xi = e_3$ and define η by $\eta = g(\xi,\cdot)$. Define ϕ by $\phi\xi = 0,\ \phi e_1 = e_2,\ \phi e_2 = -e_1$. Then we have that (ϕ,ξ,η,g) is a left invariant almost contact metric structure on G. By the *Koszul formula* we get $\nabla_{e_1}\xi = \frac{1}{2}(c_1-c_2-c_3)e_2,\ \nabla_{e_2}\xi = \frac{1}{2}(c_1-c_2+c_3)e_1,\ \nabla_\xi\xi = 0$. Then we find that $\nabla\xi$ is symmetric (which is equivalent to $d\eta = 0$) if and only if $c_3 = 0$. And moreover div $\xi = 0$. Thus, we see that (ϕ,ξ,η,g) is an almost cosymplectic structure. We compute $h = \frac{1}{2}\pounds_\xi\phi$. Then $he_1 = \frac{1}{2}(c_2-c_1)e_1$ and $he_2 = \frac{1}{2}(c_1-c_2)e_2$. From this, we see that $(G;\phi,\xi,\eta,g)$ is cosymplectic $(h = 0)$ if and only if $c_1 = c_2$. We already know that every cosymplectic three-manifold is pseudo-symmetric of constant type. So, we now assume that G is not normal. Then Poisson bracket relations (6) become

$$[e_3, e_1] = (a+\mu)e_2, \quad [e_1, e_2] = 0, \quad [e_2, e_3] = (a-\mu)e_1, \tag{15}$$

where $2a = c_1 + c_2$ and $2\mu = c_2 - c_1$. Furthermore, we compute the Ricci operator:

$$\begin{aligned} Se_1 &= -2a\mu\, e_1, \\ Se_2 &= 2a\mu\, e_2, \\ Se_3 &= \left(-2\mu^2\right) e_3. \end{aligned} \tag{16}$$

From (16), we find that pseudo-symmetry implies either $a = 0$ or $|a| = \mu$. Using J. Milnor's result ([9]), we see that G is isometric to either the group $E(1,1)$ or the Heisenberg group Nil_3 with a left invariant almost cosymplectic structure, respectively. Thus, we have:

Proposition 11. *A simply connected unimodular Lie group G with left invariant almost cosymplectic structure is pseudo-symmetric if and only if G is the universal covering $\tilde{E}(2)$ of the group of rigid motion of Euclidean 2-space, the commutative group $\mathbb{R}^3$ with a left invariant cosymplectic structure, respectively, or the group $E(1,1)$ or the Heisenberg group Nil_3 with a left invariant almost cosymplectic structure, respectively.*

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