Bull. Korean Math. Soc. **53** (2016), No. 4, pp. 1197–1211
http://dx.doi.org/10.4134/BKMS.b150601
pISSN: 1015-8634 / eISSN: 2234-3016

A REFINEMENT OF THE UNIT AND UNITARY CAYLEY GRAPHS OF A FINITE RING

Ali Reza Naghipour and Meysam Rezagholibeigi

Abstract. Let R be a finite commutative ring with nonzero identity. We define $\Gamma(R)$ to be the graph with vertex set R in which two distinct vertices x and y are adjacent if and only if there exists a unit element u of R such that $x + uy$ is a unit of R. This graph provides a refinement of the unit and unitary Cayley graphs. In this paper, basic properties of $\Gamma(R)$ are obtained and the vertex connectivity and the edge connectivity of $\Gamma(R)$ are given. Finally, by a constructive way, we determine when the graph $\Gamma(R)$ is Hamiltonian. As a consequence, we show that $\Gamma(R)$ has a perfect matching if and only if $|R|$ is an even number.

1. Introduction

Throughout this paper, R is a finite commutative ring with nonzero identity. The group of units and the Jacobson radical of R are denoted by $U(R)$ and $J(R)$, respectively. The *unit graph* $G(R)$ is the graph with vertex set R in which two distinct vertices x and y are adjacent if and only if $x + y \in U(R)$. Unit graphs were introduced in [2] and their properties were investigated in [7], [16], [17] and [19]. The *unitary Cayley graph* G_R is the graph with vertex set R such that two distinct vertices x and y are adjacent if and only if $x - y \in U(R)$. Unitary Cayley graphs were introduced in [8] and their properties were investigated in [1], [10], [11], [12] and [15]. For example, in [10] the chromatic number, clique number and independence number of G_R are given along with other results. The authors in [15] give a necessary and sufficient condition for G_R to be Ramanujan graph.

In [9], Khashayarmanesh and Khorsandi provide a generalization of the unit and unitary Cayley graphs as follows: Let G be a multiplicative subgroup of $U(R)$ and S be a non-empty subset of G such that $S^{-1} = \{s^{-1} \mid s \in S\} \subseteq S$. Then $\Gamma(R, G, S)$ is the (simple) graph with vertex set R in which two distinct elements $x, y \in R$ are adjacent if and only if there exists $s \in S$ such that $x+sy \in G$. The authors in [3] derive several bounds for the genus of $\Gamma(R, U(R), S)$. In

Received July 27, 2015; Revised November 11, 2015.
2010 *Mathematics Subject Classification.* 05C25, 05C40, 05C75, 13M05, 16U60.
Key words and phrases. Cayley graphs, Clique number, Hamiltonian graphs, finite rings, matchings, unit graphs.

this paper, we use $\Gamma(R)$ to denote the graph $\Gamma(R, U(R), U(R))$. For a subset C of R, the induced subgraph of $\Gamma(R)$ over C is denoted by $\Gamma(C)$.

We recall that a ring R is said to have *unit 1-stable range* if, whenever $Rx + Ry = R$ $(x, y \in R)$, there exists $u \in U(R)$ such that $x + uy \in U(R)$. We refer the reader to [6] and [13] for more information about unit 1-stable range rings.

In [18], Sharma and Bhatwadekar defined another graph on R, $\Omega(R)$, with vertices the elements of R, in which two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. It is easy to see that $\Gamma(R)$ is a subgraph of $\Omega(R)$. The concepts of $\Gamma(R)$ and $\Omega(R)$ give an interesting graph interpretation of unit 1-stable range rings. In fact, a commutative ring R has unit 1-stable range if and only if $\Gamma(R) \cong \Omega(R)$. This provides a motivation to introduce and study the properties of $\Gamma(R)$.

For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A graph G is called a *refinement* of a graph H if $V(G) = V(H)$ and if x, y are adjacent in H, then x, y are adjacent in G. We mention that "G is a refinement of H" has the same meaning as "H is a spanning subgraph of G". We note that $\Gamma(R)$ is a refinement of both $G(R)$ and G_R. If we omit the word "distinct", we obtain the graph $\overline{\Gamma}(R)$; this graph may have loops. Some examples of this kind of graphs are displayed in Figure 1.

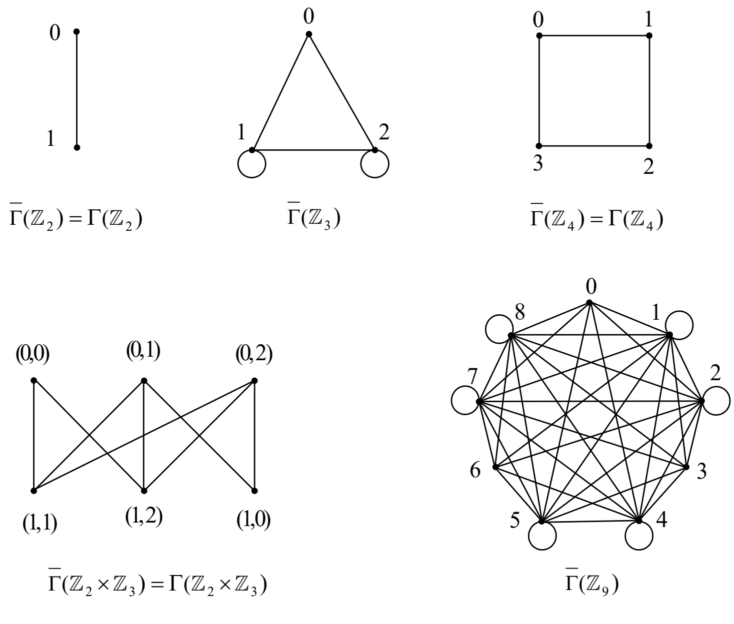

FIGURE 1. The graphs $\Gamma(R)$ and $\overline{\Gamma}(R)$ of the specific rings R.

For a local ring R, we have the following immediate result about the loops of $\overline{\Gamma}(R)$.

Proposition 1.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Then*

(1) *If $|R/\mathfrak{m}| = 2$, then $\overline{\Gamma}(R)$ has no loop (i.e., $\Gamma(R) = \overline{\Gamma}(R)$);*
(2) *If $|R/\mathfrak{m}| \neq 2$, then only the elements of $U(R)$ have a loop in $\overline{\Gamma}(R)$.*

A graph G in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We use K_n to denote the complete graph with n vertices. For a graph G and vertex $x \in V(G)$, the *degree* of x, denoted by $\deg(x)$, is the number of edges of G incident with x. The minimum degree of G is denoted by $\delta(G)$. For $x \in V(G)$, we denote by $N_G(x)$ the set of all vertices of G adjacent to x.

A graph G is called *bipartite* if $V(G)$ admits a partition into two classes such that vertices in the same partition class must not be adjacent. A simple bipartite graph in which every two vertices from different partition classes are adjacent is called a *complete bipartite graph*, denoted by $K_{m,n}$, where m and n are the sizes of the partition classes. A *clique* is a set of pairwise adjacent vertices of G (any complete subgraph). The largest integer n such that K_n is a subgraph of G is the *clique number* $\omega(G)$ of G. An *independent set* is a set of pairwise non-adjacent vertices of G. A *walk* from x to y is an ordered list of vertices (not necessarily distinct) $x = v_0, v_1, \ldots, v_{n-1}, v_n = y$ such that v_{i-1} is adjacent to v_i for $i = 1, \ldots, n$. We denote this walk by x—v_1— $\cdots$ —v_{n-1}—y. A *path* of length n is an ordered list of distinct vertices $v_0, v_1, \ldots, v_{n-1}, v_n$ such that v_{i-1} is adjacent to v_i for $i = 1, \ldots, n$. We denote this path by v_0—v_1— $\cdots$ —v_{n-1}—v_n. A *cycle* is a path v_0—v_1— $\cdots$ —v_{n-1}—v_n with an extra edge v_0—v_n. The *union* of two simple graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $V(G)$ and $V(H)$ are disjoint, we refer to $G \cup H$ as a *disjoint union*, and denote it by $G + H$. The *join* of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding edges joining every vertex of G to every vertex of H.

A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and G is called a *Hamiltonian graph* if it contains a Hamiltonian cycle. For other notions not mentioned in this paper, one can refer to [4] and [20].

The plan of this paper is as follows: In Section 2, we give some basic properties of $\Gamma(R)$. In Section 3, we determine the clique number of $\Gamma(R)$. In Section 4, by a constructive way, we determine when the graph $\Gamma(R)$ is Hamiltonian. Finally, we determine when the graph $\Gamma(R)$ has a perfect matching.

2. Basic properties of $\Gamma(R)$

In this section, we study some basic properties of $\Gamma(R)$. We begin with the following lemma.

Lemma 2.1. *Let R be a ring. Then each element of $U(R)$ is adjacent to all elements of $J(R)$.*

Proof. Let $x \in U(R)$ and $y \in J(R)$. Suppose on the contrary that x and y are not adjacent. Then $x + uy \notin U(R)$ for all $u \in U(R)$, and so $x - y \notin U(R)$. Therefore there exists a maximal ideal $\mathfrak{m}$ of R such that $x - y \in \mathfrak{m}$. This implies that $x \in \mathfrak{m}$, which is a contradiction. This completes the proof. □

Let R be a ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$. Then it is easy to see that $\Gamma(R)$ is a bipartite graph. In the next section, we show that the converse of this result is also true (see Corollary 3.2).

In the following theorem, we determine when $\Gamma(R)$ is a complete bipartite graph.

Theorem 2.2. *Let R be a ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$. Then $\Gamma(R)$ is a complete bipartite graph if and only if R is a local ring.*

Proof. Suppose that $\Gamma(R)$ is a complete bipartite graph with bipartition $\{V_1, V_2\}$. First we show that $U(R)$ is an independent set of $\Gamma(R)$. Suppose on the contrary that $U(R)$ is not an independent set of $\Gamma(R)$. Then there exist $x, y \in U(R)$ such that x is adjacent to y. So, there exists $u \in U(R)$ such that $x + uy \in U(R)$. Since $|R/\mathfrak{m}| = 2$, there are $m_1, m_2 \in \mathfrak{m}$ such that $x = 1 + m_1$ and $y = 1 + m_2$. This implies that $1 + m_1 + u + um_2 \in U(R)$. On the other hand, $1 + u \in \mathfrak{m}$, because $|R/\mathfrak{m}| = 2$. Therefore we have $1 + u + m_1 + um_2 \in \mathfrak{m}$, which is a contradiction. Since $\Gamma(R)$ is a complete bipartite graph and $U(R)$ is an independent set of $\Gamma(R)$, without loss of generality, we may assume that $U(R) \subseteq V_1$. We claim that $V_1 = U(R)$. Suppose on the contrary that there exists $v_1 \in V_1 \backslash U(R)$. Then there exists a maximal ideal $\mathfrak{n}$ of R such that $v_1 \in \mathfrak{n}$. Since the distinct elements of a maximal ideal can not be adjacent, $\mathfrak{n} \subseteq V_1$ and so $J(R) \subseteq \mathfrak{n} \subseteq V_1$, which is a contradiction, by the above lemma. Therefore, $V_1 = U(R)$. It follows that $\mathfrak{m} \subseteq V_2$. Now we show that $V_2 = \mathfrak{m}$. Suppose on the contrary that there exists $v_2 \in V_2 \setminus \mathfrak{m}$. Then $v_2 = 1 + m$ for some $m \in \mathfrak{m}$. By the assumption, 1 is adjacent to v_2, and hence there exists $u_0 \in U(R)$ such that $(1 + m) + u_0.1 = 1 + m + u_0 \in U(R)$. Hence $1 + m + u_0 = 1 + m_0$ for some $m_0 \in \mathfrak{m}$. Therefore $u_0 = m_0 - m$, which is a contradiction. Thus $V_2 = \mathfrak{m}$. It follows that R is a local ring.

The converse follows easily from [9, Propostion 3.2]. □

If R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$, then by the above theorem $\deg(x) = |U(R)|$ for each $x \in R$. In the case where $|R/\mathfrak{m}| > 2$, the following theorem determines the degree of vertices of $\Gamma(R)$.

Theorem 2.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| > 2$ and let $x \in R$. Then*

$$\deg(x) = \begin{cases} |R| - 1 & \text{if } x \in U(R), \\ |U(R)| & \text{otherwise.} \end{cases}$$

Proof. Let $\mathfrak{m}, u_1+\mathfrak{m}, \ldots, u_t+\mathfrak{m}$, be the set of all distinct cosets of $R/\mathfrak{m}$, where $u_i \in U(R)$ for $i = 1, \ldots, t$. Let $x_i \in u_i + \mathfrak{m}$ and $x_j \in u_j + \mathfrak{m}$, where i, j are two distinct elements of $\{1, \ldots, t\}$. We claim that x_i and x_j are adjacent. Suppose on the contrary that x_i and x_j are not adjacent. Therefore, $u_i + uu_j \in \mathfrak{m}$ for all $u \in U(R)$ and so $u_i - u_j \in \mathfrak{m}$, which is a contradiction. Now let $k \in \{1, \ldots, t\}$. We show that every pair of elements of the coset $u_k + \mathfrak{m}$ are adjacent. Suppose on the contrary that there exist two distinct elements $m_1, m_2 \in \mathfrak{m}$ such that $u_k + m_1$ and $u_k + m_2$ are not adjacent. Then $(u_k + m_1) + u(u_k + m_2) \in \mathfrak{m}$ for all $u \in U(R)$. We conclude that $u_k(1 + u) \in \mathfrak{m}$ for all $u \in U(R)$ and so $1 - u \in \mathfrak{m}$ for all $u \in U(R)$. This implies that $|R/\mathfrak{m}| = 2$, which is a contradiction. It is clear that the elements of $u_i + \mathfrak{m}$ are adjacent to the elements of $\mathfrak{m}$, for all $i = 1, \ldots, t$ and also no pair of elements of $\mathfrak{m}$ are adjacent. These observations complete the proof. □

Theorem 2.4. *Let R be a ring. Suppose that $\Gamma(R)$ is a complete n-partite graph. Then the following hold*:

(1) *R is a local ring*;
(2) *$n = 2$ or $n = |U(R)| + 1$.*

Proof. (1) Suppose that V is the part containing zero. We show that $V = R \setminus U(R)$. For any $x \in V$ and any $u \in U(R)$, we have $ux \notin U(R)$. Therefore $V \subseteq R \setminus U(R)$. Now let y be an element of $R \setminus U(R)$ such that $y \notin V$. So y is adjacent to zero and hence $uy \in U(R)$, for some $u \in U(R)$. This yields $y \in U(R)$, which is a contradiction. Hence $V = R \setminus U(R)$. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be two distinct maximal ideals of R. Then $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ and hence $x + y = 1$ for some $x \in \mathfrak{m}_1$ and $y \in \mathfrak{m}_2$. Therefore x and y are adjacent elements of V, which is a contradiction. This implies that R is a local ring.

(2) First suppose that $|R/\mathfrak{m}| = 2$. Then $n = 2$, by Theorem 2.2. Now let $|R/\mathfrak{m}| > 2$ and $U(R) = \{u_1, \ldots, u_t\}$. For any $1 \leq i \leq t$, we set $V_i = \{u_i\}$ and $V_{t+1} = \mathfrak{m}$. Therefore $\Gamma(R)$ is a complete $(t+1)$-partite graph by Theorem 2.3. This completes the proof. □

Theorem 2.5. *Let R be a ring, with exactly two maximal ideal, say $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then $\Gamma(R)$ is connected if and only if $|R/\mathfrak{m}_1| \neq 2$ or $|R/\mathfrak{m}_2| \neq 2$.*

Proof. Suppose that $\Gamma(R)$ is not connected. In view of Lemma 2.1 and the fact that every element of $(\mathfrak{m}_1 \setminus \mathfrak{m}_2)$ is adjacent to every element of $(\mathfrak{m}_2 \setminus \mathfrak{m}_1)$, there are two components V_1 and V_2 of $\Gamma(R)$ such that $V_1 = J(R) \cup U(R)$ and $V_2 = (\mathfrak{m}_1 \setminus \mathfrak{m}_2) \cup (\mathfrak{m}_2 \setminus \mathfrak{m}_1)$. We show that $|R/\mathfrak{m}_1| = 2$. Suppose on the contrary that $|R/\mathfrak{m}_1| \neq 2$. So there exists $x \in R \setminus \mathfrak{m}_1$ such that $1 - x \notin \mathfrak{m}_1$. Then $1 - x \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$ or $1 - x \in U(R)$. First suppose that $1 - x \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. So $x \notin \mathfrak{m}_2$. Therefore $x \in U(R) \subseteq V_1$ and $1 - x \in V_2$, which is a contradiction.

Now suppose that $1 - x \in U(R)$. Then $x \notin \mathfrak{m}_2 \setminus \mathfrak{m}_1$, for otherwise 1 is adjacent to x, which is a contradiction. Hence $x \in U(R)$. Since $R/\mathfrak{m}_1$ is a field, there is $v \in R \setminus \mathfrak{m}_1$ such that $1 - vx \in \mathfrak{m}_1$. We consider the following four cases:

Case 1: $1 - vx \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $v \in U(R)$. In this case, we have $vx + (1 - vx) \in U(R)$, which is a contradiction.
Case 2: $1 - vx \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $v \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. It follows that $1 - v \notin U(R) \cup \mathfrak{m}_2$, and hence $1 - v \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$. Now we conclude that $1 - vx - 1 + v \in \mathfrak{m}_1$ and therefore $v(1 - x) \in \mathfrak{m}_1$. Since $1 - x$ is unit, we must have $v \in \mathfrak{m}_1$, which is a contradiction.
Case 3: $1 - vx \in J(R)$ and $v \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. Then it is clear that $vx \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. But we have $1 - vx + vx \in U(R)$, which is a contradiction.
Case 4: $1-vx \in J(R)$ and $v \in U(R)$. Let a be an arbitrary element of $\mathfrak{m}_1 \setminus \mathfrak{m}_2$. Then we have $a(1-x)+vx \notin U(R)$, since $a(1-x)$ is not adjacent to v. Also if $a(1 - x) + vx \in \mathfrak{m}_1$, then we conclude that $vx \in \mathfrak{m}_1$, which is a contradiction, and therefore $a(1 - x) + vx \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. Now according to the assumption that $1 - vx \in J(R)$, we have

$$1 + a - ax \in \mathfrak{m}_2 \setminus \mathfrak{m}_1. \tag{2.1}$$

Since 1 is not adjacent to a, we have $1 - ax \notin U(R)$. Also if $1 - ax \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$, we conclude that $1 \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$, which is a contradiction. So $1 - ax \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. By (2.1), we obtain $a \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$, which is a contradiction. Hence the first assumption is not true and therefore $|R/\mathfrak{m}_1| = 2$. A similar argument shows that $|R/\mathfrak{m}_2| = 2$.

Conversely, let $|R/\mathfrak{m}_1| = |R/\mathfrak{m}_2| = 2$. It is enough to show that every element of $U(R)$ is not connected to elements of $(\mathfrak{m}_1 \setminus \mathfrak{m}_2) \cup (\mathfrak{m}_2 \setminus \mathfrak{m}_1)$. Let $z \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and u be an arbitrary element of $U(R)$. Suppose on the contrary that u is adjacent to z. Then $u + vz \in U(R)$ for some $v \in U(R)$. Since $|R/\mathfrak{m}_1| = |R/\mathfrak{m}_2| = 2$, we have $1-u-vz \in \mathfrak{m}_1 \cap \mathfrak{m}_2$. Also, since $|R/\mathfrak{m}_2| = 2$, we have $1-u \in \mathfrak{m}_2$. Hence $vz \in \mathfrak{m}_2$ and therefore $z \in \mathfrak{m}_2$, which is a contradiction. A similar argument shows that every element of $U(R)$ is not connected to elements of $\mathfrak{m}_2 \setminus \mathfrak{m}_1$. This completes the proof. □

Corollary 2.6. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a ring such that R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Then $\Gamma(R)$ is connected if and only if $R/J(R)$ has at most one $\mathbb{Z}_2$ as a summand.*

Proof. Suppose that $R/J(R)$ has at least two $\mathbb{Z}_2$ as summands. Without loss of generality, we may assume $|R_1/\mathfrak{m}_1| = |R_2/\mathfrak{m}_2| = 2$. Let $S := R_1 \times R_2$. By the above theorem $\Gamma(S)$ is disconnected and therefore it is easy to see that $\Gamma(R)$ is disconnected. Conversely, suppose that $R/J(R)$ has at most one $\mathbb{Z}_2$ as a summand. Let $(u_1, \ldots, u_n) \in U(R)$, $m_1 \in \mathfrak{m}_1$ and let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be arbitrary vertices of $\Gamma(R)$. Put $M := (m_1, u_2, \ldots, u_n)$ and $U := (u_1, \ldots, u_n)$ such that $U \notin \{X, Y\}$. We consider the following two cases:
Case 1: $|R_i/\mathfrak{m}_i| > 2$ for all $1 \le i \le n$. Then, by Theorem 2.3, X—U—Y is a path between X and Y. So $\Gamma(R)$ is connected in this case.
Case 2: $|R_1/\mathfrak{m}_1| = 2$ and $|R_i/\mathfrak{m}_i| > 2$ for all $2 \le i \le n$. First suppose that $x_1, y_1 \in \mathfrak{m}_1$. Then X—U—Y is a path from X to Y. If $x_1, y_1 \in U(R_1)$, then we have the path X—M—Y from X to Y. Now, suppose that $x_1 \in \mathfrak{m}_1$ and

$y_1 \in U(R_1)$. In this case X—U—M—Y is a path from X to Y. If $x_1 \in U(R_1)$ and $y_1 \in \mathfrak{m}_1$, a similar argument shows that X is connected to Y. Therefore $\Gamma(R)$ is connected. □

3. Clique number

The purpose of this section is to determine the clique number of $\Gamma(R)$.

Theorem 3.1. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a ring, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Then*

$$\omega(\Gamma(R)) = \begin{cases} 2 & \text{if } |R_i/\mathfrak{m}_i| = 2 \text{ for some } 1 \le i \le n, \\ |U(R)| + n & \text{otherwise.} \end{cases}$$

Proof. Let $|R_i/\mathfrak{m}_i| = 2$ for some $1 \le i \le n$. Then $M := R_1 \times \cdots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \cdots \times R_n$ is a maximal ideal of R such that $|R/M| = 2$. Therefore the remark before Theorem 2.2 implies that $\omega(\Gamma(R)) = 2$.

Now suppose that $|R_i/\mathfrak{m}_i| > 2$ for all $1 \le i \le n$. We set:

$$S_i := U(R_1) \times U(R_2) \times \cdots \times U(R_{i-1}) \times \mathfrak{m}_i \times R_{i+1} \times \cdots \times R_n, (1 \le i \le n),$$
$$S_{n+1} := U(R_1) \times U(R_2) \times \cdots \times U(R_n).$$

It is easy to see that $S_i \bigcap S_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{i=n+1} S_i = R$. By Theorem 2.3 and Proposition 1.1, S_{n+1} is a clique. Set

$$C := S_{n+1} \cup \{(0, 1, 1, \ldots, 1), (1, 0, 1, \ldots, 1), (1, 1, \ldots, 1, 0)\}.$$

It is easy to see that C is a clique of $\Gamma(R)$. Since $S_i (1 \le i \le n)$ is an independent set, every clique of $\Gamma(R)$ contains at most one element of S_i $(1 \le i \le n)$. Therefore $\omega(\Gamma(R)) = |U(R_1)| \times |U(R_2)| \times \cdots \times |U(R_n)| + n = |U(R)| + n$. □

Corollary 3.2. *Let R be a ring such that $\Gamma(R)$ is a bipartite graph. Then there is a maximal ideal $\mathfrak{m}$ of R such that $|R/\mathfrak{m}| = 2$.*

Proof. Let $R = R_1 \times R_2 \times \cdots \times R_n$ such that R_i is a local ring with maximal ideal $\mathfrak{m}_i$ for $1 \le i \le n$ (see [4, Theorem 8.7]). Suppose on the contrary that for all ideals of R, we have $|R/\mathfrak{m}| > 2$. Equivalently, $|R_i/\mathfrak{m}_i| > 2$ for all $1 \le i \le n$. In view of Theorem 3.1, we conclude that

$$|U(R_1)| \times |U(R_2)| \times \cdots \times |U(R_n)| + n = 2.$$

So we have $n = 1$ (i.e., $R = R_1$) and $|U(R_1)| = 1$. Suppose that $|R| > 2$. Let x be an element of R such that $x \notin \{0, 1\}$. Then $1 + x \notin U(R)$ and $x \notin U(R)$. So $1 = (1 + x) - x \in \mathfrak{m}$, which is a contradiction. Therefore $|R| = 2$ and hence $R = \mathbb{Z}_2$, which is again a contradiction. This completes the proof. □

4. Connectivity

In the following, we use $\kappa(G)$ and $\kappa'(G)$ to denote the vertex-connectivity and edge-connectivity of a graph G, respectively. The *local connectivity* between distinct vertices x and y is the maximum number of pairwise internally disjoint xy-paths, denoted by $p(x, y)$ (see [5, Page 206]). We begin with the following notation:

Notation. Let $S = R_1\times\cdots\times R_n$, $T = R_{n+1}\times\cdots\times R_m$ and $R = S\times T$ such that R_i is ring for all $1 \leq i \leq m$. Suppose that $X = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m) \in R$, $\widehat{X} = (x_1, x_2, \ldots, x_n) \in S$, $\widehat{Y} = (x_{n+1}, \ldots, x_m) \in T$. For convenience, we let X denote one of the following expressions:

$$(\widehat{X}, \widehat{Y}),$$
$$(\widehat{X}, x_{n+1}, \ldots, x_m),$$
$$(x_1, x_2, \ldots, x_n, \widehat{Y}).$$

Theorem 4.1. *Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a ring such that F_i is field. If $\Gamma(R)$ is connected, then $\kappa(\Gamma(R)) = \kappa'(\Gamma(R)) = \delta(\Gamma(R)) = |U(R)|$.*

Proof. Since $\Gamma(R)$ is connected, by Corollary 2.6, we have the following cases:
Case 1: $|F_i| > 2$ for all $1 \leq i \leq n$. We decompose R to the subsets S_i, as defined in Theorem 3.1. Set $S := S_1 \cup S_2 \cup \cdots \cup S_n$. It is easy to see that $\Gamma(R) \cong \Gamma(S_{n+1}) \vee \Gamma(S)$. The vertex $(0, 0, \ldots, 0) \in S_1 \subseteq S$ is an isolated vertex in $\Gamma(S)$ and therefore $\kappa(S) = 0$. Also we know that $\Gamma(S_{n+1}) \cong K_{|U(R)|}$ and hence $\kappa(\Gamma(S_{n+1})) = |U(R)| - 1$. On the other hand, it is clear that $\delta(\Gamma(R)) = \deg((0, 0, \ldots, 0)) = |U(R)|$. By using [5, Exercises 9.1.2, 9.3.2], we conclude that $\kappa(\Gamma(R)) = \kappa'(\Gamma(R)) = \delta(\Gamma(R)) = |U(R)|$. The assertion is proved.
Case 2: $|F_1| = 2$ and $|F_i| > 2$ for all $2 \leq i \leq n$. Let $X := (x, x_2, \ldots, x_n)$ and $Y := (y, y_2, \ldots, y_n)$ be arbitrary distinct elements of R. Let $\widehat{X} := (x_2, \ldots, x_n) \in F_2 \times \cdots \times F_n$ and $\widehat{Y} := (y_2, \ldots, y_n) \in F_2 \times \cdots \times F_n$. We consider the following four subcases:
Subcase 1. No entries of $\widehat{X}$ and $\widehat{Y}$ are equal to zero. Thus, $\widehat{X}$ and $\widehat{Y}$ are adjacent in $\Gamma(F_2 \times \cdots \times F_n)$. Also for each $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\widehat{X}, \widehat{Y}\}$, $\widehat{X}$—A—$\widehat{Y}$ is a path of length two between $\widehat{X}$ and $\widehat{Y}$. The number of such distinct A is $(f_2 - 1)\cdots(f_n - 1) - 2$. Now we consider the following two cases: If $x = y$, we choose $t \in \mathbb{Z}_2 \setminus \{x\}$ and construct the following pairwise internally disjoint paths from X to Y:

$$X = (x, \widehat{X})—(t, A)—Y = (x, \widehat{Y}),$$
$$X = (x, \widehat{X})—(t, \widehat{X})—Y = (x, \widehat{Y}),$$
$$X = (x, \widehat{X})—(t, \widehat{Y})—Y = (x, \widehat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\widehat{X}, \widehat{Y}\}$.

If $x \neq y$, we have the following pairwise internally disjoint paths:

$$X = (x, \widehat{X}) — Y = (y, \widehat{Y}),$$
$$X = (x, \widehat{X}) — (y, A) — (x, A) — Y = (y, \widehat{Y}),$$
$$X = (x, \widehat{X}) — (y, \widehat{X}) — (x, \widehat{Y}) — Y = (y, \widehat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\widehat{X}, \widehat{Y}\}$.

Hence, in this case, $p(X, Y) \geq (f_2 - 1) \cdots (f_n - 1) - 2 + 2 = |U(R)| = \delta(\Gamma(R))$.

Subcase 2. Both $\widehat{X}$ and $\widehat{Y}$ have at least one entry which is equal to zero. Then for any $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\})$, $\widehat{X} — A — \widehat{Y}$ is a path from $\widehat{X}$ to $\widehat{Y}$ in $\Gamma(F_2 \times \cdots \times F_n)$. The number of such distinct A, and therefore such paths, is $(f_2 - 1) \cdots (f_n - 1)$. We consider the following two cases:

If $x = y$, we construct the following paths from X to Y:

$$X = (x, \widehat{X}) — (t, A) — Y = (x, \widehat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\})$, $t \in \mathbb{Z}_2 \setminus \{x\}$.

If $x \neq y$, we provide the following internally disjoint paths:

$$X = (x, \widehat{X}) — (y, A) — (x, A) — Y = (y, \widehat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\})$.

In this case we also deduce that $p(X, Y) \geq (f_2 - 1) \cdots (f_n - 1) = |U(R)| = \delta(\Gamma(R))$.

Subcase 3. No entry of $\widehat{X}$ is equal to zero and at least one entry of $\widehat{Y}$ is zero. Hence for any $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\widehat{X}\}$, $\widehat{X} — A — \widehat{Y}$ is a path from $\widehat{X}$ to $\widehat{Y}$. Note that $\widehat{X}$ has loop and also $\widehat{X}$ is adjacent to $\widehat{Y}$. The number of such A is $(f_2 - 1) \cdots (f_n - 1) - 1$. We consider the following two cases:

If $x = y$, we provide the following paths from X to Y:

$$X = (x, \widehat{X}) — (t, A) — Y = (x, \widehat{Y}),$$
$$X = (x, \widehat{X}) — (t, \widehat{X}) — Y = (x, \widehat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\widehat{X}\}$.

If $x \neq y$, we have the following paths from X to Y:

$$X = (x, \widehat{X}) — (y, A) — (x, A) — Y = (y, \widehat{Y}),$$
$$X = (x, \widehat{X}) — (y, \widehat{X}) — (x, \widehat{Y}) — Y = (y, \widehat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\widehat{X}\}$.

Therefore, $p(X, Y) \geq (f_2 - 1) \cdots (f_n - 1) - 1 + 1 = |U(R)| = \delta(\Gamma(R))$.

Subcase 4. No entry of $\widehat{Y}$ is equal to zero and at least one entry of $\widehat{X}$ is zero. This subcase is similar to the previous subcase and so we omit the argument. Hence, for every $X, Y \in R$, we have $p(X, Y) \geq |U(R)| = \delta(\Gamma(R))$. This implies that $\kappa(\Gamma(R)) = \delta(\Gamma(R))$. This completes the proof. □

Let G be a connected graph. A non-empty subset S of vertices of G is called a *vertex cut* if $G - S$ (the removal of vertices of S from G) is not connected or has exactly one vertex. We note that by Menger's Theorem, for a finite connected graph G, $\kappa(G)$ is equal to the minimum size of vertex cuts of G (see [20, Theorem 4.2.21]).

Theorem 4.2. *Let R be a ring. Then*

$$\kappa(\Gamma(R)) = \kappa(\Gamma(R/J(R))|J(R)|.$$

Proof. Let $\kappa(\Gamma(R/J(R)) = t$ and $\{b_1 + J(R), b_2 + J(R), \ldots, b_t + J(R)\}$ be a vertex cut of $\Gamma(R/J(R))$. Then, by [14, Proposition 4.8], it is not hard to see that $\bigcup_{i=1}^{i=t} b_i + J(R)$ is a vertex cut of $\Gamma(R)$. Therefore $\kappa(\Gamma(R)) \leq \kappa(\Gamma(R/J(R))|J(R)|$.

Let $\kappa(\Gamma(R)) = n$ and C be a vertex cut of $\Gamma(R)$ such that $|C| = n$. We claim that $C = \bigcup_{i=1}^{i=m} a_i + J(R)$ for some $a_i \in R$. Let $a + j \in C$, where $a \in R$ and $j \in J(R)$. We show that $a + J(R) \subseteq C$. Suppose on the contrary that $a + j_0 \notin C$ for some $j_0 \in J(R)$. Since C is a vertex cut, there are $x, y \in R$ such that x is not connected to y in $\Gamma(R)\setminus C$. On the other hand, $\Gamma(R)\setminus(C\setminus\{a+j\})$ is a connected graph. So we have the following walk in $\Gamma(R) \setminus (C \setminus \{a + j\})$:

$$x = x_1 — x_2 — \cdots — x_{i-1} — (a + j) — x_i — \cdots — x_n = y,$$

where $x_i \in G \setminus C$. Since $a + j_0 \notin C$ and $N_{\Gamma(R)}(a + j) = N_{\Gamma(R)}(a + j_0)$, we have the following walk in $\Gamma(R) \setminus C$:

$$x = x_1 — x_2 — \cdots — x_{i-1} — (a + j_0) — x_i — \cdots — x_n = y,$$

which is a contradiction. Therefore $C = \bigcup_{i=1}^{m} a_i + J(R)$ for some $a_i \in R$ and hence $n = m|J(R)|$. By [14, Proposition 4.8], it is easy to see that $\{a_1 + J(R), a_2 + J(R), \ldots, a_m + J(R)\}$ is a vertex cut of $\Gamma(R/J(R))$. So

$$\kappa(\Gamma(R/J(R))) \leq m = n/|J(R)| = \kappa(\Gamma(R))/|J(R)|.$$

This completes the proof. □

The following theorem is one of our main results in this paper.

Theorem 4.3. *Let R be a ring. Then $\kappa(\Gamma(R)) = \kappa'(\Gamma(R)) = \delta(\Gamma(R)) = |U(R)|$.*

Proof. Let $R = R_1 \times \cdots \times R_n$ be a ring such that R_i is a local ring with maximal ideal $\mathfrak{m}_i$. By Theorems 4.1 and 4.2, we have

$$\begin{aligned}\kappa(\Gamma(R)) &= \kappa(\Gamma(R/J(R))|J(R)| \\ &= \kappa(\Gamma(R_1/\mathfrak{m}_1 \times \cdots \times R_n/\mathfrak{m}_n))|\mathfrak{m}_1| \cdots |\mathfrak{m}_n| \\ &= (|R_1/\mathfrak{m}_1| - 1) \cdots (|R_n/\mathfrak{m}_n| - 1)|\mathfrak{m}_1| \cdots |\mathfrak{m}_n| \\ &= (|R_1| - |\mathfrak{m}_1|) \cdots (|R_n| - |\mathfrak{m}_n|) \\ &= |U(R)|.\end{aligned}$$

This completes the proof. □

5. Hamiltonian cycle and matching

Let $R \neq \mathbb{Z}_2$ be a ring. Since $\Gamma(R)$ is a refinement of the unit graph $G(R)$, [17, Theorem 2.1] implies that $\Gamma(R)$ is Hamiltonian. In this section, by a simple and constructive method, we show that $\Gamma(R)$ is Hamiltonian if and only if it is connected. As a consequence of this result, we show that $\Gamma(R)$ has a perfect matching if and only if $|R|$ is an even number. We begin with the following lemma.

Lemma 5.1. *Let R be a ring. If $\Gamma(R/J(R))$ is Hamiltonian, then $\Gamma(R)$ is also Hamiltonian.*

Proof. Let $J(R) = \{j_1, \ldots, j_n\}$ and $a_1 + J(R)—\cdots—a_k + J(R)$ be a Hamiltonian cycle in $\Gamma(R/J(R))$. By [14, Proposition 4.8], we have the following path in $\Gamma(R)$:

$$P_i := j_i + a_1—j_i + a_2—\cdots—j_i + a_k \ , \ (1 \leq i \leq n).$$

Now we construct the following Hamiltonian cycle in $\Gamma(R)$:

$$P_1—P_2—\cdots—P_n.$$

This completes the proof. □

Remark 5.2. We note that the converse of the above lemma is false. For example, let $R \neq \mathbb{Z}_2$ be a ring such that $R/J(R) = \mathbb{Z}_2$. Then $\Gamma(R/J(R))$ is not Hamiltonian. But it is easy to see that R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$. Therefore $\Gamma(R)$ is a complete bipartite graph, by Theorem 2.2. Hence $\Gamma(R)$ is Hamiltonian.

Theorem 5.3. *Let R be a ring such that $R \neq \mathbb{Z}_2$. Then $\Gamma(R)$ is a connected graph if and only if $\Gamma(R)$ is Hamiltonian.*

Proof. Suppose $\Gamma(R)$ is a connected graph. In view of [14, Theorem 3.5], we may assume that $R/J(R) = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a field. Since $\Gamma(R)$ is connected, by Corollary 2.6, we have the following cases:
Case 1: $|F_i| > 2$, for all $1 \leq i \leq n$. In this case, we claim that $\Gamma(R)$ is a Hamiltonian graph. More generally, we show that there is a Hamiltonian cycle $\widehat{X}_1—\widehat{X}_2—\cdots—\widehat{X}_s$ such that no entries of $\widehat{X}_1$ and $\widehat{X}_s$ are zero. We use induction on n. Suppose that $n = 1$ and $F_1 = \{a_1 = 0, a_2, \ldots, a_{|F_1|}\}$. Then it is easy to see that $a_2—0—a_3—a_4—\cdots—a_{|F_1|}$ is a Hamiltonian cycle in $\Gamma(F_1)$. Now suppose that $n > 1$. By the induction hypothesis there is a Hamiltonian cycle $\widehat{X}_1—\widehat{X}_2—\cdots—\widehat{X}_s$ in $\Gamma(F_1 \times F_2 \times \cdots \times F_{n-1})$ such that no entries of $\widehat{X}_1$ and $\widehat{X}_s$ are zero. Let $F_n = \{c_1 = 0, c_2, \ldots, c_{|F_n|}\}$. In view of Proposition 1.1, we define the following path:

$$\begin{aligned}P_{i,i+1} := (\widehat{X}_i, c_2)—(\widehat{X}_{i+1}, 0)—(\widehat{X}_i, c_3)—(\widehat{X}_{i+1}, c_2)—(\widehat{X}_i, 0)—(\widehat{X}_{i+1}, c_3)\\ —(\widehat{X}_i, c_4)—(\widehat{X}_{i+1}, c_4)—\cdots—(\widehat{X}_i, c_{|F_n|})—(\widehat{X}_{i+1}, c_{|F_n|}).\end{aligned}$$

Now we have the following two cases:

If s is an even number we construct the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$P_{1,2}—P_{3,4}—\cdots—P_{s-1,s}.$$

If s is an odd number we construct the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$P_{1,2}—P_{3,4}—\cdots—P_{s-2,s-1}—(\widehat{X}_s,0)—(\widehat{X}_s,c_2)—(\widehat{X}_s,c_3)—\cdots—(\widehat{X}_s,c_{|F_n|}).$$

Case 2: $R/J(R)=\mathbb{Z}_2$. In this case $\Gamma(R)$ is Hamiltonian, by Remark 5.2.
Case 3: $n>1$ and $F_1=\mathbb{Z}_2$ and $F_i\neq\mathbb{Z}_2$ for all $2\le i\le n$. By Case 1, $\Gamma(F_2\times F_3\times\cdots\times F_n)$ has a Hamiltonian cycle, say $\widehat{Y}_1—\widehat{Y}_2—\cdots—\widehat{Y}_h$, such that no entries of $\widehat{Y}_1$ and $\widehat{Y}_h$ are zero. We have the following two cases: If h is an even number, we construct the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$(1,\widehat{Y}_1)—(0,\widehat{Y}_2)—(1,\widehat{Y}_3)—(0,\widehat{Y}_4)—\cdots—(1,\widehat{Y}_{h-1})—(0,\widehat{Y}_h)$$
$$—(1,\widehat{Y}_h)—(0,\widehat{Y}_{h-1})—\cdots—(1,\widehat{Y}_2)—(0,\widehat{Y}_1).$$

If h is an odd number, we have the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$(1,\widehat{Y}_1)—(0,\widehat{Y}_2)—(1,\widehat{Y}_3)—(0,\widehat{Y}_4)\cdots—(0,\widehat{Y}_{h-1})—(1,\widehat{Y}_h)$$
$$—(0,\widehat{Y}_h)—(1,\widehat{Y}_{h-1})\cdots—(1,\widehat{Y}_2)—(0,\widehat{Y}_1).$$

Now Lemma 5.1 implies that $\Gamma(R)$ is a Hamiltonian graph. The converse is trivial. □

A *matching* in a graph G is a set of edges no two of which share an endpoint. The vertices incident to the edges of a matching M are *saturated* by M. A *perfect matching* in a graph is a matching that saturates every vertex.

Lemma 5.4. *Let R be a ring. If $\Gamma(R/J(R))$ has a perfect matching, then $\Gamma(R)$ also has a perfect matching.*

Proof. Suppose that $J(R)=\{j_1,\dots,j_m\}$ and let $a_1+J(R),\dots,a_k+J(R)$ be all distinct elements of $R/J(R)$. Let $\{e_1,\dots,e_{k/2}\}$ be a perfect matching for $\Gamma(R/J(R))$. Without loss of generality, we may assume that e_i is the edge between vertices $a_{2i-1}+J(R)$ and $a_{2i}+J(R)$, for all $1\le i\le k/2$. According to this assumption and [14, Proposition 4.8], we conclude that $a_{2i-1}+j_t$ is adjacent to $a_{2i}+j_t$ in $\Gamma(R)$ by some edge, say $e_{i,t}$, for all $1\le i\le k/2$ and all $1\le t\le m$. Now it is easy to see that $\{e_{i,t}|1\le i\le k/2, 1\le t\le m\}$ is a perfect matching for $\Gamma(R)$. □

Remark 5.5. The converse of the above lemma is also true (see Corollary 5.7).

Theorem 5.6. *Let R be a ring. Then $\Gamma(R)$ has a perfect matching if and only if $|R|$ is an even number.*

Proof. Suppose that $|R|$ is an even number. First assume that $\Gamma(R)$ is connected. If $R = \mathbb{Z}_2$, obviously R has a perfect matching. So let $R \neq \mathbb{Z}_2$. By Theorem 5.3, $\Gamma(R)$ has the following Hamiltonian cycle:

$$v_1 \text{---} v_2 \text{---} \cdots \text{---} v_n.$$

Let e_i be the edge between the vertices v_i and v_{i+1} for all $1 \leq i \leq n-1$. Set $M := \{e_1, e_3, \ldots, e_{n-1}\}$. Then M is a perfect matching.

Now let $\Gamma(R)$ be a disconnected graph. By Corollary 2.6, we may assume that $R/J(R) = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}} \times F_1 \times F_2 \times \cdots \times F_t$, such that $n \geq 2$, where F_i is a field and $F_i \neq \mathbb{Z}_2$, for all $1 \leq i \leq t$. First consider the ring $S = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}}$. For $x \in \{0, 1\}$, we define:

$$x^c := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1. \end{cases}$$

If $\widehat{X} = (x_1, x_2, \ldots, x_n)$ is an arbitrary element of S, we define $\widehat{X}^c := ({x_1}^c, {x_2}^c, \ldots, {x_n}^c)$. It is clear that $\widehat{X}^c$ is the unique neighborhood of $\widehat{X}$ and hence every element of $\Gamma(S)$ has degree 1. Therefore $\Gamma(S)$ has $2^n/2$ connected components that are isomorphic to K_2. Now we consider the ring $R/J(R)$. We have $R/J(R) = \{(\widehat{X}, \widehat{Y}) |\ \widehat{X} \in S \text{ and } \widehat{Y} \in F_1 \times \cdots \times F_t\}$. Suppose that $\widehat{X}$ is an arbitrarily fixed element of S and set

$$C := \{(\widehat{X}, \widehat{Y}) |\ \widehat{Y} \in F_1 \times \cdots \times F_t\} \cup \{(\widehat{X}^c, \widehat{Y}) |\ \widehat{Y} \in F_1 \times \cdots \times F_t\}.$$

Clearly, if $\widehat{Z} \in S$ and $\widehat{Z} \notin \{\widehat{X}, \widehat{X}^c\}$, then $(\widehat{Z}, \widehat{Y})$ is not adjacent to any element of C. We claim that C is a connected component of $\Gamma(R/J(R))$ and has a perfect matching. Define the following map:

$$h : \Gamma(C) \longrightarrow \Gamma(\mathbb{Z}_2 \times F_1 \times \cdots \times F_t),$$

where $h(\widehat{X}, \widehat{Y}) = (0, \widehat{Y})$ and $h(\widehat{X}^c, \widehat{Y}) = (1, \widehat{Y})$. It is easy to see that any two vertices of $\Gamma(C)$, say c_1, c_2, are adjacent if and only if $h(c_1)$ is adjacent to $h(c_2)$. So $\Gamma(C)$ is isomorphic to $\Gamma(\mathbb{Z}_2 \times F_1 \times \cdots \times F_t)$. The graph $\Gamma(\mathbb{Z}_2 \times F_1 \times \cdots \times F_t)$ has a Hamiltonian cycle, by Theorem 5.3, and has even vertices. Therefore it has a perfect matching. This implies that $\Gamma(C)$ also has a perfect matching. On the other hand, all connected components of $\Gamma(R/J(R))$ are isomorphic to $\Gamma(C)$ and hence $\Gamma(R/J(R))$ has a perfect matching. Now Lemma 5.4 implies that $\Gamma(R)$ has a perfect matching.

The converse is trivial. □

Corollary 5.7. *Let R be a ring. Then $\Gamma(R)$ has a perfect matching if and only if $\Gamma(R/J(R))$ has a perfect matching.*

Proof. Suppose that $R = R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Suppose $\Gamma(R)$ has a perfect matching. By Theorem 5.6, $|R|$ is an even number. Therefore there is $1 \leq i \leq n$, such that $|R_i|$ is an even number.

Hence, by [1, Proposition 2.1], $|R_i/\mathfrak{m}_i|$ is even. So we deduce that $|R/J(R)| = |R_1/\mathfrak{m}_1| \times \cdots \times |R_n/\mathfrak{m}_n|$ is an even number. By the above Theorem, we conclude that $\Gamma(R/J(R))$ has a perfect matching.

The converse follows easily from Lemma 5.4. □

Acknowledgments. The research of the first author was in part supported by a grant from Shahrekord University (No. 94GRD1M531). We would like to thank Ghodratollah Aalipour for many useful comments and suggestions. Furthermore, we would like to thank the referee for the careful review and the valuable comments.

References

[1] R. Akhtar, M. Boggess, T. Jackson-Henderson, I. Jimenez, R. Karpman, A. Kinzel, and D. Pritikin, *On the unitary Cayley graph of a finite ring*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 117, 13 pp.

[2] N. Ashrafi, H. R. Maimani, M. R. Pournaki, and S. Yassemi, *Unit graphs associated with rings*, Comm. Algebra **38** (2010), no. 8, 2851–2871.

[3] T. Asir and T. T. Chelvam, *On the genus of generalized unit and unitary Cayley graphs of a commutative ring*, Acta Math. Hungar. **142** (2014), no. 2, 444–458.

[4] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, 1969.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, 244 Springer, New York, 2008.

[6] H. Chen, *Related to Stable Conditions*, World Scientic (Series in Algebra 11), Hakensack, NJ, 2011.

[7] A. K. Das, H. R. Maimani, M. R. Pournaky, and S. Yassemi, *Nonplanarity of unit graphs and classification of the toroidal ones*, Pacific J. Math. **268** (2014), no. 2, 371–387.

[8] E. Fuchs, *Longest induced cycles in circulant graphs*, Electron. J. Combin. **14** (2005), no. 1, Research Paper 52, 12 pp.

[9] K. Khashyarmanesh and M. R. Khorsandi, *A generalization of the unit and unitary Cayley graphs of a commutative ring*, Acta Math. Hungar. **137** (2012), no. 4, 242–253.

[10] D. Kiani and M. M. H. Aghaei, *On the unitary Cayley graphs of a ring*, Electron. J. Combin. **19** (2012), no. 2, Research paper 10, 10 pp.

[11] D. Kiani, M. M. H. Aghaei, Y. Meemark, and B. Suntornpoch, *Energy of unitary Cayley graphs and gcd-graphs*, Linear Algebra Appl. **435** (2011), no. 6, 1336–1343.

[12] W. Klotz and T. Sander, *Some properties of unitary Cayley graphs*, Electron. J. Combin. **14** (2007), no. 1, Research Paper 45, 12 pp.

[13] T. Y. Lam, *Bass's work in ring theory and projective modules*, Algebra K-theory, groups, and education (New York, 1997), 83–124, Contemp. Math., 243, Amer. Math. Soc., Providence, RI, 1999.

[14] ———, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, Inc. 2001.

[15] X. Liu and S. Zhou, *Spectral properties of unitary Cayley graphs of finite commutative rings*, Electron. J. Combin. **19** (2012), no. 4, Research paper 13, 19 pp.

[16] H. R. Maimani, M. R. Pournaki, A. Tehranian, and S. Yassemi, *Graphs attached to rings revisited*, Arab. J. Sci. Eng. **36** (2011), no. 6, 997–1011.

[17] H. R. Maimani, M. R. Pournaki, and S. Yassemi, *Necessary and sufficient conditions for unit graphs to be Hamiltonian*, Pacific J. Math. **249** (2011), no. 2, 419–429.

[18] P. K. Sharma and S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra **176** (1995), no. 1, 124–127.

[19] H. Su and Y. Zhou, *On the girth of the unit graph of a ring*, J. Algebra Appl. **13** (2014), no. 2, 1350082, 12 pp.

[20] D. B. West, *Introduction to Graph Theory*, second ed., Prentice-Hall, 2000.

Ali Reza Naghipour
Department of Mathematical Sciences
Shahrekord University
P.O. Box 115, Shahrekord, Iran
E-mail address: naghipour@sci.sku.ac.ir

Meysam Rezagholibeigi
Department of Mathematical Sciences
Shahrekord University
P.O. Box 115, Shahrekord, Iran
E-mail address: qolibeigi.meysam@gmail.com