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MEROMORPHIC FUNCTIONS SHARING FOUR VALUES WITH THEIR DIFFERENCE OPERATORS OR SHIFTS

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Abstract. We prove a uniqueness theorem of nonconstant meromorphic functions sharing three distinct values IM and a fourth value CM with their shifts, and prove a uniqueness theorem of nonconstant entire functions sharing two distinct small functions IM with their shifts, which respectively improve Corollary 3.3(a) and Corollary 2.2(a) from [12], where the meromorphic functions and the entire functions are of hyper order less than 1. An example is provided to show that the above results are the best possible. We also prove two uniqueness theorems of nonconstant meromorphic functions sharing four distinct values with their difference operators.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We use the standard notation of Nevanlinna theory as explained in [11, 17, 22]. We will denote by $E \subset \mathbb{R}^+$ a set of finite logarithmic measure (instead of the usual linear measure), not necessarily the same at each occurrence. Then by the error term $S(r, f)$ we mean any quantity which is of the growth $o(T(r, f))$ as r tends to infinity outside of E.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (cf. [22]). Next we denote by $\overline{N}_0(r, a, f, g)$ the reduced counting function of common a-points of f and g, and denote $\overline{N}_{12}(r, a, f, g)$ by

$$\overline{N}_{12}(r, a, f, g) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_0(r, a, f, g).$$

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where and in what follows, $\overline{N}\left(r,1/(f-\infty)\right)$ means $\overline{N}(r,f)$. If $\overline{N}_{12}(r,a,f,g)=S(r,f)+S(r,g)$, we say that f and g share a IM*. Let $\overline{N}_E(r,a)$ "count" those points in $\overline{N}\left(r,1/(f-a)\right)$, where each point in $\overline{N}_E(r,a)$ is taken by f and g with the same multiplicity, and each such point is counted only once. We say that f and g share the value a CM*, if

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_E(r,a)=S(r,f)+S(r,g).$$

We say that α is a small function of f, if α is a meromorphic function satisfying $T(r,\alpha)=S(r,f)$ (cf. [22]). Suppose that $\alpha\not\equiv\infty$ is a small function of f and g. If $f-\alpha$ and $g-\alpha$ share 0 CM (IM), then we say that f, g share the small function α CM (IM). Throughout this paper, we denote by $\rho(f)$ the order of f, by $\rho_2(f)$ the hyper order of f, and by $\lambda(f)$ the exponent of convergence of zeros of f (cf. [11, 17, 22]). We also need the following two definitions:

Definition 1.1 ([21])**.** Let f be a nonconstant meromorphic function. We define difference operators as

$$\Delta_\eta f(z)=f(z+\eta)-f(z)\text{ and }\Delta_\eta^n f(z)=\Delta_\eta^{n-1}(\Delta_\eta f(z)),$$

where η is a nonzero complex number, $n\geq 2$ is a positive integer. If $\eta=1$, we denote $\Delta_\eta f(z)=\Delta f(z)$.

Remark 1.1. Definition 1.1 implies $\Delta_\eta^n f(z)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}f(z+j\eta)$.

Definition 1.2 ([16, Definition 1])**.** Let p be a positive integer and $a\in\mathbb{C}\cup\{\infty\}$. Then by $N_{p)}(r,1/(f-a))$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r,1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r,1/(f-a))$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r,1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities), where and what follows, $N_{p)}(r,1/(f-a))$, $\overline{N}_{p)}(r,1/(f-a))$, $N_{(p}(r,1/(f-a))$ and $\overline{N}_{(p}(r,1/(f-a))$ mean $N_{p)}(r,f)$, $\overline{N}_{p)}(r,f)$, $N_{(p}(r,f)$ and $\overline{N}_{(p}(r,f)$ respectively, if $a=\infty$.

Recently the value distribution theory of difference polynomials, Nevanlinna characteristic of $f(z+\eta)$, Nevanlinna theory for the difference operator and the difference analogue of the lemma on the logarithmic derivative has been established (cf. [5, 7, 8, 18, 19]). Using these theories, uniqueness questions of meromorphic functions sharing values with their shifts have been recently treated as well (cf. [12, 13, 24]). In this paper, we will consider uniqueness questions of meromorphic functions sharing four values with their shifts or difference operators.

We first recall two theorems from [12]:

Theorem A ([12, Corollary 3.3]). *Let f be a nonconstant entire function such that $\rho(f) < \infty$, and let a, b and c be three distinct finite values. If $f(z)$ and $f(z+\eta)$ share a, b, c IM, where η is a nonzero complex number, then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

Theorem B ([12, Theorem 2.1]). *Let f be a nonconstant meromorphic function of finite order, and let a_1, a_2, a_3 be three distinct values in the extended complex plane. If $f(z)$ and $f(z+\eta)$ share a_1, a_2, a_3 CM, where η is a nonzero complex number, then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

We will prove the following results:

Theorem 1.1. *Let f be a nonconstant meromorphic function such that $\rho_2(f) < 1$, and let η be a nonzero complex number. Suppose that f and $\Delta_\eta f$ share a_1, a_2, a_3 IM, and share ∞ CM, where a_1, a_2, a_3 are three distinct finite values. Then $2f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

Theorem 1.2. *Let f be a nonconstant meromorphic function such that $\rho(f) < \infty$, and let η be a nonzero complex number. Suppose that f and $\Delta_\eta f$ share a_1, a_2, a_3, a_4 IM, where a_1, a_2, a_3, a_4 are four distinct finite values. Then $2f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

By Theorem 1.1 we get the following result:

Corollary 1.1. *Let f be a nonconstant entire function such that $\rho_2(f) < 1$, and let η be a nonzero complex number. Suppose that f and $\Delta_\eta f$ share a_1, a_2, a_3 IM, where a_1, a_2, a_3 are three distinct finite values. Then $2f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

Theorem 1.3. *Let f be a nonconstant meromorphic function of hyper order $\rho_2(f) < 1$, and let η be a nonzero complex number. Suppose that $f(z)$ and $f(z+\eta)$ share 0, 1, c IM, and share ∞ CM, where c is a finite value such that $c \neq 0, 1$. Then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

We also recall the following result from [12]:

Theorem C ([12, Corollary 2.2]). *Let f be a nonconstant entire function such that $\rho(f) < \infty$, and let a_1, a_2 be two distinct finite values. If $f(z)$ and $f(z+\eta)$ share a_1, a_2 CM, where η is a nonzero complex number, then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

We will prove the following result, which improves Theorem C:

Theorem 1.4. *Let f be a nonconstant entire function such that $\rho_2(f) < 1$, and let $a(z)$ and $b(z)$ be two distinct small functions of $f(z)$ such that $a(z), b(z) \not\equiv \infty$. Suppose that $f(z) - a(z)$ and $f(z+\eta) - a(z)$ share 0 IM, $f(z) - b(z)$ and $f(z+\eta) - b(z)$ share 0 IM, where η is a nonzero complex number. Then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

We give the following example:

Example 1.1 ([12]). Let $f(z) = \exp\{\sin z\}$ and $\eta = \pi$. Then $f(z)$ and $f(z+\eta)$ share 0, 1, -1, ∞ CM and $\rho_2(f) = 1$. But $f(z) \not\equiv f(z+\eta)$. This example shows that the condition "$\rho_2(f) < 1$" in Theorems 1.3 and 1.4 is best possible.

2. Preliminaries

In this section, we will give some lemmas to prove the main results of this paper, where Lemmas 2.1-2.12 play an important role in proving Theorems 1.1-1.3, Lemmas 2.13-2.15 play an important role in proving Theorem 1.4. Moreover, Lemmas 2.16-2.20 are used to prove Theorems 2.1-2.4, which can be found in this section.

Lemma 2.1 ([6, Lemma 1]). *Let f and g be two distinct nonconstant meromorphic functions, and let a_1, a_2, a_3 and a_4 be four distinct values in the extended complex plane. If f and g share a_1, a_2, a_3 and a_4 IM, then*

(i) $T(r,f) = T(r,g) + O(\log(rT(r,f)))$ *as $r \notin E$ and $r \to \infty$.*

(ii) $2T(r,f) = \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f-a_j}\right) + O(\log(rT(r,f)))$ *as $r \notin E$ and $r \to \infty$,*

where $\overline{N}(r, 1/(f-\infty))$ means $\overline{N}(r,f)$.

Lemma 2.2 ([1, Theorem 3]). *Let f and g be two nonconstant rational functions. If f and g share four distinct values a_1, a_2, a_3, a_4 IM, then $f = g$.*

Lemma 2.3 ([9, Theorem 5.1]). *Let f be a nonconstant meromorphic function and $\eta \in \mathbb{C}$. If f is of finite order, then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O\left(\frac{T(r,f)\log r}{r}\right)$$

for all r outside of a set $E \subset (1, +\infty)$ satisfying

$$\limsup_{r\longrightarrow\infty} \frac{\int_{E\cap[1,r)} dt/t}{\log r} = 0,$$

i.e., outside of a set $E \subset (1, +\infty)$ of zero logarithmic density. If $\rho_2(f) = \rho_2 < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_2-\varepsilon}}\right)$$

for all r outside of a finite logarithmic measure.

Lemma 2.4 ([6]). *Let f and g be two nonconstant meromorphic functions that share a_1, a_2, a_3 IM and a_4 CM, where a_1, a_2, a_3, a_4 are four distinct values in the extended complex plane. Suppose that there exists some real constant $\mu > 4/5$ and some set $I \subset \mathbb{R}^+$ that has infinite linear measure such that $N(r, a_4, f)/T(r,f) \geq \mu$ for all $r \in I$. Then f and g share all four values CM.*

Lemma 2.5 ([14, Lemma 7] or [22, Theorem 4.1]). *Let F and G be two distinct nonconstant meromorphic functions such that F and G share* 0, 1, c, ∞ *CM, where c is a finite value such that $c \neq 0, 1$. Then F, G satisfy one of the following* *six relations*: (i) $F+G=0$, (ii) $F+G=2$, (iii) $(F-1/2)(G-1/2)=1/4$, (iv) $FG=1$, (v) $(F-1)(G-1)=1$ *and* (vi) $F+G=1$, *where* $c\in\{-1,2,1/2\}$, $c=-1$ *in* (i) *and* (iv), $c=2$ *in* (ii) *and* (v), $c=1/2$ *in* (iii) *and* (vi).

Lemma 2.6 ([22, Theorem 1.62]). *Let* $f_1,f_2,\dots,f_n$ *be nonconstant meromorphic functions, and let* $f_{n+1}\not\equiv 0$ *be a meromorphic function such that* $\sum_{j=1}^{n+1} f_j=1$. *If there exists a subset* $I\subseteq\mathbb{R}^+$ *satisfying* $mesI=\infty$ *such that*

$$\sum_{i=1}^{n+1} N\left(r,\frac{1}{f_i}\right)+n\sum_{\substack{i=1\\ i\neq j}}^{n+1}\overline{N}(r,f_i)<(\mu+o(1))T(r,f_j),\quad j=1,2,\dots,n,$$

as $r\to\infty$ *and* $r\in I$, *where* $\mu<1$. *Then* $f_{n+1}=1$.

Lemma 2.7 ([23, Theorem 5.1]). *Let* f *and* g *be two distinct nonconstant meromorphic functions, and let* a_1, a_2, a_3, a_4, a_5 *be five distinct values in the extended complex plane. If* f *and* g *share* a_1, a_2, a_3, a_4 *IM*, then*

$$\overline{N}_0(r,a_5,f,g)\le\overline{N}_{12}(r,a_4,f,g)+S(r,f).$$

Lemma 2.8 ([5, Theorem 2.2]). *Let* f *be a meromorphic function with exponent of convergence of poles* $\lambda(1/f)=\lambda<+\infty$, *and let* $\eta\neq 0$ *be a complex number. Then, for each* $\varepsilon>0$, *we have*

$$N(r,f(z+\eta))=N(r,f(z))+O(r^{\lambda-1+\varepsilon})+O(\log r).$$

We next introduce the term ε-set (cf. [4]), which is used in the following lemma. We define an ε-set to be a countable union of discs

$$E=\bigcup_{j=1}^{\infty}B(b_j,r_j)\quad\text{such that}\qquad \lim_{j\to\infty}|b_j|=\infty\quad\text{and}\quad\sum_{j=1}^{\infty}\frac{r_j}{|b_j|}<\infty.$$

Here $B(a,r)$ denotes the open disc of center a and radius r, and $S(a,r)$ will denote the corresponding boundary circle. Note that if E is an ε-set, then the set of $r\ge 1$ for which the circle $S(0,r)$ meets E has a finite logarithmic measure and hence a zero logarithmic density.

The term ε-set was introduced in the context of the following theorem, which was proved by Hayman [10] for entire functions, and by Anderson-Clunie [2] for meromorphic functions with deficient poles.

Lemma 2.9 ([4, Lemma 3.3]). *Let* g *be a function transcendental and meromorphic in the plane of order less than 1, and let* $h>0$. *Then there exists an* ε*-set* E *such that*

$$\frac{g'(z+\eta)}{g(z+\eta)}\longrightarrow 0\quad\textit{and}\quad\frac{g(z+\eta)}{g(z)}\longrightarrow 1\quad\textit{as } z\to\infty\quad\textit{in } \mathbb{C}\setminus E,$$

uniformly in η *for* $|\eta|\le h$. *Further,* E *may be chosen so that for large* $|z|\notin E$ *the function* g *has no zeros or poles in* $|\zeta-z|\le h$.

Next we introduce Wiman-Valiron theory: Let $f = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. We define by $\mu(r) = \max\{|a_n|r^n : n = 0, 1, 2, \ldots\}$ the maximum term of f, and define by $\nu(r, f) = \max\{m : \mu(r) = |a_m|r^m\}$ the central index of f (cf. [15, pp. 187–199]).

Lemma 2.10 ([15, pp. 187–199]). *Let g be a transcendental entire function, let $0 < \delta < 1/4$ and z be such that $|z| = r$ and that $|g(z)| > M(r, g)\nu(r, g)^{-\frac{1}{4}+\delta}$ holds. Then there exists a set $E \subset (0, +\infty)$ of finite logarithmic measure, i.e., $\int_E dt/t < +\infty$, such that $g^{(m)}(z) = \left(\frac{\nu(r,g)}{z}\right)^m (1 + o(1))\, g(z)$ holds for all $m \geq 0$ and $r \notin E$.*

Lemma 2.11 ([6, Lemma 3]). *Let f and g be distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3 and a_4 IM, where $a_4 = \infty$. Then the following statements hold*:

(i) *$N_1(r, 0, f') = O(\log(rT(r, f)))$ and $N_1(r, 0, g') = O(\log(rT(r, f)))$ as $r \notin E$ and $r \to \infty$, where $N_1(r, 0, f')$ and $N_1(r, 0, g')$ "count" respectively only those points in $N(r, 0, f')$ and $N(r, 0, g')$ which do not occur when $f(z) = g(z) = a_j$ for some $j = 1, 2, 3, 4$.*

(ii) *For $j = 1, 2, 3, 4$, let $N_2(r, a_j)$ refer only to those a_j-points that are multiple for both f and g and "count" each such point the number of times of the smaller of the two multiplicities. Then $\sum_{j=1}^{4} N_2(r, a_j) = O(\log(rT(r, f)))$ as $r \notin E$ and $r \to \infty$.*

Lemma 2.12 ([6]). *Suppose that f and g are two distinct nonconstant meromorphic functions that share a_1, a_2, a_3, ∞ IM, where a_1, a_2, a_3 are three distinct finite values. Set*

$$\phi = \frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)}.$$

Then ϕ is an entire function such that $T(r, \phi) = S(r, f)$.

Lemma 2.13 ([1, Theorem 1]). *Let P_1 and P_2 be two nonconstant polynomials, and let a and b be two distinct finite values. If P_1 and P_2 share a and b IM, then $P_1 = P_2$.*

Lemma 2.14. *Let f be a nonconstant entire function such that $\rho_2(f) < 1$, and let a and b be two distinct small functions of f such that $a \not\equiv \infty$ and $b \not\equiv \infty$. Set*

$$\varphi(z) = \frac{\Theta(f(z))\{f(z) - f(z+\eta)\}}{(f(z) - a(z))(f(z) - b(z))} \tag{2.1}$$

and

$$\chi(z) = \frac{\Theta(f(z+\eta))\{f(z) - f(z+\eta)\}}{(f(z+\eta) - a(z))(f(z+\eta) - b(z))}, \tag{2.2}$$

where

$$\Theta(f(z)) = \begin{vmatrix} f(z) - a(z) & a(z) - b(z) \\ f'(z) - a'(z) & a'(z) - b'(z) \end{vmatrix}$$

$$= \begin{vmatrix} f(z)-b(z) & a(z)-b(z) \\ f'(z)-b'(z) & a'(z)-b'(z) \end{vmatrix}$$

$$= \begin{vmatrix} f(z)-a(z) & f'(z)-a'(z) \\ f(z)-b(z) & f'(z)-b'(z) \end{vmatrix}. \tag{2.3}$$

Suppose that $f(z)-a(z)$ and $f(z+\eta)-a(z)$ share 0 IM, $f(z)-b(z)$ and $f(z+\eta)-b(z)$ share 0 IM. Then $T(r,\varphi(z))+T(r,\chi(z)) = S(r,f(z))$.

Proof. By (2.3) we know that (2.1) can be rewritten respectively as

$$\varphi(z) = \left(\frac{f'(z)-b'(z)}{f(z)-b(z)} - \frac{f'(z)-a'(z)}{f(z)-a(z)}\right)\cdot\{f(z)-f(z+\eta)\} \tag{2.4}$$

$$= \frac{\Theta(f(z))\{(f(z)-a(z))+a(z)\}}{(f(z)-a(z))(f(z)-b(z))}\cdot\left(1-\frac{f(z+\eta)}{f(z)}\right)$$

$$= \varphi_1(z)\left(1-\frac{f(z+\eta)}{f(z)}\right) \tag{2.5}$$

for all $z \in \mathbb{C}$, where

$$\varphi_1(z) = \frac{\begin{vmatrix} f(z)-b(z) & a(z)-b(z) \\ f'(z)-b'(z) & a'(z)-b'(z) \end{vmatrix}}{f(z)-b(z)} + \frac{a(z)\begin{vmatrix} f(z)-a(z) & f'(z)-a'(z) \\ f(z)-b(z) & f'(z)-b'(z) \end{vmatrix}}{(f(z)-a(z))(f(z)-b(z))}. \tag{2.6}$$

Similarly, (2.2) can be rewritten respectively as

$$\chi(z) = \left(\frac{f'(z+\eta)-b'(z)}{f(z+\eta)-b(z)} - \frac{f'(z+\eta)-a'(z)}{f(z+\eta)-a(z)}\right)\cdot\{f(z)-f(z+\eta)\} \tag{2.7}$$

$$= \frac{\Theta(f(z+\eta))f(z+\eta)}{(f(z+\eta)-a(z))(f(z+\eta)-b(z))}\cdot\left(\frac{f(z)}{f(z+\eta)}-1\right)$$

$$= \chi_1(z)\left(\frac{f(z)}{f(z+\eta)}-1\right) \tag{2.8}$$

for all $z \in \mathbb{C}$, where

$$\chi_1(z) = \frac{\begin{vmatrix} f(z+\eta)-b(z) & a(z)-b(z) \\ f'(z+\eta)-b'(z) & a'(z)-b'(z) \end{vmatrix}}{f(z+\eta)-b(z)}$$

$$+ \frac{a(z)\begin{vmatrix} f(z+\eta)-a(z) & f'(z+\eta)-a'(z) \\ f(z+\eta)-b(z) & f'(z+\eta)-b'(z) \end{vmatrix}}{(f(z+\eta)-a(z))(f(z+\eta)-b(z))}. \tag{2.9}$$

Noting that $f(z)-a(z)$ and $f(z+\eta)-a(z)$ share 0 IM, $f(z)-b(z)$ and $f(z+\eta)-b(z)$ share 0 IM, we get by (2.4) and (2.7) that

$$\begin{aligned} N(r,\varphi(z))+N(r,\chi(z)) &\le 2N(r,a(z))+2N(r,b(z)) \\ &\le 2T(r,a(z))+2T(r,b(z)) \\ &= S(r,f(z)). \end{aligned} \tag{2.10}$$

By (2.5), (2.6), (2.8), (2.9), Lemma 2.3 and the lemma of logarithmic derivatives (cf. [17, Theorem 2.3.3]) we deduce

$$m(r,\varphi(z))+m(r,\chi(z))\le S(r,f(z))+S(r,f(z+\eta)). \tag{2.11}$$

By Nevanlinna's three small functions theorem (cf. [22, Theorem 1.36]) we get

$$\begin{aligned} T(r,f(z)) &\le \overline{N}\left(r,\frac{1}{f(z)-a(z)}\right)+\overline{N}\left(r,\frac{1}{f(z)-b(z)}\right)+S(r,f(z))\\ &= \overline{N}\left(r,\frac{1}{f(z+\eta)-a(z)}\right)+\overline{N}\left(r,\frac{1}{f(z+\eta)-b(z)}\right)+S(r,f(z))\\ &\le 2T(r,f(z+\eta))+S(r,f(z)). \end{aligned} \tag{2.12}$$

Similarly

$$T(r,f(z+\eta))\le 2T(r,f(z))+S(r,f(z+\eta)). \tag{2.13}$$

By (2.12) and (2.13) we deduce

$$S(r,f(z))=S(r,f(z+\eta)). \tag{2.14}$$

By (2.11) and (2.14) we get

$$m(r,\varphi(z))+m(r,\chi(z))=S(r,f(z)). \tag{2.15}$$

By (2.10) and (2.15) we get the conclusion of Lemma 2.14. □

Lemma 2.15 ([18, Proof of Theorem 2.3]). *Let f be a transcendental meromorphic solution of a difference equation of the form*

$$U(z,f)P(z,f)=Q(z,f) \tag{2.16}$$

such that $\rho_2(f)<1$, where $U(z,f)$, $P(z,f)$, $Q(z,f)$ are difference polynomials such that the total degree $\deg U(z,f)=n$ in $f(z)$ and its shifts $f(z+\eta_1),\ldots,f(z+\eta_k)$, and $\deg Q(z,f)\le n$. Moreover, assume that all coefficients b_λ in (2.16) *are small in the sense that $T(r,b_\lambda)=S(r,f)$ and that $U(z,f)$ contains exactly one term of maximal total degree in $f(z)$ and its shifts. Then we have $m(r,P(z,f))=S(r,f)$.*

Lemma 2.16 ([3, Theorem 1.1]). *Let f be a nonconstant zero order meromorphic function, and $q\in\mathbb{C}\setminus\{0\}$. Then $m\left(r,f(qz)/f(z)\right)=o(T(r,f))$ on a set of logarithmic density* 1.

Lemma 2.17 ([3, Theorem 2.1] or [18, Theorem 2.5]). *Let f be a transcendental meromorphic solution of order zero of a q-difference equation of the form $U_q(z,f)P_q(z,f)=Q_q(z,f)$, where $U_q(z,f)$, $P_q(z,f)$ and $Q_q(z,f)$ are q-difference polynomials such that the total degree $\deg U_q(z,f)=n$ in $f(z)$ and its q-shifts, where $\deg Q_q(z,f)\le n$. Moreover, we assume that $U_q(z,f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then $m(r,P_q(z,f))=o(T(r,f))$ on a set of logarithmic density* 1.

Lemma 2.18 ([25, Theorem 1.1]). *Let $f(z)$ be a nonconstant zero order meromorphic function, and let $q \in \mathbb{C} \setminus \{0\}$. Then $T(r, f(qz)) = (1 + o(1))T(r, f(z))$ on a set of lower logarithmic density* 1.

Lemma 2.19 ([25, Theorem 1.1]). *Let $f(z)$ be a nonconstant zero order meromorphic function, and let $q \in \mathbb{C} \setminus \{0\}$. Then $N(r, f(qz)) = (1 + o(1))N(r, f(z))$ on a set of lower logarithmic density* 1.

Applying Lemmas 2.16-2.19 and then proceeding as in the proof of Lemma 2.14, we can get the following result:

Lemma 2.20. *Let $f(z)$ be a nonconstant zero order entire function, and let $q \in \mathbb{C} \setminus \{0\}$, and let $a(z)$, $b(z)$ be two distinct small functions of $f(z)$ such that $a(z) \not\equiv \infty$ and $b(z) \not\equiv \infty$. Set*

$$\varphi_2(z) = \frac{\Theta(f(z))\{f(z) - f(qz)\}}{(f(z) - a(z))(f(z) - b(z))}$$

and

$$\chi_2(z) = \frac{\Theta(f(qz))\{f(z) - f(qz)\}}{(f(qz) - a(z))(f(qz) - b(z))},$$

where $\Theta(f(z))$ is defined as in (2.3). *Suppose that $f(z) - a(z)$ and $f(qz) - a(z)$ share* 0 *IM, $f(z) - b(z)$ and $f(qz) - b(z)$ share* 0 *IM. Then*

$$m(r, \varphi_2(z)) + m(r, \chi_2(z)) = o(T(r, f))$$

on a set of logarithmic density 1.

Applying Lemmas 2.16-2.20, we can prove the following uniqueness results of zero order meromorphic functions sharing four values with their q-difference operators or q-shifts, the proofs are just the same as the proofs of Theorems 1.1-1.4:

Theorem 2.1. *Let $f(z)$ be a nonconstant zero order meromorphic function, and let $q \in \mathbb{C} \setminus \{0\}$. Suppose that $f(z)$ and $f(qz) - f(z)$ share* 0, 1, *c IM, and share ∞ CM, where c is a complex number such that $c \neq 0, 1, \infty$. Then $2f(z) = f(qz)$ for all $z \in \mathbb{C}$.*

Theorem 2.2. *Let $f(z)$ be a nonconstant zero order meromorphic function, and let $q \in \mathbb{C} \setminus \{0\}$. Suppose that $f(z)$ and $f(qz) - f(z)$ share a_1, a_2, a_3, a_4 IM, where a_1, a_2, a_3, a_4 are four distinct finite values. Then $2f(z) = f(qz)$ for all $z \in \mathbb{C}$.*

Theorem 2.3. *Let $f(z)$ be a nonconstant zero order meromorphic function, and let $q \in \mathbb{C} \setminus \{0\}$. Suppose that $f(z)$ and $f(qz)$ share* 0, 1, *c IM, and share ∞ CM, where c is a finite value such that $c \neq 0, 1$. Then $f(z) = f(qz)$ for all $z \in \mathbb{C}$.*

Theorem 2.4. *Let f be a nonconstant zero order entire function, let $q \in \mathbb{C} \setminus \{0\}$, and let a and b be two distinct small functions of f such that $a, b \not\equiv \infty$. Suppose that $f(z) - a(z)$ and $f(qz) - a(z)$ share* 0 *IM, $f(z) - b(z)$ and $f(qz) - b(z)$ share* 0 *IM. Then $f(z) = f(qz)$ for all $z \in \mathbb{C}$.*

3. Proof of theorems

Proof of Theorem 1.1. We first suppose that f is a transcendental meromorphic function, while $\Delta_\eta f$ is a rational function. Then, by Lemma 2.1 we get

$$T(r,f(z))=T(r,\Delta_\eta f(z))+O(\log(rT(r,f(z))))=O(\log(rT(r,f(z))))$$

as $r\to\infty$ and $r\notin E$, this implies that $f(z)$ is a rational function, which is impossible. Secondly we suppose that $f(z)$ and $\Delta_\eta f(z)$ are nonconstant rational functions. Then, by Lemma 2.2 we get the conclusion of Theorem 1.1. Finally we suppose that $f(z)$ and $\Delta_\eta f(z)$ are transcendental meromorphic functions and suppose that $f\not\equiv\Delta_\eta f$.

Let $z_\infty\in\mathbb{C}$ be a pole of $f(z)$. Then, by the condition that $f(z)$ and $\Delta_\eta f(z)$ share ∞ CM we know that either $f(z+\eta)$ is analytic at z_∞ or z_∞ is a pole of $f(z+\eta)$ such that the multiplicity of z_∞ related to $f(z+\eta)$ is not greater than the multiplicity of z_∞ related to $f(z)$. Combining Lemmas 2.1 and 2.3, and the condition that $f(z)$ and $\Delta_\eta f(z)$ share a_1, a_2, a_3 IM, we get

$$\begin{aligned}
2T(r,f(z))&=\overline{N}\left(r,\frac{1}{f(z)-a_1}\right)+\overline{N}\left(r,\frac{1}{f(z)-a_2}\right)+\overline{N}\left(r,\frac{1}{f(z)-a_3}\right)\\
&\quad+\overline{N}\left(r,f(z)\right)+S(r,f(z))\\
&\le\overline{N}\left(r,\frac{1}{f(z+\eta)-2f(z)}\right)+\overline{N}\left(r,f(z)\right)+S(r,f(z))\\
&\le T(r,f(z+\eta)-2f(z))+\overline{N}\left(r,f(z)\right)+S(r,f(z))\\
&=m(r,f(z+\eta)-2f(z))+N(r,f(z+\eta)-2f(z))+\overline{N}\left(r,f(z)\right)\\
&\quad+S(r,f(z))\\
&\le m(r,f(z))+m\left(r,\frac{f(z+\eta)}{f(z)}-2\right)+N(r,f(z))+\overline{N}\left(r,f(z)\right)\\
&\quad+S(r,f(z))\\
&\le T(r,f(z))+N(r,f(z))+S(r,f(z)),
\end{aligned}$$

i.e.,

$$T(r,f(z))=N(r,f(z))+S(r,f(z)). \tag{3.1}$$

By (3.1) we know that there exists a subset $I\subset\mathbb{R}^+$ with logarithmic measure $\log mesI=+\infty$ such that

$$\lim_{\substack{r\to\infty\\ r\in I}}\frac{N(r,f(z))}{T(r,f(z))}=1. \tag{3.2}$$

By (3.2) and Lemma 2.4 we know that $f(z)$ and $\Delta_\eta f(z)$ share a_1, a_2, a_3, ∞ CM, and so $F(z)$ and $G(z)$ share 0, 1, c, ∞ CM, where

$$F(z)=\frac{f(z)-a_1}{a_2-a_1},\quad G(z)=\frac{\Delta_\eta f(z)-a_1}{a_2-a_1} \tag{3.3}$$

and

$$c = \frac{a_3 - a_1}{a_2 - a_1}. \tag{3.4}$$

By Lemma 2.5 we know that $F(z)$, $G(z)$ satisfy one of the six relations of Lemma 2.5. We discuss the following six cases:

Case 1. Suppose that $F(z)$ and $G(z)$ satisfy (i) of Lemma 2.5. Then $F(z) + G(z) = 0$ for all $z \in \mathbb{C}$, and so we get by (3.3) that $f(z + \eta) = 2a_1$ for all $z \in \mathbb{C}$, which is impossible.

Case 2. Suppose that $F(z)$ and $G(z)$ satisfy (ii) of Lemma 2.5. Then $F(z) + G(z) = 2$ for all $z \in \mathbb{C}$, and so we get by (3.3) that $f(z + \eta) = 2a_2$ for all $z \in \mathbb{C}$, which is impossible.

Case 3. Suppose that $F(z)$ and $G(z)$ satisfy (iii) of Lemma 2.5. Then $1/2$, ∞ are Picard exceptional values of $F(z)$ and $G(z)$. Hence

$$F(z) = \frac{1 + e^{\gamma_1(z)}}{2}, \quad G(z) = \frac{1 + e^{-\gamma_1(z)}}{2} \tag{3.5}$$

for all $z \in \mathbb{C}$, where γ_1 is a nonconstant entire function. By substituting (3.3) into (3.5) we get

$$e^{\gamma_1(z+\eta)} - e^{\gamma_1(z)} - e^{-\gamma_1(z)} = \frac{a_2 + a_1}{a_2 - a_1} \tag{3.6}$$

for all $z \in \mathbb{C}$. Noting that γ_1 is a non-constant entire function, we can get by (3.6) and Lemma 2.6 that $(a_2+a_1)/(a_2-a_1) = 0$, and so (3.6) can be rewritten as

$$e^{\gamma_1(z+\eta)+\gamma_1(z)} - e^{2\gamma_1(z)} = 1 \tag{3.7}$$

for all $z \in \mathbb{C}$. By (3.7) and Lemma 2.6 we can get a contradiction.

Case 4. Suppose that $F(z)$ and $G(z)$ satisfy (iv) of Lemma 2.5. Then 0, ∞ are Picard exceptional values of $F(z)$ and $G(z)$. Hence

$$F(z) = e^{\gamma_2(z)}, \quad G(z) = e^{-\gamma_2(z)}, \tag{3.8}$$

where γ_2 is a nonconstant entire function. By (3.8) we get

$$e^{\gamma_2(z+\eta)} - e^{\gamma_2(z)} - e^{-\gamma_2(z)} = \frac{a_1}{a_2 - a_1} \tag{3.9}$$

for all $z \in \mathbb{C}$. Noting that γ_2 is a nonconstant entire function, we can get by (3.9) and Lemma 2.6 that $a_1/(a_2 - a_1) = 0$, and so (3.9) can be rewritten as

$$e^{\gamma_2(z+\eta)+\gamma_2(z)} - e^{2\gamma_2(z)} = 1 \tag{3.10}$$

for all $z \in \mathbb{C}$. By (3.10) and Lemma 2.6 we can get a contradiction.

Case 5. Suppose that $F(z)$ and $G(z)$ satisfy (v) of Lemma 2.5. Then

$$F(z) = 1 + e^{\gamma_3(z)}, \quad G(z) = 1 + e^{-\gamma_3(z)}, \tag{3.11}$$

where γ_3 is a nonconstant entire function. Then, by (3.3) and (3.11) we get

$$e^{\gamma_3(z+\eta)} - e^{\gamma_3(z)} - e^{-\gamma_3(z)} = \frac{a_2}{a_2 - a_1} \tag{3.12}$$

for all $z \in \mathbb{C}$. Proceeding as in Case 3, we can get a contradiction by (3.12).

Case 6. Suppose that $F(z)$ and $G(z)$ satisfy (vi) of Lemma 2.5. Then $F(z)+G(z) = 1$ for all $z \in \mathbb{C}$. This together with (3.3) gives $f(z+\eta) = a_1 + a_2$, which is impossible. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. First of all, we suppose that $f(z)$ is a transcendental meromorphic function, while $\Delta_\eta f(z)$ is a rational function. Then, by Lemma 2.1 we get

$$T(r, f(z)) = T(r, \Delta_\eta f(z)) + O(\log(rT(r, f(z)))) = O(\log(rT(r, f(z)))),$$

which implies that $f(z)$ is a rational function, which is impossible. Secondly we suppose that $f(z)$ and $\Delta_\eta f(z)$ are nonconstant rational functions. Then it follows by Lemma 2.2 that $f = \Delta_\eta f$, which reveals the conclusion of Theorem 1.2. Finally we suppose that f and $\Delta_\eta f$ are transcendental meromorphic functions such that $f \not\equiv \Delta_\eta f$. Combining this with Lemma 2.7 and the condition that $f(z)$ and $\Delta_\eta f(z)$ share a_1, a_2, a_3, a_4 IM, we have

$$\overline{N}_0(r, \infty, f, \Delta_\eta f) \le \overline{N}_{12}(r, a_4, f, \Delta_\eta f) + S(r, f(z)) = S(r, f(z)). \tag{3.13}$$

Noting that $\Delta_\eta f(z) = f(z+\eta) - f(z)$, we get by (3.13) that

$$\begin{aligned} \overline{N}(r, |f(z) = \infty, f(z+\eta) \ne \infty) &+ \overline{N}(r, |f(z) = f(z+\eta) = \Delta_\eta f(z) = \infty) \\ &\le \overline{N}_0(r, \infty, f, \Delta_\eta f) \\ &= S(r, f(z)), \end{aligned} \tag{3.14}$$

where $\overline{N}(r, |f(z) = f(z+\eta) = \Delta_\eta f(z) = \infty)$ denotes the reduced counting function of common poles of $f(z)$, $f(z+\eta)$ and $\Delta_\eta f(z)$ in $\{z : |z| < r\}$, $\overline{N}(r, |f(z) = \infty, f(z+\eta) \ne \infty)$ denotes the reduced counting function of those points in $\overline{N}(r, f(z))$, which are not poles of $f(z+\eta)$. Therefore, by (3.14) we get

$$\begin{aligned} \overline{N}(r, \Delta_\eta f(z)) &= \overline{N}(r, |f(z) = \infty, f(z+\eta) \ne \infty) \\ &\quad + \overline{N}(r, |f(z+\eta) = \infty, f(z) \ne \infty) \\ &\quad + \overline{N}(r, |f(z) = f(z+\eta) = \Delta_\eta f(z) = \infty) \\ &= \overline{N}(r, |f(z+\eta) = \infty, f(z) \ne \infty) + S(r, f(z)) \\ &\le \overline{N}(r, f(z+\eta)) + S(r, f(z)), \end{aligned} \tag{3.15}$$

where $\overline{N}(r, |f(z) = f(z+\eta) = \infty)$ denotes the reduced counting function of common poles of $f(z)$ and $f(z+\eta)$ in $|z| < r$, $\overline{N}(r, |f(z+\eta) = \infty, f(z) \ne \infty)$

denotes the reduced counting function of those points in $\overline{N}(r, f(z+\eta))$, which are not poles of $f(z)$. Similarly

$$\overline{N}(r, f(z+\eta)-2f(z)) \leq \overline{N}(r, f(z+\eta)) + S(r, f(z)). \tag{3.16}$$

By Lemma 2.1 and the condition that $f(z)$, $\Delta_\eta f(z)$ share a_1, a_2, a_3, a_4 IM we get

$$S(r, f(z)) = S(r, \Delta_\eta f(z)). \tag{3.17}$$

By Lemma 2.1 and the second fundamental theorem we get

$$\begin{aligned} 3T(r, f(z)) &\leq \overline{N}(r, f(z)) + \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f(z)-a_j}\right) + S(r, f(z)) \\ &= \overline{N}(r, f(z)) + 2T(r, f(z)) + S(r, f(z)), \end{aligned}$$

which implies that

$$T(r, f(z)) = N(r, f(z)) + S(r, f(z)) = \overline{N}(r, f(z)) + S(r, f(z)), \tag{3.18}$$

which implies that

$$N_{1)}(r, f(z)) = N(r, f(z)) + S(r, f(z)) = T(r, f(z)) + S(r, f(z)). \tag{3.19}$$

Similarly, we can get

$$T(r, \Delta_\eta f(z)) = N(r, \Delta_\eta f(z)) + S(r, f(z)) = \overline{N}(r, \Delta_\eta f(z)) + S(r, f(z)). \tag{3.20}$$

and

$$N_{1)}(r, \Delta_\eta f(z)) = N(r, \Delta_\eta f(z)) + S(r, f(z)) = T(r, \Delta_\eta f(z)) + S(r, f(z)). \tag{3.21}$$

By (3.19) and (3.21) we get

$$\begin{aligned} N_{(2}(r, f(z+\eta)) &= N_{(2}(r, \Delta_\eta f(z)) + S(r, f(z)) \\ &\leq N(r, \Delta_\eta f(z)) - N_{1)}(r, \Delta_\eta f(z)) \\ &\leq S(r, f(z)). \end{aligned} \tag{3.22}$$

By (3.22) we get

$$\begin{aligned} N(r, f(z+\eta)) &= N_{1)}(r, f(z+\eta)) + N_{(2}(r, f(z+\eta)) \\ &= N_{1)}(r, f(z+\eta)) + S(r, f(z)). \end{aligned} \tag{3.23}$$

By (3.14), (3.15), (3.16), (3.17), (3.19), (3.21), (3.23), Lemma 2.3, Lemma 2.8, the second fundamental theorem and the condition that $f(z)$, $\Delta_\eta f(z)$ share a_1, a_2, a_3, a_4 IM we get

$$\begin{aligned} 3T(r, \Delta_\eta f(z)) &\leq \overline{N}\left(r, \Delta_\eta f(z)\right) + \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{\Delta_\eta f(z)-a_j}\right) + S(r, \Delta_\eta f(z)) \\ &\leq \overline{N}(r, f(z+\eta)) + \overline{N}\left(r, \frac{1}{\Delta_\eta f(z)-f(z)}\right) + S(r, f(z)) \\ &\leq \overline{N}(r, f(z+\eta)) + T\left(r, f(z+\eta)-2f(z)\right) + S(r, f(z)) \end{aligned}$$

$$
\begin{aligned}
&= \overline{N}(r, f(z+\eta)) + N\left(r, f(z+\eta) - 2f(z)\right) \\
&\quad + m\left(r, f(z+\eta) - 2f(z)\right) + S(r, f(z)) \\
&\le \overline{N}(r, f(z+\eta)) + \overline{N}\left(r, f(z+\eta) - 2f(z)\right) + m\left(r, f(z)\right) \\
&\quad + m\left(r, \frac{f(z+\eta)}{f(z)} - 2\right) + S(r, f(z)) \\
&\le 2\overline{N}(r, f(z+\eta)) + m\left(r, f(z)\right) + S(r, f(z)) \\
&\le 2N\left(r, f(z+\eta)\right) + m(r, f(z)) + S(r, f(z)) \\
&= 2N(r, f(z)) + m(r, f(z)) + O(r^{\rho(f)-1}) + O(\log r) + S(r, f(z)) \\
&\le 2T(r, f(z)) + O(r^{\rho(f)-1}) + O(\log r) + S(r, f(z)),
\end{aligned}
$$

i.e.,

$$3T(r, \Delta_\eta f(z)) \le 2T(r, f(z)) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r) + S(r, f(z)). \tag{3.24}$$

By Lemma 2.1 and the condition that $f(z)$, $\Delta_\eta f(z)$ share a_1, a_2, a_3, a_4 IM we get

$$T(r, \Delta_\eta f(z)) = T(r, f(z)) + S(r, f(z)). \tag{3.25}$$

By (3.24) and (3.25) we get

$$T(r, f(z)) \le O(r^{\rho(f)-1+\varepsilon}) + O(\log r) + S(r, f(z)). \tag{3.26}$$

Noting that $S(r, f(z)) = o(T(r, f(z)))$ as $r \to \infty$ and $r \notin E$, where $E \subset \mathbb{R}^+$ is some subset of a finite logarithmic measure, we get

$$T(r, f(z)) = O(r^{\rho(f)-1+\varepsilon}) + O(\log r) \tag{3.27}$$

as $r \to \infty$ and $r \notin E$. By (3.27), the supposition that $f(z)$ is a transcendental meromorphic function we deduce $\rho(f) \ge 1$. This together with the standard reasoning of removing exceptional set (cf. [17, Lemma 1.1.2]) gives

$$T(r, f(z)) = O((2r)^{\rho(f)-1+\varepsilon}) + O(\log r + \log 2) \tag{3.28}$$

as $r \to \infty$. By (3.28) we get $\rho(f) \le \rho(f) - 1$, which is impossible. This proves Theorem 1.2. □

Proof of Theorem 1.3. First of all, let

$$\psi(z) = \frac{\phi_1(z)}{\phi_1(z+\eta)} \tag{3.29}$$

for all $z \in \mathbb{C}$, where

$$\phi_1(z) = \frac{f'(z)}{f(z)(f(z)-1)(f(z)-c)} \tag{3.30}$$

for all $z \in \mathbb{C}$. Then it follows by (3.29) and (3.30) that

$$\psi(z) = \frac{f(z+\eta)(f(z+\eta)-1)(f(z+\eta)-c)f'(z)}{f(z)(f(z)-1)(f(z)-c)f'(z+\eta)} \tag{3.31}$$

for all $z \in \mathbb{C}$. By Lemma 2.11(ii) and the assumptions of Theorem 1.3 we get

$$N(r,\psi) + N\left(r,\frac{1}{\psi}\right) = S(r,f(z)). \tag{3.32}$$

Noting that $\rho_2(f) = \rho_2(f') < 1$, by (3.29)-(3.32) and Lemma 2.3 we get

$$m(r,\psi) = S(r,f(z)). \tag{3.33}$$

By (3.32) and (3.33) we get

$$T(r,\psi) = S(r,f(z)). \tag{3.34}$$

By rewriting (3.31) we get

$$\frac{f'(z)}{f(z)(f(z)-1)(f(z)-c)} = \psi(z)\cdot\frac{f'(z+\eta)}{f(z+\eta)(f(z+\eta)-1)(f(z+\eta)-c)} \tag{3.35}$$

for all $z \in \mathbb{C}$. Next we suppose that $f(z) \not\equiv f(z+\eta)$ and let $a_1 = 0$, $a_2 = 1$, $a_3 = c$, $a_4 = \infty$. By Lemma 2.11(ii), we can see that

$$\sum_{n\geq 2, m\geq 2} \overline{N}_{(n,m)}(r,a_j) = S(r,f(z)).$$

By this equality we discuss the following four cases:

Case 1. Suppose that there exists some a_j $(1 \leq j \leq 3)$ such that

$$\overline{N}_{(1,1)}(r,a_j) \neq S(r,f(z)), \tag{3.36}$$

where $\overline{N}_{(1,1)}(r,a_j)$ denotes the reduced counting function of common simple a_j-points of $f(z)$ and $f(z+\eta)$ in $|z| < r$. By (3.36) we know that at least one of $\overline{N}_{(1,1)}(r,0)$, $\overline{N}_{(1,1)}(r,1)$ and $\overline{N}_{(1,1)}(r,c)$, say $\overline{N}_{(1,1)}(r,0)$, satisfies $\overline{N}_{(1,1)}(r,0) \neq S(r,f(z))$. Let $z_0 \in \{z : |z| < r\}$ be a common simple 1-point of $f(z)$ and $f(z+\eta)$. Then, by the Laurent expansions of the left side and the right side of (3.35) in a punctured disk about z_0 we deduce $\psi(z_0) = 1$, this together with (3.34) and $\overline{N}_{(1,1)}(r,0) \neq S(r,f(z))$ implies that $\psi = 1$, and so (3.35) can be rewritten as

$$\frac{f'(z)}{f(z)(f(z)-1)(f(z)-c)} = \frac{f'(z+\eta)}{f(z+\eta)(f(z+\eta)-1)(f(z+\eta)-c)} \tag{3.37}$$

for all $z \in \mathbb{C}$. By (3.37) and the condition that $f(z)$ and $f(z+\eta)$ share 0, 1, c IM we deduce that $f(z)$ and $f(z+\eta)$ share 0, 1, c CM. Combining Lemma 2.5, we consider the following two subcases:

Subcase 1.1. Suppose that $f(z)$ and $f(z+\eta)$ satisfy one of $f(z)+f(z+\eta) = 0$, $f(z)+f(z+\eta) = 2$ and $f(z)+f(z+\eta) = 1$, say

$$f(z) + f(z+\eta) = 0 \tag{3.38}$$

for all $z \in \mathbb{C}$, where $c = -1$. Then 1 and -1 are Picard exceptional values of $f(z)$ and $f(z+\eta)$. Hence

$$f(z) - 1 = (f(z)+1)e^{\gamma_3(z)} \tag{3.39}$$

for all $z \in \mathbb{C}$, where γ_3 is a nonconstant entire function. By (3.39) we have

$$f(z) = \frac{1 + e^{\gamma_3(z)}}{1 - e^{\gamma_3(z)}} \tag{3.40}$$

for all $z \in \mathbb{C}$. By substituting (3.40) into (3.38) we have $e^{\gamma_3(z)+\gamma_3(z+\eta)} = 1$ for all $z \in \mathbb{C}$. Hence

$$\gamma_3(z) + \gamma_3(z+\eta) = c \tag{3.41}$$

for all $z \in \mathbb{C}$, where c is some finite complex constant. On the other hand, by (3.40) we have $\rho_2(f) = \rho_2(e^{\gamma_3}) = \rho(\gamma_3) < 1$. This implies that γ_3 is a nonconstant entire function of order $\rho(\gamma_3) < 1$. Therefore, by Lemma 2.9 we know that for a positive number h satisfying $|\eta| < h$, there exists an ε-set E_1, such that as $z \notin E_1$ and $|z| \to \infty$, we have

$$\frac{\gamma_3(z+\eta)}{\gamma_3(z)} \longrightarrow 1, \tag{3.42}$$

uniformly in η. Next we let E_r be the set of $r \geq 1$ for which the circle $S(0,r)$ meets E_1. Then, E_r has a finite logarithmic measure (cf. [10]). Therefore, by (3.42) and Lemma 2.10 we know that there exist some infinite sequence of points $z_{r_k} = r_k e^{i\theta_k}$, where and in what follows, $r_k \notin E_r \cup E$ and $\theta_k \in [0, 2\pi)$, $E \subset (1, +\infty)$ is a subset of finite logarithmic measure, i.e., $\int_E dt/t < +\infty$, such that

$$|\gamma_3(z_{r_k})| = M(r_k, \gamma_3) \longrightarrow \infty \quad \text{and} \quad \frac{\gamma_3(z_{r_k}+\eta)}{\gamma_3(z_{r_k})} \longrightarrow 1 \tag{3.43}$$

as $r_k \to \infty$. By (3.41) and (3.43) we have

$$2 = 1 + \lim_{r_k \to \infty} \frac{\gamma_3(z_{r_k}+\eta)}{\gamma_3(z_{r_k})} = \lim_{r_k \to \infty} \frac{c}{\gamma_3(z_{r_k})} = 0, \tag{3.44}$$

which is a contradiction.

Subcase 1.2. Suppose that $f(z)$ and $f(z+\eta)$ satisfy one of $f(z)f(z+\eta) = 1$, $(f(z) - 1/2)(f(z+\eta) - 1/2) = 1/4$ and $(f(z) - 1)(f(z+\eta) - 1) = 1$, say

$$(f(z) - 1/2)(f(z+\eta) - 1/2) = 1/4 \tag{3.45}$$

for all $z \in \mathbb{C}$, where $c = 1/2$. Then ∞ and $1/2$ are Picard exceptional values of $f(z)$ and $f(z+\eta)$. Hence $f(z) = e^{\gamma_4(z)}/2 + 1/2$ for all $z \in \mathbb{C}$, where γ_4 is a nonconstant entire function. This together with (3.45) gives $e^{\gamma_4(z)+\gamma_4(z+\eta)} = 1$ for all $z \in \mathbb{C}$. Next in the same manner as in Subcase 1.1 we can get a contradiction.

Case 2. Suppose that there exists some a_j $(1 \leq j \leq 3)$ such that

$$\overline{N}_{(1,m)}(r, a_j) \neq S(r, f(z)), \tag{3.46}$$

where $m \geq 2$ is a positive integer, $\overline{N}_{(1,m)}(r, a_j)$ denotes the reduced counting function of those zeros of $f(z) - a_j$ with multiplicity 1, and of $f(z+\eta) - a_j$

with multiplicity m. Then, by (3.34), (3.35) and (3.46) we deduce $\psi = 1/m$, and so (3.35) can be rewritten as

$$\frac{mf'(z)}{f(z)(f(z)-1)(f(z)-c)} = \frac{f'(z+\eta)}{f(z+\eta)(f(z+\eta)-1)(f(z+\eta)-c)} \tag{3.47}$$

for all $z \in \mathbb{C}$. Set

$$\phi_2(z) = \frac{f(z)'f'(z+\eta)(f(z)-f(z+\eta))^2}{f(z)(f(z)-1)(f(z)-c)f(z+\eta)(f(z+\eta)-1)(f(z+\eta)-c)}. \tag{3.48}$$

Then, by (3.48) and Lemma 2.12 we know that $\phi_2(z) \not\equiv 0$ is an entire function such that

$$T(r, \phi_2(z)) = S(r, f(z)). \tag{3.49}$$

By (3.47) and (3.48) we get

$$(f(z)-c)^2 = \frac{m}{\phi_2(z)} \cdot \left(\frac{f(z+\eta)-1}{f(z)-1} - 1\right)^2 \cdot \left(\frac{f'(z)}{f(z)}\right)^2. \tag{3.50}$$

By (3.50), Lemma 2.3 and the lemma of logarithmic derivatives (cf. [17, Theorem 2.3.3]) we get

$$\begin{aligned}
&2m(r, f(z)) + O(1) \\
&= m(r, (f(z)-c)^2) \\
&\le m\left(r, \frac{m}{\phi_2(z)}\right) + 2m\left(r, \frac{f(z+\eta)-1}{f(z)-1} - 1\right) + 2m\left(r, \frac{f'(z)}{f(z)}\right) \\
&\le T(r, \phi_2(z)) + S(r, f(z)) \le S(r, f(z)),
\end{aligned}$$

i.e., $m(r, f(z)) = S(r, f(z))$, which implies that there exists some subset $I \subset \mathbb{R}^+$ with its linear measure $mesI = +\infty$ such that

$$\lim_{\substack{r\to\infty \\ r\in I}} \frac{N(r, f(z))}{T(r, f(z))} = 1. \tag{3.51}$$

By (3.51) and Lemma 2.4 we know that $f(z)$ and $f(z+\eta)$ share 0, 1, c, ∞ CM. Next in the same manner as in Case 1 we can get a contradiction.

Case 3. Suppose that there exists some a_j $(1 \le j \le 3)$ such that

$$\overline{N}_{(m,1)}(r, a_j) \ne S(r, f(z)), \tag{3.52}$$

where $m \ge 2$ is a positive integer, $\overline{N}_{(m,1)}(r, a_j)$ denotes the reduced counting function of those zeros of $f(z)-a_j$ with multiplicity m, and of $f(z+\eta)-a$ with multiplicity 1. Next in the same manner as in Case 2 we can get a contradiction.

Case 4. Suppose that

$$\sum_{j=1}^{3} \overline{N}_{(1,1)}(r, a_j) + \sum_{j=1}^{3} \overline{N}_{(m,1)}(r, a_j) + \sum_{j=1}^{3} \overline{N}_{(1,m)}(r, a_j) = S(r, f(z)), \tag{3.53}$$

where $m \geq 1$ is any positive integer. Then, by (3.53) we get

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{f(z)}\right) &= \sum_{n\geq 1}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,0) \\
&\leq \sum_{n\leq 15}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,0) + \sum_{n\geq 1}\sum_{m\leq 15}\overline{N}_{(n,m)}(r,0) \\
&\quad + \sum_{n\geq 16}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,0) + \sum_{n\geq 1}\sum_{m\geq 16}\overline{N}_{(n,m)}(r,0) \\
&= 2\sum_{n\leq 15}\sum_{m\leq 15}\overline{N}_{(n,m)}(r,0) + \sum_{n\leq 15}\sum_{m\geq 16}\overline{N}_{(n,m)}(r,0) \\
&\quad + \sum_{n\geq 16}\sum_{m\leq 15}\overline{N}_{(n,m)}(r,0) + \sum_{n\geq 16}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,0) \\
&\quad + \sum_{n\geq 1}\sum_{m\geq 16}\overline{N}_{(n,m)}(r,0) \\
&\leq \frac{2}{16}\left(N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+\eta)}\right)\right) + S(r, f(z)) \\
&\leq \frac{1}{8}T(r, f(z)) + \frac{1}{8}T(r, f(z+\eta)) + S(r, f(z)). \qquad (3.54)
\end{aligned}$$

Similarly, by (3.53) we get

$$\overline{N}\left(r, \frac{1}{f(z)-1}\right) \leq \frac{1}{8}T(r, f(z)) + \frac{1}{8}T(r, f(z+\eta)) + S(r, f(z)) \qquad (3.55)$$

and

$$\overline{N}\left(r, \frac{1}{f(z)-c}\right) \leq \frac{1}{8}T(r, f(z)) + \frac{1}{8}T(r, f(z+\eta)) + S(r, f(z)). \qquad (3.56)$$

By Lemma 2.1 we get

$$\begin{aligned}
2T(r, f(z)) &= \overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-1}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{f(z)-c}\right) + S(r, f(z)). \qquad (3.57)
\end{aligned}$$

By (3.54)-(3.57) we get

$$2T(r, f(z)) \leq T(r, f(z)) + \frac{3}{8}T(r, f(z)) + \frac{3}{8}T(r, f(z+\eta)) + S(r, f(z)), \qquad (3.58)$$

i.e.,

$$5T(r, f(z)) \leq 3T(r, f(z+\eta)) + S(r, f(z)). \qquad (3.59)$$

Similarly

$$5T(r, f(z+\eta)) \leq 3T(r, f(z)) + S(r, f(z)),$$

this together with (3.59) gives

$$2T(r, f(z)) + 2T(r, f(z+\eta)) \leq S(r, f(z)),$$

which is impossible. This complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. Suppose that $f(z)$ and $f(z+\eta)$ are nonconstant polynomials. Then a and b are two distinct complex number, this together with Lemma 2.13 reveals the conclusion of Theorem 1.4. Next we suppose that $f(z)$ and $f(z+\eta)$ are transcendental entire functions and $f(z) \not\equiv f(z+\eta)$. Set (2.1), (2.2) and (2.3). Then we have (2.4) and (2.7). By Lemma 2.14 we have

$$T(r, \varphi(z)) = S(r, f(z)) \tag{3.60}$$

and

$$T(r, \chi(z)) = S(r, f(z)). \tag{3.61}$$

Set

$$H_{(m,n)}(z) = m\varphi(z) - n\chi(z), \tag{3.62}$$

where m and n are some two positive integers. We discuss the following two cases:

Case 1. Suppose that there exist some two integers m_0 and n_0 such that $H_{(m_0,n_0)} = 0$. This together with (2.4), (2.7) and (3.62) gives

$$\begin{aligned} &m_0\left(\frac{f'(z)-a'(z)}{f(z)-a(z)} - \frac{f'(z)-b'(z)}{f(z)-b(z)}\right) \\ &= n_0\left(\frac{f'(z+\eta-a'(z)}{f(z+\eta)-a(z)} - \frac{f'(z+\eta)-b'(z)}{f(z+\eta)-b(z)}\right) \end{aligned}$$

for all $z \in \mathbb{C}$, which implies that

$$\left(\frac{f(z)-b(z)}{f(z)-a(z)}\right)^{m_0} = A_0\left(\frac{f(z+\eta)-b(z)}{f(z+\eta)-a(z)}\right)^{n_0} \tag{3.63}$$

for all $z \in \mathbb{C}$, where A_0 is a nonzero constant. By (3.63) and the standard Valiron-Mokhon'ko lemma (cf. [20]) we deduce

$$m_0 T(r, f(z)) = n_0 T(r, f(z+\eta)) + S(r, f(z)). \tag{3.64}$$

By (3.64) we have

$$S(r, f(z)) = S(r, f(z+\eta)). \tag{3.65}$$

By Lemma 2.3, the assumptions of Theorem 1.4 and Nevanlinna's three small functions theorem we get

$$\begin{aligned} T(r, f(z)) &\le \overline{N}\left(r, \frac{1}{f(z)-a(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-b(z)}\right) + S(r, f(z)) \\ &\le N\left(r, \frac{1}{f(z)-f(z+\eta)}\right) + S(r, f(z)) \\ &\le T(r, f(z)-f(z+\eta)) + S(r, f(z)) \\ &\le m(r, f(z+\eta)) + m\left(r, \frac{f(z)}{f(z+\eta)} - 1\right) + S(r, f(z)) \end{aligned}$$

$$\leq T(r, f(z+\eta)) + S(r, f(z)),$$

i.e.,

$$T(r, f(z)) \leq T(r, f(z+\eta)) + S(r, f(z)). \tag{3.66}$$

Similarly,

$$T(r, f(z+\eta)) \leq T(r, f(z)) + S(r, f(z)) + S(r, f(z+\eta)). \tag{3.67}$$

By (3.65)-(3.67) we get

$$T(r, f(z)) = T(r, f(z+\eta)) + S(r, f(z)). \tag{3.68}$$

By (3.64) and (3.68) we get $m_0 = n_0$, and so it follows by (3.63) that

$$\frac{f(z) - b(z)}{f(z) - a(z)} = A_1 \cdot \frac{f(z+\eta) - b(z)}{f(z+\eta) - a(z)}, \tag{3.69}$$

where A_1 is a nonzero constant satisfying $A_1^{m_0} = A_0$. By (3.69) and the supposition $f(z) \not\equiv f(z+\eta)$ we have $A_1 \neq 1$, and so (3.69) can be rewritten as

$$\begin{aligned} &f(z)((A_1 - 1)f(z+\eta) + a(z) - A_1 b(z)) \\ &= (A_1 a(z) - b(z))f(z+\eta) + (1 - A_1)a(z)b(z). \end{aligned} \tag{3.70}$$

By (3.70) and Lemma 2.15 we get

$$m(r, (A_1 - 1)f(z+\eta) + a(z) - A_1 b(z)) = S(r, f(z)). \tag{3.71}$$

By (3.68) and (3.71) we get

$$T(r, f(z)) = S(r, f(z)), \tag{3.72}$$

which is impossible.

Case 2. Suppose that for any two positive integers m and n, we have

$$H_{(m,n)}(z) \not\equiv 0. \tag{3.73}$$

Let z_a be a zero of $f(z) - a(z)$ with multiplicity n, and a zero of $f(z+\eta) - a(z)$ with multiplicity m, such that $a(z_a) \neq \infty$, $b(z_a) \neq \infty$ and $a(z_a) - b(z_a) \neq 0$. Then, by (2.4), (2.7), (3.62) and by a calculating we can get $H_{(m,n)}(z_a) = 0$. Similarly, if z_b be a zero of $f(z) - b(z)$ with multiplicity n, and a zero of $f(z+\eta) - b(z)$ with multiplicity m such that $a(z_b) \neq \infty$, $b(z_b) \neq \infty$ and $a(z_b) - b(z_b) \neq 0$, then $H_{(m,n)}(z_b) = 0$. This together with (3.60), (3.61) and (3.73) gives

$$\begin{aligned} \overline{N}_{(n,m)}(r, a) + \overline{N}_{(n,m)}(r, b) &\leq N\left(r, \frac{1}{H_{(m,n)}(z)}\right) + N(r, a(z)) + N(r, b(z)) \\ &\quad + N\left(r, \frac{1}{a(z) - b(z)}\right) \\ &\leq T(r, H_{(m,n)}(z)) + S(r, f(z)) \\ &\leq T(r, \varphi(z)) + T(r, \chi(z)) + S(r, f(z)) \end{aligned}$$

$$= S(r, f(z)), \tag{3.74}$$

where and in what follows, $\overline{N}_{(n,m)}(r,a)$ ($\overline{N}_{(n,m)}(r,b)$, respectively) denotes the reduced counting function of those zeros of $f(z)-a(z)$ ($f(z)-b(z)$, respectively) with multiplicity n, and of $f(z+\eta)-a(z)$ ($f(z+\eta)-b(z)$, respectively) with multiplicity m. By (3.74) we can get

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{f(z)-a(z)}\right) &= \sum_{n\geq 1}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,a) \\
&\leq \sum_{n\leq 15}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,a) + \sum_{n\geq 1}\sum_{m\leq 15}\overline{N}_{(n,m)}(r,a) \\
&\quad + \sum_{n\geq 16}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,a) + \sum_{n\geq 1}\sum_{m\geq 16}\overline{N}_{(n,m)}(r,a) \\
&= 2\sum_{n\leq 15}\sum_{m\leq 15}\overline{N}_{(n,m)}(r,a) + \sum_{n\leq 15}\sum_{m\geq 16}\overline{N}_{(n,m)}(r,a) \\
&\quad + \sum_{n\geq 16}\sum_{m\leq 15}\overline{N}_{(n,m)}(r,a) + \sum_{n\geq 16}\sum_{m\geq 1}\overline{N}_{(n,m)}(r,a) \\
&\quad + \sum_{n\geq 1}\sum_{m\geq 16}\overline{N}_{(n,m)}(r,a) \\
&\leq \frac{2}{16}\left(N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+\eta)}\right)\right) + S(r, f(z)) \\
&\leq \frac{1}{8}T(r, f(z)) + \frac{1}{8}T(r, f(z+\eta)) + S(r, f(z)).
\end{aligned} \tag{3.75}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{f(z)-b(z)}\right) \leq \frac{1}{8}T(r, f(z)) + \frac{1}{8}T(r, f(z+\eta)) + S(r, f(z)). \tag{3.76}$$

Noting that f is a nonconstant entire function of hyper-order $\rho_2(f) < 1$, we can get by (3.75), (3.76), Lemma 2.3 and Nevanlinna's three small functions theorem that

$$\begin{aligned}
T(r, f(z)) &\leq \overline{N}\left(r, \frac{1}{f(z)-a(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-b(z)}\right) + S(r, f(z)) \\
&\leq \frac{1}{4}T(r, f(z)) + \frac{1}{4}T(r, f(z+\eta)) + S(r, f(z)) \\
&\leq \frac{1}{4}T(r, f(z)) + \frac{1}{4}m(r, f(z)) + \frac{1}{4}m\left(r, \frac{f(z+\eta)}{f(z)}\right) + S(r, f(z)) \\
&\leq \frac{1}{2}T(r, f(z)) + o\left(\frac{T(r,f)}{r^{1-\rho_2-\varepsilon}}\right) + S(r, f(z)) \\
&\leq \frac{1}{2}T(r, f(z)) + S(r, f(z)),
\end{aligned}$$

i.e., $T(r, f(z)) = S(r, f(z))$, for all r outside of a finite logarithmic measure, which is impossible. This completes the proof of Theorem 1.4. □

4. Concluding remarks

Regarding Theorems 1.1 and 1.2, we give the following two conjectures:

Conjecture 4.1. If the condition "$\rho_2(f) < 1$" of Theorem 1.1 is replaced with "$\rho(f) = \infty$", then the conclusion of Theorem 1.1 still holds.

Conjecture 4.2. If the condition "$\rho(f) < \infty$" of Theorem 1.2 is replaced with "$\rho(f) = \infty$", then the conclusion of Theorem 1.2 still holds.

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