Bull. Korean Math. Soc. **53** (2016), No. 4, pp. 1185–1196
http://dx.doi.org/10.4134/BKMS.b150600
pISSN: 1015-8634 / eISSN: 2234-3016

ON FUNCTIONAL EQUATIONS OF THE FERMAT-WARING TYPE FOR NON-ARCHIMEDEAN VECTORIAL ENTIRE FUNCTIONS

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Abstract. We show a class of homogeneous polynomials of Fermat-Waring type such that for a polynomial P of this class, if $P(f_1,\ldots,f_{N+1}) = P(g_1,\ldots,g_{N+1})$, where $f_1,\ldots,f_{N+1}$; $g_1,\ldots,g_{N+1}$ are two families of linearly independent entire functions, then $f_i = cg_i,\ i = 1,2,\ldots,N+1$, where c is a root of unity. As a consequence, we prove that if X is a hypersurface defined by a homogeneous polynomial in this class, then X is a unique range set for linearly non-degenerate non-Archimedean holomorphic curves.

1. Introduction

The function equation $P(f) = P(g)$, where P is a polynomial, f, g are functions in some classes, has a long history, dating back to Ritt ([24]). In recent years the problem of existence or non-existence of solutions to the equation has investigated by many authors (see [1], [2], [6], [8], [10], [20], [21], [22], [23]). For the case of entire functions of one variable in a non-Archimedean field, many interesting results are obtained ([4], [5], [6], [9], [11], [12], [13], [16], [17]).

In this paper we investigated the case of the Fermat-Waring type for non-Archimedean vectorial entire functions. Namely, we consider the equation:

$$P(f_1, f_2, \ldots, f_{N+1}) = P(g_1, g_2, \ldots, g_{N+1}),$$

where P is a polynomial of Fermat-Waring type, and f_i, g_i are entire functions in a non-Archimedean field. We show if $f_1,\ldots,f_{N+1}$; $g_1,\ldots,g_{N+1}$ are two families of linearly independent entire functions, then $f_i = cg_i$, $i = 1,2,\ldots, N+1$, where c is a root of unity. As a consequence, we obtained a class of unique range sets for linearly non-degenerate non-Archimedean holomorphic curves.

Received July 27, 2015; Revised November 3, 2015.

2010 *Mathematics Subject Classification.* 11D88, 30D35.

Key words and phrases. Diophantine equations, non-Archimedean field, unique range sets, holomorphic curves.

The work was supported by a NAFOSTED grant.

Throughout this paper, $\mathbb{K}$ will denote an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value denoted by $|\cdot|$. We assume that the reader is familiar with the notations in the non-Archimedean Nevanlinna theory (see [14]).

Let f be a non-constant meromorphic function on $\mathbb{K}$. For every $a \in \mathbb{K}$, define the function $\mu_f^a : \mathbb{K} \to \mathbb{N}$ by

$$\mu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ d & \text{if } f(z) = a \text{ with multiplicity } d. \end{cases}$$

A non-Archimedean holomorphic map f is a map $f = [f_1, \ldots, f_{N+1}] : \mathbb{K} \to \mathbb{P}^N(\mathbb{K})$, where $f_1, \ldots, f_{N+1}$ are non-Archimedean entire functions without common zeros. The map $\tilde{f} = (f_1, \ldots, f_{N+1}) : \mathbb{K} \to \mathbb{K}^{N+1} - \{0\}$ is called a reduced representation of f (see [25]).

Let H be a hypersurface of $\mathbb{P}^N(\mathbb{K})$ such that the image of f is not contained in H, and H is defined by the equation $F = 0$. For every $z \in \mathbb{K}$ set

$$\mu_f(H, z) = \mu_{F\circ\tilde{f}}(z),\ \mu_f(H) = \mu_{F\circ\tilde{f}}.$$

Let us first describe the class of polynomials of Fermat-Waring type considered in this paper.

A family of q polynomials of $N+1$ variables are said to be *in general position* if no set of $N+1$ polynomials in this family has common zeros in $\mathbb{K}^{N+1} - \{0\}$.

Now let given q linear forms of $N+1$ variables ($q > N+1$) in general position:

$$L_i = L_i(z_1, \ldots, z_{N+1}) = \alpha_{i,1}z_1 + \alpha_{i,2}z_2 + \cdots + \alpha_{i,N+1}z_{N+1},\ i = 1, 2, \ldots, q.$$

Let n, m, be positive integers, $m < n$, $a, b \in \mathbb{K}$, $a, b \neq 0$.

The following polynomial is called a *Yi (m,n)-polynomial*:

$$Y_{(m,n)}(z_1, z_2) = z_1^n - az_1^{n-m}z_2^m + bz_2^n.$$

Now consider q homogeneous polynomials:

$$P_1 = P_1(z_1, \ldots, z_{N+1}) = Y_{(m,n)}(L_1, L_2) = L_1^n - aL_1^{n-m}L_2^m + bL_2^n,$$

and for $q \geq i \geq 2$, set:

$$P_i = P_i(z_1, \ldots, z_{N+1}) = Y_{(m,n)}(P_{i-1}, L_{i+1}^{n^{i-1}}).$$

Then we consider the following polynomial of Fermat-Waring type of degree n^q:

$$P(z_1, z_2, \ldots, z_{N+1}) = P_q(z_1, \ldots, z_{N+1}). \tag{1.1}$$

The polynomial $P(z_1, z_2, \ldots, z_{N+1})$ is called a *q-iteration of Yi (m,n)-polynomials.*

For entire functions $f_1, \ldots, f_{N+1}$; $g_1, \ldots, g_{N+1}$ over $\mathbb{K}$ we consider the following equation:

$$P(f_1, \ldots, f_{N+1}) = P(g_1, \ldots, g_{N+1}). \tag{1.2}$$

Denote by X the hypersurface of Fermat-Waring type in $\mathbb{P}^N(\mathbb{K})$, which is defined by the equation

$$P(z_1, \ldots, z_{N+1}) = 0. \tag{1.3}$$

We shall prove the following theorems.

Theorem 1.1. *Let $P(z_1, z_2, \ldots, z_{N+1})$ be a q-iteration of Yi (m,n)-polynomials $n \geq 2m+8$, $m \geq 3$, and $f_1, \ldots, f_{N+1}$; $g_1, \ldots, g_{N+1}$ be two families of linearly independent entire functions over $\mathbb{K}$, satisfying the equation $P(f_1, \ldots, f_{N+1}) = P(g_1, \ldots, g_{N+1})$. Then $g_i = cf_i$, $c^{n^q} = 1$, $i = 1, \ldots, N+1$.*

Theorem 1.2. *Let f and g be two linearly non-degenerate holomorphic mappings from $\mathbb{K}$ to $\mathbb{P}^N(\mathbb{K})$. Let X be the Fermat-Waring hypersurface defined by the equation $P(z_1, \ldots, z_{N+1}) = 0$, where $P(z_1, \ldots, z_{N+1})$ is a q-iteration of Yi (m,n)-polynomials, and $n \geq 2m+8$, $m \geq 3$. Then $\mu_f(X) = \mu_g(X)$ implies $f \equiv g$.*

The main tool to be used is the non-Archimedean Nevanlinna theory, so we first recall some basic facts of the theory. More details can be found in [3], [14], [15], [17], [19].

The authors would like to thank the referee for his/her valuable suggestions.

2. Preliminaries

Let f be a non-constant meromorphic function on $\mathbb{K}$.

The following lemma were proved in [3], see also [14].

Lemma 2.1. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and let $a_1, a_2, \ldots, a_q$, $q \geq 2$, be distinct points of $\mathbb{K}$. Then*

$$(q-1)T(r,f) \leq \overline{N}(r,f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) - \log r + O(1).$$

Let f be a holomorphic curve from $\mathbb{K}$ to $\mathbb{P}^N(\mathbb{K})$ with reduced representation $\tilde{f} = (f_1, \ldots, f_{N+1})$. Define the *characteristic function of* f, by

$$T_f(r) = \log ||f||_r, \quad \text{where} \quad ||f||_r = \max_{1 \leq i \leq N+1} |f_i|_r,$$

where for an entire function f, denote by $|f|_r$ the maximum of $|f(z)|$ for $|z| \leq r$.

Let H be a hypersurface of $\mathbb{P}^N(\mathbb{K})$ such that the image of f is not contained in H, and H is defined by the equation $F = 0$. Set

$$N_f(H,r) = N(r, \frac{1}{F(\tilde{f})}), \ N_{k,f}(H,r) = N_k(r, \frac{1}{F(\tilde{f})}).$$

Let f be a holomorphic curve from $\mathbb{K}$ to $\mathbb{P}^N(\mathbb{K})$. Then f is called *linearly non-degenerate* if there is not any linear form L of variables $z_1, \ldots, z_{N+1}$ such that $L(\tilde{f}) = 0$, i.e., the image of f is not contained in any hyperplane of $\mathbb{P}^N(\mathbb{K})$.

Let q, N be positive integers with $q \geq N+1$. We say that the hypersurfaces $H_1, \ldots, H_q$ of $\mathbb{P}^N(\mathbb{K})$ are *in general position* if $\bigcap_{i=1}^{N+1} H_{j_i} = \emptyset$ for every subset

$\{j_1, \ldots, j_{N+1}\} \subset \{1, \ldots, q\}$. The following lemmas were proved in [19].

Lemma 2.2. *Let f be a linearly non-degenerate holomorphic curve from $\mathbb{K}$ to $\mathbb{P}^N(\mathbb{K})$ and $H_1, \ldots, H_q$ be hyperplanes of $\mathbb{P}^N(\mathbb{K})$ in general position. Then*

$$(q - N - 1)T_f(r) \leq \sum_{i=1}^{q} N_{N,f}(H_i, r) - \frac{N(N+1)}{2} \log r + O(1).$$

Lemma 2.3. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and let $a_1, a_2, \ldots, a_q$, $q \geq 3$, be distinct points of $\mathbb{K} \cup \{\infty\}$. Suppose either $f - a_i$ has no zeros, or all the zeros of the functions $f - a_i$ have multiplicity at least $m_i, i = 1, \ldots, q$. Then*

$$\sum_{i=1}^{q}(1 - \frac{1}{m_i}) < 2.$$

3. Functional equations and unique range sets

We first need the following lemmas:

Lemma 3.1. *Let $d, N \in \mathbb{N}^*, q_i \in \mathbb{N}$ and $z_i^{d-q_i} D_i(z_1, z_2, \ldots, z_{N+1})$ be a family in general position of homogeneous polynomials with coefficients in $\mathbb{K}$ of degree d such that $f_i^{d-q_i} D_i(f_1, \ldots, f_{N+1}) \not\equiv 0$, $1 \leq i \leq N+1$. Suppose*

$$\sum_{i=1}^{N+1} f_i^{d-q_i} D_i(f_1, \ldots, f_{N+1}) = 0,\ d \geq N^2 - 1 + \sum_{i=1}^{N+1} q_i, N > 1.$$

Then $f_1^{d-q_1} D_1(f_1, \ldots, f_{N+1}), \ldots, f_N^{d-q_N} D_N(f_1, \ldots, f_{N+1})$ are linearly dependent on $\mathbb{K}$.

Proof. We consider the following possible cases:
Case 1: $f_1, \ldots, f_{N+1}$ have no common zeros.

By the hypothesis, $z_i^{d-q_i} D_i(z_1, \ldots, z_{N+1})$ is a family in general position, we then get

$$\tilde{F} = \left(f_1^{d-q_1} D_1(f_1, f_2, \ldots, f_{N+1}), \ldots, f_N^{d-q_N} D_N(f_1, f_2, \ldots, f_{N+1})\right)$$

which is a reduced representation of the holomorphic curve

$$F = \left[f_1^{d-q_1} D_1(f_1, f_2, \ldots, f_{N+1}) : \cdots : f_N^{d-q_N} D_N(f_1, f_2, \ldots, f_{N+1})\right]$$

from $\mathbb{K}$ to $\mathbb{P}^{N-1}(\mathbb{K})$. Assume that F is linearly non-degenerate. By the hypothesis we have

$$\sum_{i=1}^{N+1} f_i^{d-q_i} D_i(f_1, f_2, \ldots, f_{N+1}) = 0. \tag{3.1}$$

We first prove $dT_f(r) = T_F(r) + O(1)$. Set

$$R_i(z_1, \ldots, z_{N+1}) = z_i^{d-q_i} D_i(z_1, z_2, \ldots, z_{N+1}),\ i = 1, \ldots, N+1.$$

From the hypothesis of general position and the Hilbert Nullstellensatz [26] it implies that for any integer $k, 1 \leq k \leq N+1$, there is an integer $m_k \geq d$ such that

$$z_k^{m_k} = \sum_{i=1}^{N+1} a_{i_k}(z_1, \ldots, z_{N+1}) R_i(z_1, \ldots, z_{N+1}),$$

where $a_{i_k}(z_1, \ldots, z_{N+1})$, $1 \leq i \leq N+1$, are homogeneous polynomials with coefficients in $\mathbb{K}$ of degree $m_k - d$. Therefore

$$f_k^{m_k} = \sum_{i=1}^{N+1} a_{i_k}(f_1, \ldots, f_{N+1}) R_i(f_1, \ldots, f_{N+1}),\ k = 1, \ldots, N+1.$$

It implies that

$$T_{f_k^{m_k}}(r) = m_k T_{f_k}(r) \leq (m_k - d) T_f(r) + \max_{1 \leq i \leq N+1} T_{R_i(f_1, \ldots, f_{N+1})}(r) + O(1),$$

$$dT_f(r) \leq \max_{1 \leq i \leq N+1} T_{R_i(f_1, \ldots, f_{N+1})}(r) + O(1). \tag{3.2}$$

On the other hand,

$$T_{R_i(f_1, \ldots, f_{N+1})}(r) = T_{f_i^{d-q_i} D_i(f_1, f_2, \ldots, f_{N+1})}(r) \leq dT_f(r) + O(1) \tag{3.3}$$

for all $i = 1, \ldots, N+1$.

By (3.2) and (3.3) we have $dT_f(r) = \max_{1 \leq i \leq N+1} T_{R_i(f_1, \ldots, f_{N+1})}(r) + O(1)$. Therefore $dT_f(r) = T_F(r) + O(1)$. Consider the following hyperplanes in general position in $\mathbb{P}^{N-1}$:

$$H_1 : x_1 = 0;\ H_2 : x_2 = 0;\ \ldots; H_N : x_N = 0;\ \ H_{N+1} : x_1 + x_2 + \cdots + x_N = 0.$$

Using Lemma 2.2, and noting that $d - q_i \geq N - 1$,

$$N_{N-1,F}(H_{N+1}, r) = N_{N-1}(r, \frac{1}{f_{N+1}^{d-q_{N+1}} D_{N+1}(f_1, f_2, \ldots, f_{N+1})}),$$

we have

$$\begin{aligned} dT_f(r) = T_F(r) + O(1) &\leq \sum_{i=1}^{N+1} N_{N-1,F}(H_i, r) - \frac{N(N-1)}{2} \log r + O(1) \\ &\leq (N-1) \sum_{i=1}^{N+1} N(r, \frac{1}{f_i}) + \sum_{i=1}^{N+1} N(r, \frac{1}{D_i(f_1, f_2, \ldots, f_{N+1})}) \\ &\quad - \frac{N(N-1)}{2} \log r + O(1) \\ &\leq (N-1)(N+1) T_f(r) + \sum_{i=1}^{N+1} q_i T_f(r) - \frac{N(N-1)}{2} \log r + O(1) \\ &\leq \big(N^2 - 1 + \sum_{i=1}^{N+1} q_i\big) T_f(r) - \frac{N(N-1)}{2} \log r + O(1), \end{aligned}$$

and

$$\big(d-(N^2-1)-\sum_{i=1}^{N+1} q_i\big)T_f(r)+\frac{N(N-1)}{2}\log r \le O(1).$$

Because $d \ge N^2-1+\sum_{i=1}^{N+1} q_i$, we have a contradiction.

So $f_1^{d-q_1}D_1(f_1,\dots,f_{N+1}),\dots,f_N^{d-q_N}D_N(f_1,\dots,f_{N+1})$ are linearly dependent on $\mathbb{K}$.

Case 2: $f_1,\dots,f_{N+1}$ have common zeros. Let l be a greatest common divisor of $f_1,f_2,\dots,f_{N+1}$. Write $f_i=lh_i$, $i=1,\dots,N+1$. Then $h_1,\dots,h_{N+1}$ have no common zeros. From (3.1) we obtain

$$\begin{aligned} &l^d\sum_{i=1}^{N+1} h_i^{d-q_i}D_i(h_1,h_2,\dots,h_{N+1})=0, \text{ and}\\ &\sum_{i=1}^{N+1} h_i^{d-q_i}D_i(h_1,\dots,h_{N+1})=0. \end{aligned} \tag{3.4}$$

By a similar argument as in the proof of Case 1 for (3.4) we get that

$$h_1^{d-q_1}D_1(h_1,\dots,h_{N+1}),\dots,h_N^{d-q_N}D_N(h_1,\dots,h_{N+1})$$

are linearly dependent on $\mathbb{K}$. So

$$f_1^{d-q_1}D_1(f_1,\dots,f_{N+1}),\dots,f_N^{d-q_N}D_N(f_1,\dots,f_{N+1})$$

are linearly dependent on $\mathbb{K}$.

Lemma 3.1 is proved. □

Lemma 3.2. *Let $n,n_1,n_2,\dots,n_q,q\in\mathbb{N}^*$, $a_1,\dots,a_q,c\in\mathbb{K}$, $c\neq 0$, and $q\ge 2+\sum_{i=1}^{q}\frac{n_i}{n}$. Then the functional equation*

$$(f-a_1)^{n_1}(f-a_2)^{n_2}\cdots(f-a_q)^{n_q}=cg^n$$

has no non-constant meromorphic solutions (f,g).

Proof. Suppose that (f,g) is a non-constant meromorphic solution of the equation:

$$(f-a_1)^{n_1}(f-a_2)^{n_2}\cdots(f-a_q)^{n_q}=cg^n.$$

From this we see that if $z_0\in\mathbb{K}$ is a zero of $f-a_i$ for some $1\le i\le q$, then z_0 is a zero of g and $n_i\mu_f^{a_i}(z_0)=n\mu_g^0(z_0)$. So

$$\overline{N}(r,\frac{1}{f-a_i})\le\frac{n_i}{n}N(r,\frac{1}{f-a_i})\le\frac{n_i}{n}T(r,f)+O(1).$$

From this and by Lemma 2.1,

$$\begin{aligned}(q-2)T(r,f)&\le\sum_{i=1}^{q}\overline{N}(r,\frac{1}{f-a_i})-\log r+O(1)\\ &\le\sum_{i=1}^{q}\frac{n_i}{n}N(r,\frac{1}{f-a_i})-\log r+O(1)\end{aligned}$$

$$\leq \sum_{i=1}^{q} \frac{n_i}{n} T(r,f) - \log r + O(1); (q - 2 - \sum_{i=1}^{q} \frac{n_i}{n}) T(r,f) + \log r \leq O(1).$$

Since $q \geq 2 + \sum_{i=1}^{q} \frac{n_i}{n}$, we obtain a contradiction. □

Lemma 3.3. *Let $n, m \in \mathbb{N}^*$, $n \geq 2m+8$, $a_1, b_1, a_2, b_2, c \in \mathbb{K}$, $a_1 \neq 0$, $b_1 \neq 0$, $a_2 \neq 0$, $b_2 \neq 0$, $c \neq 0$, and let f_1, f_2, g_1, g_2 be non-zero entire functions.*

1. *Suppose that $\frac{f_1}{f_2}$ is a non-constant meromorphic function, and*

$$f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = b_2 g_2^n. \tag{3.5}$$

Then there exists $c_1 \neq 0$ such that $c_1 b_2 g_2^n = b_1 f_2^n$, $g_2 = h f_2$ with $b_1 = c_1 b_2 h^n$, $h \in \mathbb{K}$.

2. *Suppose that $\frac{f_1}{f_2}$ and $\frac{g_1}{g_2}$ are non-constant meromorphic functions, and*

$$f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = c(g_1^n + a_2 g_1^{n-m} g_2^m + b_2 g_2^n). \tag{3.6}$$

i. *If $m \geq 2$, then*

$$c b_2 g_2^n = b_1 f_2^n, \ g_2 = h f_2 \ \text{with} \ b_1 = c b_2 h^n, \ h \in \mathbb{K}.$$

ii. *If $m \geq 3$, then*

$$g_1 = l f_1, g_2 = h f_2 \ \text{with} \ 1 = c l^n, a_1 = c a_2 l^{n-m} h^m, b_1 = c b_2 h^n, \ l, h \in \mathbb{K}.$$

Proof. 1. From (3.5) we have

$$f_1^{n-m}(f_1^m + a_1 f_2^m) + b_1 f_2^n - b_2 g_2^n = 0. \tag{3.7}$$

Note that $x_1^{n-m}(x_1^m + a_1 x_2^m), b_1 x_2^n, -b_2 x_3^n$ are the homogeneous polynomials of degree n in general position. Since $n \geq 2m+8$ and by Lemma 3.1, there exists $c_1 \neq 0$ such that $c_1 b_2 g_2^n = b_1 f_2^n$. Therefore $g_2 = h f_2$ with $b_1 = c_1 b_2 h^n$, $h \in \mathbb{K}$.

2.i. We consider the possible cases:

Case 1: $c = 1$. Then

$$f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = g_1^n + a_2 g_1^{n-m} g_2^m + b_2 g_2^n, \tag{3.8}$$

i.e.,

$$b_1 f_2^n + f_1^{n-m}(f_1^m + a_1 f_2^m) - b_2 g_2^n - g_1^{n-m}(g_1^m + a_2 g_2^m) = 0. \tag{3.9}$$

Note that $b_1 x_1^n$, $x_2^{n-m}(x_2^m + a_1 x_1^m)$, $-b_2 x_3^n$, $-x_4^{n-m}(x_4^m + a_2 x_3^m)$ are the homogeneous polynomials of degree n in general position. Since $n \geq 2m+8$ and by Lemma 3.1, there exist constants C_1, C_2, C_3, $(C_1, C_2, C_3) \neq (0,0,0)$, such that

$$C_1 b_1 f_2^n + C_2 f_1^{n-m}(f_1^m + a_1 f_2^m) + C_3 b_2 g_2^n = 0. \tag{3.10}$$

We consider the following possible subcases:

Subcase 1: $C_3 = 0$. Then from (3.10) we have

$$C_1 b_1 f_2^n + C_2 f_1^{n-m}(f_1^m + a_1 f_2^m) = 0.$$

Since f_2 is a non-zero entire function, we have $C_2 \neq 0$. If $C_1 = 0$, then $\frac{f_1}{f_2}$ is a constant, a contradiction. So $C_1, C_2 \neq 0$. Then $\frac{f_1}{f_2}$ is a constant, a contradiction. So $C_3 \neq 0$.

Subcase 2: $C_2 = 0$. Then from (3.10) we have $C_1 b_1 f_2^n + C_3 b_2 g_2^n = 0$. Because f_2, g_2 are non-zero entire functions, we have $C_1 \neq 0$, $C_3 \neq 0$. From this and (3.9) it follows that $g_2^n = -\frac{C_1 b_1}{C_3 b_2} f_2^n$, $\frac{g_2}{f_2} = h$, $h \in \mathbb{K}$, $h \neq 0$, and

$$b_1\Big(1 + \frac{C_1}{C_3}\Big) f_2^n + f_1^{n-m}(f_1^m + a_1 f_2^m) - g_1^{n-m}(g_1^m + a_2 g_2^m) = 0,$$

$$(3.11)\quad -g_1^n + f_1^{n-m}(f_1^m + a_1 f_2^m) + \Big(b_1(1 + \frac{C_1}{C_3}) f_2^{n-m} - a_2 h^m g_1^{n-m}\Big) f_2^m = 0.$$

Suppose that $1 + \frac{C_1}{C_3} \neq 0$. Then, from the similarity of (3.11) and (3.9), by a similar argument as in (3.9), there exist constants C_1', C_2', $(C_1', C_2') \neq (0,0)$, such that

$$(3.12)\qquad C_2' g_1^n + C_1' f_1^{n-m}(f_1^m + a_1 f_2^m) = 0.$$

Since g_1 is a non-zero entire function and $\frac{f_1}{f_2}$ is not a constant, by (3.12) we obtain $C_1' \neq 0$, $C_2' \neq 0$. We have

$$C_1' f_1^{n-m}(f_1^m + a_1 f_2^m) = -C_2' g_1^n, C_1'\Big(\frac{f_1}{f_2}\Big)^n + C_1' a_1 \Big(\frac{f_1}{f_2}\Big)^{n-m} = -C_2'\Big(\frac{g_1}{f_2}\Big)^n,$$

$$(3.13)\qquad C_1'\Big(\frac{f_1}{f_2}\Big)^{n-m}\Big((\frac{f_1}{f_2})^m + a_1\Big) = -C_2'\Big(\frac{g_1}{f_2}\Big)^n.$$

Note that the equation $z^m + a_1 = 0$ has m distinct roots $d_1, d_2, \ldots, d_m$. Set $f = \frac{f_1}{f_2}$, $g = \frac{g_1}{f_2}$. Consequently, by (3.13) we have

$$(3.14)\qquad f^{n-m}(f - d_1)\cdots(f - d_m) = C g^n,\ C \neq 0.$$

Since $\frac{f_1}{f_2}$ is not a constant, neither is $\frac{g_1}{f_2}$. By $m \geq 2$, $n \geq 2m + 8$ we have $m+1 \geq 2 + \frac{n-m}{n} + \sum_{i=1}^m \frac{1}{n}$. Then applying Lemma 3.2 to (3.14) with $q = m+1$, $n = n$, $n_1 = n - m$, $n_2 = 1 = n_3 = \cdots = n_m$, we have a contradiction. So $1 + \frac{C_1}{C_3} = 0$. Therefore $c b_2 g_2^n = b_1 f_2^n$, and $g_2 = h f_2$ with $b_1 = c b_2 h^n$.

Subcase 3. $C_1 = 0$. From (3.10) we have $C_2 f_1^{n-m}(f_1^m + a_1 f_2^m) + C_3 b_2 g_2^n = 0$. Then, from the similarity of this equation and (3.12), by a similar argument as in (3.12) we have a contradiction.

Subcase 4. $C_1 \neq 0$, $C_2 \neq 0$, $C_3 \neq 0$.

By a similar argument as in (3.7) we obtain a contradiction. So $b_2 g_2^n = b_1 f_2^n$, $g_2 = h f_2$, $h \in \mathbb{K}$, $h \neq 0$, with $b_1 = b_2 h^n$.

Case 2. $c \neq 1$. Set $b^n = c$, $e_1 = b g_1$, $e_2 = b g_2$. From this and (3.6) we get

$$f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = e_1^n + a_2 e_1^{n-m} e_2^m + b_2 e_2^n.$$

Applying the case with $c = 1$ here we obtain $b_2 e_2^n = b_2 b^n g_2^n = b_2 c g_2^n = b_1 f_2^n$, $g_2 = h f_2$ with $b_1 = c b_2 h^n$.

*2.*ii. $m \geq 3$. From (3.6) we have

$$b_1 f_2^n (\frac{1}{b_1} f^n + \frac{a_1}{b_1} f^{n-m} + 1) = c b_2 g_2^n (\frac{1}{b_2} g^n + \frac{a_2}{b_2} g^{n-m} + 1),$$

where $f = \frac{f_1}{f_2}$, $g = \frac{g_1}{g_2}$. Set $\frac{1}{b_1} = a_3$, $\frac{a_1}{b_1} = b_3$, $\frac{1}{b_2} = a_4$, $\frac{a_2}{b_2} = b_4$. Since $cb_2 g_2^n = b_1 f_2^n$,

$$a_3 f^n + b_3 f^{n-m} = a_4 g^n + b_4 g^{n-m}.$$

Set $h_1 = \frac{g}{f}$. From this we obtain

$$a_3 f^m + b_3 = a_4 (\frac{g}{f})^n f^m + b_4 (\frac{g}{f})^{n-m}, \quad a_3 f^m + b_3 = a_4 h_1^n f^m + b_4 h_1^{n-m},$$

$$f^m (a_3 - a_4 h_1^n) = b_4 h_1^{n-m} - b_3, -\frac{a_4 (h_1^n - \frac{a_3}{a_4})}{b_4 (h_1^{n-m} - \frac{b_3}{b_4})} = (\frac{1}{f})^m. \tag{3.15}$$

Assume that h_1 is not a constant. Note that the equation $z^n - \frac{a_3}{a_4} = 0$ has n simple roots, the equation $z^{n-m} - \frac{b_3}{b_4} = 0$ has $n - m$ simple roots. Then the equations $z^n - \frac{a_3}{a_4} = 0$, $z^{n-m} - \frac{b_3}{b_4} = 0$ have at most $n - m$ common simple roots. Therefore the equation $z^n - \frac{a_3}{a_4} = 0$ has at least m distinct roots, which are not roots of the equation $z^{n-m} - \frac{b_3}{b_4} = 0$. Let $r_1, r_2, \ldots, r_m$ be all these roots. Then, from (3.15) we see that all the simple zeros of the equations $h_1 - r_j$, $j = 1, \ldots, m$, have multiplicities $\geq m$. By Lemma 2.3 we have $m(1 - \frac{1}{m}) < 2$. Therefore $0 < m < 3$. From $m \geq 3$, we obtain a contradiction. Thus h_1 is constant and so is $g_1 = l f_1$. Consequently, $g_1 = l f_1$, $g_2 = h f_2$. From that and since $\frac{f_1}{f_2}$ is not a constant we obtain $1 = c l^n$, $a_1 = c a_2 l^{n-m} h^m$, $b_1 = c b_2 h^n$. $\square$

Now we use the above lemmas to prove the main result of the paper.

Proof of Theorem 1.1. Set $L_i(\tilde{f}) = L_i(f_1, \ldots, f_{N+1})$, $L_i(\tilde{g}) = L_i(g_1, \ldots, g_{N+1})$, $i = 1, \ldots, q$, $P_i(\tilde{f}) = P_i(f_1, \ldots, f_{N+1}), P_i(\tilde{g}) = P_i(g_1, \ldots, g_{N+1})$, $i = 1, \ldots, q$. We first prove $P_i(\tilde{f}) \not\equiv 0$, $i = 1, 2, \ldots, q$; $q > N$, by induction on i. With $i = 1$ assume that

$$P_1(\tilde{f}) = L_1^n(\tilde{f}) - a L_1^{n-m}(\tilde{f}) L_2^m(\tilde{f}) + b L_2^n(\tilde{f}) \equiv 0.$$

It follows from this and $L_2^n(\tilde{f}) \not\equiv 0$ that $\frac{L_1(\tilde{f})}{L_2(\tilde{f})}$ is a constant, and we have a contradiction to the linearly independence of $f_1, \ldots, f_{N+1}$. With $i = 2$, assume that

$$P_2(\tilde{f}) = P_1^n(\tilde{f}) - a P_1^{n-m}(\tilde{f}) L_3^{nm}(\tilde{f}) + b L_3^{n^2}(\tilde{f}) \equiv 0.$$

Since $P_1(\tilde{f}) \not\equiv 0$, $L_3^n(\tilde{f}) \not\equiv 0$ we see that $\frac{P_1(\tilde{f})}{L_3^n(\tilde{f})}$ is a constant. Hence

$$L_1^n(\tilde{f}) - a L_1^{n-m}(\tilde{f}) L_2^m(\tilde{f}) + b L_2^n(\tilde{f}) - A L_3^n(\tilde{f}) \equiv 0, \ A \neq 0.$$

Since $L_1(\tilde{f}) \not\equiv 0$, $L_2(\tilde{f}) \not\equiv 0$, $L_3(\tilde{f}) \not\equiv 0$ and $n \geq 2m + 8$, $m \geq 3$, we deduce from Lemma 3.3 that $\frac{L_2(\tilde{f})}{L_3(\tilde{f})}$ is a constant, and we have a contradiction to the linearly independence of $f_1, \ldots, f_{N+1}$.

Now we consider $P_i(\tilde{f}) \equiv 0$. Then

$$P_{i-1}^n(\tilde{f}) - aP_{i-1}^{n-m}(\tilde{f})L_{i+1}^{n^{i-1}m}(\tilde{f}) + bL_{i+1}^{n^i}(\tilde{f}) \equiv 0. \tag{3.16}$$

Applying the induction hypothesis and by a similar argument as above we have a contradiction.

Next we consider

$$P_i(\tilde{f}) = A_i P_i(\tilde{g}),\ A_i \neq 0,\ i = 1, 2, \dots, q. \tag{3.17}$$

We will show that $L_j(\tilde{g}) = c_j L_j(\tilde{f}),\ c_j \neq 0,\ j = 1, \dots, i+1$, by induction on i. With $i = 1$ we get $P_1(\tilde{f}) = A_1 P_1(\tilde{g})$,

$$L_1^n(\tilde{f}) - aL_1^{n-m}(\tilde{f})L_2^m(\tilde{f}) + bL_2^n(\tilde{f}) = A_1(L_1^n(\tilde{g}) - aL_1^{n-m}(\tilde{g})L_2^m(\tilde{g}) + bL_2^n(\tilde{g})).$$

Since $L_1(\tilde{f}) \not\equiv 0$, $L_2(\tilde{f}) \not\equiv 0$, $L_1(\tilde{g}) \not\equiv 0$, $L_2(\tilde{g}) \not\equiv 0$ and $n \geq 2m + 8$, $m \geq 3$, we deduce from Lemma 3.3 and the above equation that $L_j(\tilde{g}) = c_j L_j(\tilde{f}),\ c_j \neq 0, j = 1, 2$. Now we consider (3.17). Then

$$\begin{aligned} &P_{i-1}^n(\tilde{f}) - aP_{i-1}^{n-m}(\tilde{f})L_{i+1}^{n^{i-1}m}(\tilde{f}) + bL_{i+1}^{n^i}(\tilde{f}) \\ &= A_i(P_{i-1}^n(\tilde{g}) - aP_{i-1}^{n-m}(\tilde{g})L_{i+1}^{n^{i-1}m}(\tilde{g}) + bL_{i+1}^{n^i}(\tilde{g})). \end{aligned} \tag{3.18}$$

Since $P_{i-1}(\tilde{f}) \not\equiv 0$, $L_{i+1}(\tilde{f}) \not\equiv 0$, $P_{i-1}(\tilde{g}) \not\equiv 0$, $L_{i+1}(\tilde{g}) \not\equiv 0$ and $n \geq 2m + 8$, $m \geq 3$, we deduce from Lemma 3.3 and (3.18) that

$$P_{i-1}(\tilde{f}) = B_{i-1}P_{i-1}(\tilde{g}),\ L_{i+1}^{n^{i-1}}(\tilde{g}) = C_{i+1}L_{i+1}^{n^{i-1}}(\tilde{f}).$$

Applying the induction hypothesis here we have $L_j(\tilde{g}) = c_j L_j(\tilde{f}),\ c_j \neq 0$, $j = 1, 2, \dots, i+1$.

Now we can return to the proof of Theorem 1.1. Consider

$$P(\tilde{f}) = P(\tilde{g}),\ q > N. \tag{3.19}$$

From (3.17) we get $L_i(\tilde{g}) = c_i L_i(\tilde{f}),\ c_i \neq 0,\ i = 1, \dots, q+1$. Since $L_i, i = 1, \dots, N+1$, are linearly independent and $L_1, \dots, L_{N+1}, L_j,\ j \in \{N+2, \dots, q+1\}$ are linearly dependent we get

$$\begin{aligned} L_j &= b_{1j}L_1 + b_{2j}L_2 + \cdots + b_{N+1j}L_{N+1}, b_{kj} \neq 0, && k = 1, \dots, N+1, \\ & && j = N+2, \dots, q+1; \\ L_j(\tilde{f}) &= b_{1j}L_1(\tilde{f}) + b_{2j}L_2(\tilde{f}) + \cdots + b_{N+1j}L_{N+1}(\tilde{f}), && j = N+2, \dots, q+1; \\ L_j(\tilde{g}) &= b_{1j}L_1(\tilde{g}) + b_{2j}L_2(\tilde{g}) + \cdots + b_{N+1j}L_{N+1}(\tilde{g}), && j = N+2, \dots, q+1. \end{aligned}$$

From this and $L_i(\tilde{g}) = c_i L_i(\tilde{f})$, $c_i \neq 0$, $i = 1, 2, \dots, N+1$; $L_j(\tilde{g}) = c_j L_j(\tilde{f})$, we obtain

$$L_j(\tilde{g}) = c_1 b_{1j}L_1(\tilde{f}) + c_2 b_{2j}L_2(\tilde{f}) + \cdots + c_{N+1}b_{N+1j}L_{N+1}(\tilde{f});$$

$$\begin{aligned} &c_1 b_{1j}L_1(\tilde{f}) + c_2 b_{2j}L_2(\tilde{f}) + \cdots + c_{N+1}b_{N+1j}L_{N+1}(\tilde{f}) \\ &= c_j b_{1j}L_1(\tilde{f}) + c_j b_{2j}L_2(\tilde{f}) + \cdots + c_j b_{N+1j}L_{N+1}(\tilde{f}),\ j = N+2, \dots, q+1. \end{aligned}$$

By the linear independence of $f_1, \ldots, f_{N+1}$ we obtain $c_j = c_1 = c_j = c_2 = \cdots = c_{N+1}$, $j = N+2, \ldots, q+1$. Set $c = c_i$, $i = 1, \ldots, q+1$. Then $L_j(\tilde{g}) = cL_j(\tilde{f}), j = 1, \ldots, q+1$. Then $g_i = cf_i$, $i = 1, \ldots, N+1$, $c^{n^q} = 1$. □

Now we are going to complete the proof of Theorem 1.2

Proof of Theorem 1.2. Let $\tilde{f} = (f_1, \ldots, f_{N+1})$ and $\tilde{g} = (g_1, \ldots, g_{N+1})$ be reduced representations of f and g, respectively.

Since $\mu_f(X) = \mu_g(X)$, it is easy to see that there exists a non-zero constant c such that $P(\tilde{f}) = cP(\tilde{g})$. Set $l^{n^q} = c$ and $\tilde{h} = (lg_1, \ldots, lg_{N+1})$. Then $\tilde{h}$ is a reduced representation of g and $P_q(\tilde{f}) = P_q(\tilde{h})$. By Theorem 1.1, $f \equiv g$. □

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