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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE KAEHLER MANIFOLD WITH A SYMMETRIC METRIC CONNECTION OF TYPE $(\ell,\, m)$

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Abstract. We define a new connection on semi-Riemannian manifolds, which is called a *symmetric connection of type* $(\ell,\, m)$. Semi-symmetric connection and quarter-symmetric connection are two examples of this connection such that $(\ell,\, m) = (1,\, 0)$ and $(\ell,\, m) = (0,\, 1)$ respectively. In this paper, we study lightlike hypersurfaces of an indefinite Kaehler manifold endowed with a symmetric metric connection of type $(\ell,\, m)$.

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *symmetric connection of type* $(\ell,\, m)$ if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X,Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)JX - \theta(X)JY\}, \tag{1.1}$$

for any vector fields X and Y on $\bar{M}$, where ℓ and m are smooth functions, J is a tensor field of type $(1,1)$ and θ is a 1-form associated with a non-vanishing smooth non-null unit vector field ζ, which is called the *characteristic vector field* of $\bar{M}$, by $\theta(X) = \bar{g}(X, \zeta)$. Moreover, if $\bar{\nabla}$ satisfies $\bar{\nabla}\bar{g} = 0$, then it is called a *symmetric metric connection of type* $(\ell,\, m)$.

Two special cases are important for both the mathematical study and the applications to physics: (1) In case $(\ell, m) = (1, 0)$: $\bar{\nabla}$ is called a *semi-symmetric metric connection*. The notion of semi-symmetric metric connection on a Riemannian manifold was introduced by H. A. Hayden [7] and later studied by several authors [15]. (2) In case $(\ell, m) = (0, 1)$: $\bar{\nabla}$ is called a *quarter-symmetric metric connection*. The notion of the quarter-symmetric metric connection was introduced by K. Yano-T. Imai [16], and since then it have been studied by S. C. Rastogi [13, 14], D. Kamilya-U. C. De [9], R. S. Mishra-S. N. Pandey [10], S. Golab [6], N. Pušić [12], J. Nikić-N. Pušić [11] and some others.

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The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by K. L. Duggal-A. Bejancu [3] and later studied by many authors [4, 5]. The lightlike version of Riemannian manifolds with semi-symmetric or quarter-symmetric metric connections have been studied by some authors.

In this paper, we study the geometry of lightlike hypersurfaces of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, J)$ endowed with a symmetric metric connection of type (ℓ, m), in which the tensor field J, defined by (1.1), is identical with the indefinite almost complex structure tensor J of $\bar{M}$.

2. Lightlike hypersurfaces

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be an even dimensional indefinite Kaeler manifold with a symmetric metric connection $\bar{\nabla}$ of type (ℓ, m), where $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$ and J is an indefinite almost complex structure such that

$$J^2 = -I, \qquad \bar{g}(JX, JY) = \bar{g}(X, Y), \qquad (\bar{\nabla}_X J)Y = 0 \tag{2.1}$$

for any vector fields X and Y of $\bar{M}$.

Let (M, g) be a lightlike hypersurface of $\bar{M}$. It is known that the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM, of rank 1. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM), \tag{2.2}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the three equations in (2.1). We use same notations for any others. For any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ [3] satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$ respectively. Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.3}$$

In the following, let X, Y, Z and W be the smooth vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.2). From (2.2) and (2.3), the local Gauss and Weingartan formulas of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.7}$$

respectively, where ∇ and ∇^* are the induced connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of $S(TM)$ and satisfies

$$B(X, \xi) = 0. \tag{2.8}$$

The above second fundamental forms are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X,Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.9}$$

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{2.10}$$

In case $B = 0$, *i.e.*, $A^*_\xi = 0$, we say that M is *totally geodesic*. In case $C = 0$, *i.e.*, $A_N = 0$, we say that $S(TM)$ is *totally geodesic* in M.

The induced connection ∇ on M is not a metric one and satisfies

$$(\nabla_X g)(Y, Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.11}$$

where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N).$$

3. Symmetric metric connection of type (ℓ, m)

Due to [3, Section 6.2], for a lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$, $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1, such that $J(TM^\perp) \cap J(tr(TM)) = \{0\}$. It follow that $J(TM^\perp) \oplus J(tr(TM))$ is a subbundle of $S(TM)$, of rank 2. Thus there exist two non-degenerate almost complex distributions D_o and D on M with respect to the indefinite almost complex structure J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^\perp) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = \{TM^\perp \oplus_{orth} J(TM^\perp)\} \oplus_{orth} D_o.$$

In this case, the decomposition (2.2) of TM is reduced to

$$TM = D \oplus J(tr(TM)). \tag{3.1}$$

Consider two null vector fields U and V and two 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{3.2}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{3.3}$$

where F is a tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Applying J to (3.3) and using (2.1) and (3.2), we have

$$F^2 X = -X + u(X)U. \tag{3.4}$$

As $u(U) = 1$ and $FU = 0$, the set $(F,\, u,\, U)$ defines an indefinite almost contact structure on M and F is called the *structure tensor field* of M.

Applying $\bar{\nabla}_X$ to (3.2) and (3.3) by turns, and using (2.1), (2.4), (2.5), (2.7), (2.9), (2.10), (2.11), (3.2) and (3.3), we have

$$B(X, U) = C(X, V) \equiv \sigma(X), \tag{3.5}$$

$$\nabla_X U = F(A_{_N} X) + \tau(X)U, \tag{3.6}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V, \tag{3.7}$$

$$(\nabla_X F)(Y) = \; u(Y)A_{_N} X - B(X, Y)U, \tag{3.8}$$

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X, FY), \tag{3.9}$$

$$(\nabla_X v)(Y) = v(Y)\tau(X) - g(A_{_N} X, FY). \tag{3.10}$$

Let $\bar{M}$ be an indefinite Kaehler manifold with a symmetric metric connection of type $(\ell,\, m)$. Substituting (2.4) and (3.3) into (1.1) and then, comparing the tangent and transversal components of the resulting equation, we get

$$T(X, Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{3.11}$$

$$B(X, Y) - B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}, \tag{3.12}$$

where T is the torsion tensor with respect to the induced connection ∇.

In the entire discussion of this article, we shall assume that the characteristic vector field ζ of $\bar{M}$ to be unit spacelike, without loss of generality. From the decomposition (2.3), ζ is decomposed as

$$\zeta = \omega + \alpha\xi + \beta N,$$

where ω is a vector field on $S(TM)$ and α and β are smooth functions given by $\alpha = \theta(N)$ and $\beta = \theta(\xi)$. Replacing X by ξ to (3.12) and using (2.8), we have

$$B(\xi, X) = -m\beta u(X). \tag{3.13}$$

From this, (2.10) and the fact that $S(TM)$ is non-degenerate, we have

$$A^*_\xi \xi = -m\beta V. \tag{3.14}$$

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a symmetric metric connection of type $(\ell,\, m)$. Then the second fundamental form B of M is symmetric if and only if $m = 0$.*

Proof. If $m = 0$, then, from (3.12), we see that B is symmetric. Conversely, if B is symmetric, then taking $X = \xi$, $Y = U$ and $X = V$, $Y = U$ to (3.12) by turns, we obtain $m\beta = 0$ and $m\theta(V) = 0$ respectively. As $u(\omega) = \theta(V)$ and $m\theta(V) = 0$, we get $mu(\omega) = 0$. Taking $X = \omega$ and $Y = U$ to (3.12), we get $m\theta(\omega) = mg(\omega, \omega) = 0$. Therefore, $m = m\bar{g}(\zeta, \zeta) = m\{g(\omega, \omega) + 2\alpha\beta\} = 0$. □

Denote by $\bar{R}$, R and R^* the curvature tensors of the symmetric metric connection $\bar{\nabla}$ of type (ℓ, m) on $\bar{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten formulas and (3.11), we obtain the Gauss equations for M and $S(TM)$ such that

$$
\begin{aligned}
\bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_N Y - B(Y,Z)A_N X \\
&+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\
&\quad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\
&\quad - \ell[\theta(X)B(Y,Z) - \theta(Y)B(X,Z)] \\
&\quad - m[\theta(X)B(FY,Z) - \theta(Y)B(FX,Z)]\}N,
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
R(X,Y)PZ = R^*(X,Y)PZ &+ C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\
&+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\
&\quad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\
&\quad - \ell[\theta(X)C(Y,PZ) - \theta(Y)C(X,PZ)] \\
&\quad - m[\theta(X)C(FY,PZ) - \theta(Y)C(FX,PZ)]\}\xi.
\end{aligned} \tag{3.16}
$$

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}.$$

In general, $R^{(0,2)}$ is not symmetric. It is well known that $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, *i.e.*, $d\tau = 0$ on TM [3, 4]. Therefore it has no geometric or physical meaning similar to the Ricci curvature of the non-degenerate submanifolds and it is just a tensor quantity. Hence we need the following definition: $R^{(0,2)}$ is called the *induced Ricci tensor* [4] of M if it is symmetric. The symmetric $R^{(0,2)}$ tensor will be denoted by Ric.

Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$. Let $\dim \bar{M} = n+2$ and $\epsilon_a = g(W_a, W_a)$. Using this quasi-orthonormal frame field, we obtain

$$R^{(0,2)}(X,Y) = \sum_{a=1}^{n} \epsilon_a \, g(R(W_a,X)Y,\, W_a) + \bar{g}(R(\xi,X)Y,\, N). \tag{3.17}$$

4. Indefinite complex space forms

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$
\begin{aligned}
\bar{R}(X,Y)Z \;=\; \frac{c}{4}\{&\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX \\
&- \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\}
\end{aligned} \tag{4.1}
$$

for any vector fields X, Y and Z of $\bar{M}$.

Comparing the tangential and transversal components of the two equations (3.15) and (4.1), and using (3.3), we get

$$R(X,Y)Z \;=\; \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \bar{g}(JY,Z)FX \tag{4.2}$$

$$- \bar{g}(JX, Z)FY + 2\bar{g}(X, JY)FZ\}$$
$$- B(X, Z)A_{N}Y + B(Y, Z)A_{N}X,$$

$$\begin{aligned}(4.3)\qquad & (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ & + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\ & - \ell\{\theta(X)B(Y, Z) - \theta(Y)B(X, Z)\} \\ & - m\{\theta(X)B(FY, Z) - \theta(Y)B(FX, Z)\} \\ = \ & \frac{c}{4}\{u(X)\bar{g}(JY, Z) - u(Y)\bar{g}(JX, Z) + 2u(Z)\bar{g}(X, JY)\}.\end{aligned}$$

Taking the scalar product with N to (3.16) and then, substituting (4.2) into the resulting equation and using $(2.10)_2$, we obtain

$$\begin{aligned}(4.4)\qquad & (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ & - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ) \\ & - \ell\{\theta(X)C(Y, PZ) - \theta(Y)C(X, PZ)\} \\ & - m\{\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)\} \\ = \ & \frac{c}{4}\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ) + v(X)g(FY, PZ) \\ & \quad - v(Y)g(FX, PZ) + 2v(PZ)\bar{g}(X, JY)\}.\end{aligned}$$

Definition 1. A lightlike hypersurface M is said to be *screen conformal* [1] if there exists a non-vanishing smooth function φ on any coordinate neighborhood $\mathcal{U}$ in M such that $A_{N} = \varphi A^{*}_{\xi}$, or equivalently,

$$(4.5)\qquad C(X, PY) = \varphi B(X, Y).$$

Theorem 4.1. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a symmetric metric connection of type (ℓ, m). If M is screen conformal, then $c = 0$ and $m\beta = 0$. Moreover, if $m = 0$, then the Ricci type tensor $R^{(0,2)}$ is a symmetric induced Ricci tensor of M.*

Proof. Assume that M is screen conformal. From (3.13), we get

$$B(\xi, U) = -m\beta, \qquad B(\xi, V) = 0.$$

From these two equations, (3.5) and (4.5), we have

$$-m\beta = B(\xi, U) = C(\xi, V) = \varphi B(\xi, V) = 0.$$

Put $\mu = U - \varphi V$. From (3.5) and (4.5), we show that

$$(4.6)\qquad B(X, \mu) = 0.$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this into (4.4) and using (4.3) with $Z = PZ$, we have

$$\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ)$$

$$= \frac{c}{4}\{\eta(X)g(Y,PZ) - \eta(Y)g(X,PZ) + 2[v(PZ) - \varphi u(PZ)]\bar{g}(X,JY) + [v(X) - \varphi u(X)]\bar{g}(FY,PZ) - [v(Y) - \varphi u(Y)]\bar{g}(FX,PZ)\}.$$

Taking $Y = \xi$ and $PZ = \mu$ and using (3.13), (4.6) and $m\beta = 0$, we have

$$\frac{c}{2}\{v(X) - 3\varphi u(X)\} = 0.$$

Replacing X by V to this equation and using (3.2), we obtain $c = 0$.

Substituting (4.2) into (3.17) and using (2.9) and (3.12), we have

$$R^{(0,2)}(X,Y) = B(X,Y)tr\,A_N - \varphi g(A^*_\xi X, A^*_\xi Y) + m\{u(A_N X)\theta(Y) - \theta(A_N X)u(Y)\}.$$

From this and Theorem 3.1, we see that if $m = 0$, then $R^{(0,2)}$ is symmetric. □

Definition 2. A screen distribution $S(TM)$ is said to be *totally umbilical* [3] if there exists a smooth function γ on $\mathcal{U}$ such that $A_N X = \gamma PX$, or equivalently,

$$C(X,PY) = \gamma g(X,Y). \tag{4.7}$$

In case $\gamma \neq 0$, we say that $S(TM)$ is *proper totally umbilical* in M.

Theorem 4.2. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a symmetric metric connection of type (ℓ, m). If $S(TM)$ is totally umbilical, then $c = 0$ and $m\beta = 0$. Moreover if $m\gamma = 0$, then M is flat and $S(TM)$ is totally geodesic. In case $S(TM)$ is proper totally umbilical, the local second fundamental form B of M is of the form*

$$B(X,Y) = m\{\theta(V)g(X,Y) - \theta(X)u(Y)\}. \tag{4.8}$$

Proof. Replacing X by U to (3.13) and using (3.5) and (4.7), we have

$$-m\beta = B(\xi,U) = C(\xi,V) = \gamma g(\xi,V) = 0.$$

Applying ∇_Z to (4.7) and using (2.11), we obtain

$$(\nabla_X C)(Y,PZ) = (X\gamma)g(Y,PZ) + \gamma B(X,PZ)\eta(Y).$$

Substituting this equation and (4.7) into (4.4), we have

$$\begin{aligned}
&\{X\gamma - \gamma\tau(X) - \gamma\ell\theta(X)\}g(Y,PZ) \\
&\quad - \{Y\gamma - \gamma\tau(Y) - \gamma\ell\theta(Y)\}g(X,PZ) \\
&\quad + \gamma\{B(X,PZ)\eta(Y) - B(Y,PZ)\eta(X)\} \\
&\quad + m\gamma\{\theta(Y)g(FX,PZ) - \theta(X)g(FY,PZ)\} \\
= {}& \frac{c}{4}\{\eta(X)g(Y,PZ) - \eta(Y)g(X,PZ) + v(X)g(FY,PZ) \\
&\quad - v(Y)g(FX,PZ) + 2v(PZ)\bar{g}(X,JY)\}.
\end{aligned}$$

Replacing Y by ξ this and using (3.2) and the fact that $m\beta = 0$, we have

$$\gamma B(X,PY) = \{\xi\gamma - \gamma\tau(\xi) - \gamma\ell\beta - \frac{c}{4}\}g(X,PY) \tag{4.9}$$

$$-\frac{c}{4}\{v(X)u(PY)+2u(X)v(PY)\}-m\gamma\theta(X)u(PY).$$

Taking $X=U$, $PY=V$ and $X=V$, $PY=U$ to (4.9) by turns, we have

$$\gamma B(U,V)=\xi\gamma-\gamma\tau(\xi)-\gamma\ell\beta-\frac{3}{4}c,$$

$$\gamma\{B(V,U)+m\theta(V)\}=\xi\gamma-\gamma\tau(\xi)-\gamma\ell\beta-\frac{2}{4}c. \tag{4.10}$$

On the other hand, taking $X=U$ and $Y=V$ to (3.12), we have

$$B(U,V)=B(V,U)+m\theta(V).$$

From the last three equations, we have $c=0$. From (3.5) and (4.7), we obtain

$$B(X,U)=\gamma u(X). \tag{4.11}$$

Replacing X by V, we obtain $B(V,U)=0$. From this and (4.10), we get

$$m\gamma\theta(V)=\xi\gamma-\gamma\tau(\xi)-\gamma\ell\beta. \tag{4.12}$$

From (4.9) and (4.12), we obtain

$$\gamma B(X,Y)=m\gamma\{\theta(V)g(X,Y)-\theta(X)u(Y)\}. \tag{4.13}$$

Substituting (4.13) into (4.2) with $c=0$, we have

$$\begin{aligned} R(X,Y)Z = \; & m\gamma\{\theta(V)[g(Y,Z)PX-g(X,Z)PY] \\ & +u(Z)[\theta(X)PY-\theta(Y)PX]\}. \end{aligned} \tag{4.14}$$

Taking $X=Y=U$ to (4.13) and using (4.11), we have

$$\gamma^2=-\,m\gamma\theta(U). \tag{4.15}$$

From (4.14) and (4.15), we see that if $m\gamma=0$, then M is flat and $S(TM)$ is totally geodesic. In case $S(TM)$ is proper totally umbilical. From (4.13), we have (4.8). Thus we have our theorem. □

From (4.14), we show that if M is screen totally geodesic, then M is flat. If $m=0$, then, from (4.8), (4.14) and (4.15), we see that $R=0$ and $\gamma=0$. Thus we have the following result.

Corollary 4.3. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a symmetric metric connection of type (ℓ, m) such that $S(TM)$ is totally umbilical. If B is symmetric, then M is flat and totally geodesic, and $S(TM)$ is also totally geodesic.*

Theorem 4.4. *Let M be a lightlike hypersurfaces of an indefinite almost complex space form with a symmetric metric connection of type (ℓ, m). If $S(TM)$ is proper totally umbilical, then $R^{(0,2)}$ is symmetric if and only if $m=0$.*

Proof. Substituting (4.14) into (3.17) and using (4.8), we obtain

$$R^{(0,2)}(X,Y)=\gamma(n-1)B(X,Y),$$

due to $m\beta=0$. This result implies that $R^{(0,2)}$ is symmetric if and only if B is symmetric. Thus, by Theorem 3.1, we have our theorem. □

5. Recurrent and Lie recurrent lightlike hypersurfaces

Definition 3. The structure tensor field F of M is said to be *recurrent* [8] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY. \tag{5.1}$$

A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 5.1. *Let M be a recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a symmetric metric connection of type (ℓ, m). Then*

1. *F is parallel with respect to the induced connection ∇ on M,*
2. *D and $J(tr(TM))$ are parallel distributions on M,*
3. *M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of the distribution D.*
4. *If M is screen conformal, then M and $S(TM)$ are totally geodesic.*
5. *If $\bar{M} = \bar{M}(c)$, then $c = 0$, i.e., $\bar{M}(c)$ is flat, and M is also flat.*

Proof. (1) From (3.8) and (5.1), we get

$$\varpi(X)FY = u(Y)A_N X - B(X, Y)U. \tag{5.2}$$

Replacing Y by ξ and using (2.8), (3.2) and the fact that $F\xi = -V$, we get $\varpi(X)V = 0$. Taking the scalar product with U to this, we obtain $\varpi = 0$. It follows that $\nabla_X F = 0$. Therefore, F is parallel with respect to ∇.

(2) Replacing Y by U to (5.2) such that $\varpi = 0$, we get $A_N X = \sigma(X)U$. Taking the scalar product with V to (5.2), we have $B(X, Y) = u(Y)\sigma(X)$, *i.e.*,

$$g(A^*_\xi X, Y) = g(\sigma(X)V, Y).$$

As $S(TM)$ is non-degenerate, we get $A^*_\xi X = \sigma(X)V$. Therefore,

$$A^*_\xi X = \sigma(X)V, \qquad A_N X = \sigma(X)U. \tag{5.3}$$

In general, by using (2.1), (2.4), (2.9), (3.3) and (3.7), we derive

$$g(\nabla_X \xi, V) = -B(X, V), \; g(\nabla_X V, V) = 0, \; g(\nabla_X Z, V) = B(X, FZ) \tag{5.4}$$

for all $X \in \Gamma(TM)$ and $Z \in \Gamma(D_o)$. Taking the scalar product with V and $Z \in \Gamma(D_o)$ to $(5.3)_1$ by turns, we have $B(X, V) = 0$ and $B(X, Z) = 0$ for all $X \in \Gamma(TM)$ respectively. It follow from (5.4) that

$$\nabla_X Y \in \Gamma(D), \quad \forall\, X \in \Gamma(TM), \;\; \forall\, Y \in \Gamma(D),$$

due to $FZ \in \Gamma(D_o)$ for $Z \in \Gamma(D_o)$. Thus D is a parallel distribution on M.

Applying F to $(5.3)_2$ and using the fact that $FU = 0$, we get

$$F(A_N X) = \sigma(X)FU = 0.$$

Thus, from (3.6), we obtain

$$\nabla_X U \in \Gamma(J(tr(TM))), \quad \forall\, X \in \Gamma(TM).$$

Thus $J(tr(TM))$ is also a parallel distribution on M.

(3) As D and $J(tr(TM))$ are parallel distributions satisfying (3.1). By the decomposition theorem [2], M is locally a product manifold $\mathcal{C}_u \times M^\sharp$, where $\mathcal{C}_u$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is the leaf of D.

(4) If M is screen conformal, then, from $(5.3)_{1,2}$ and $A_N = \varphi A^*_\xi$, we have

$$\sigma(X)U = \varphi\sigma(X)V.$$

Taking the scalar product with V to this, we have $\sigma = 0$. Thus, by $(5.3)_{1,2}$, we get $A^*_\xi = 0$ and $A_N = 0$. Thus M and $S(TM)$ are totally geodesic.

(5) Taking the scalar product with U to $(5.3)_2$, we have

$$C(Y, U) = 0.$$

Applying ∇_X to this and using (3.6), $(5.3)_2$ and the fact that $FU = 0$, we have

$$(\nabla_X C)(Y, U) = 0.$$

Replacing PZ by U to (4.4) and using the last two equations, we obtain

$$\frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$ to this, we have $c = 0$ and $\bar{M}(c)$ is flat.

Substituting $(5.3)_{1,2}$ into (4.2) satisfying $c = 0$, we get

$$R(X, Y)Z = \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}u(Z)U = 0.$$

Therefore $R = 0$ and M is also flat. □

Definition 4. The structure tensor field F of M is said to be *Lie recurrent* [8] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY, \tag{5.5}$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{5.6}$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$. A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 5.2. *Let M be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a symmetric metric connection of type (ℓ, m). Then*

(1) *F is Lie parallel.*
(2) *If $\ell = 0$ or $m = 1$, then $\tau = 0$.*
(3) *In case $\bar{M} = \bar{M}(c)$, if one of the lengths of $A_N\xi$, $A_N V$ and $A^*_\xi U$ is zero, then $c = 0$ and $\bar{M}(c)$ is flat.*

Proof. (1) Using (3.4), (3.8), (3.11), (5.5) and (5.6), we get

$$\begin{aligned} \vartheta(X)FY = \ & u(Y)A_N X - B(X, Y)U - \nabla_{FY}X + F\nabla_Y X \\ & + \ell\{\theta(Y)FX - \theta(FY)X\} \\ & - m\{\theta(Y)X + \theta(FY)FX - \theta(Y)u(X)U\}. \end{aligned} \tag{5.7}$$

Taking $Y = \xi$ to (5.7) and using (2.8), (3.2) and $F\xi = -V$, we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \ell\{\beta FX + \theta(V)X\} - m\{\beta X - \theta(V)FX - \beta u(X)U\}. \tag{5.8}$$

Taking the scalar product with V to (5.8) and using $g(FX, V) = 0$, we have

$$u(\nabla_V X) + \ell\theta(V)u(X) = 0. \tag{5.9}$$

Replacing X by U to this and using (3.6), we obtain

$$\tau(V) + \ell\theta(V) = 0. \tag{5.10}$$

Replacing Y by V to (5.7) and using the fact that $FV = \xi$, we have

$$\vartheta(X)\xi = -B(X, V)U - \nabla_\xi X + F\nabla_V X + \ell\{\theta(V)FX - \beta X\} - m\{\theta(V)X + \beta FX - \theta(V)u(X)U\}. \tag{5.11}$$

Applying F to (5.11) and using (3.4) and (5.9), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \ell\{\beta FX + \theta(V)X\} - m\{\beta X - \theta(V)FX - \beta u(X)U\}.$$

Comparing this with (5.8), we obtain $\vartheta = 0$. Thus F is Lie parallel.

(2) Replacing X by ξ to (5.8) and using (2.7) and (3.14), we have

$$A^*_\xi V = -\{\tau(V) - \ell\theta(V)\}\xi + \{\tau(\xi) - \ell\beta - m\theta(V)\}V.$$

Taking the scalar product with U and N by turns, we get

$$C(V, V) = \tau(\xi) - \ell\beta - m\theta(V), \qquad \tau(V) - \ell\theta(V) = 0, \tag{5.12}$$

respectively. From (5.10) and $(5.12)_2$, we see that

$$\tau(V) = 0, \qquad \ell\theta(V) = 0. \tag{5.13}$$

Taking the scalar product with V to (5.11) and using $g(FX, V) = 0$, we get

$$B(X, V) + g(\nabla_\xi X, V) + \ell\beta u(X) = 0.$$

Replacing X by U to this and using (3.6), we obtain

$$B(U, V) = -\tau(\xi) - \ell\beta.$$

Taking $X = U$ and $Y = V$ to (3.12) and using (3.5) and $(5.12)_1$, we have

$$B(U, V) = \tau(\xi) - \ell\beta.$$

Comparing the last two equations, we obtain $\tau(\xi) = 0$. We have

$$A^*_\xi V = -\{\ell\beta + m\theta(V)\}V. \tag{5.14}$$

Taking $X = U$ and $Y = \xi$ to (5.7) and using (3.4), (3.6) and (3.12), we get

$$A_N\xi = F(A_N V) - m\beta U. \tag{5.15}$$

Taking $X = U$ to (5.7) and using (3.4), (3.5), (3.6) and (3.12), we get

$$u(Y)A_N U - F(A_N FY) - A_N Y - \{\tau(FY) + \ell\theta(FY) + m\theta(Y) - m\theta(U)u(Y)\}U = 0. \tag{5.16}$$

Taking $Y = V$ to (5.16) and using the fact that $FV = \xi$, we obtain

$$A_{N}V = -F(A_{N}\xi) - \{\ell\beta + m\theta(V)\}U. \tag{5.17}$$

Taking the scalar product with U to this and using, we have

$$C(V, U) = 0. \tag{5.18}$$

Replacing Y by U to (5.7) and using the fact $FU = 0$, we have

$$A_{N}X = B(X, U)U - F\nabla_{U}X + \theta(U)\{mX - mu(X)U - \ell FX\}. \tag{5.19}$$

Taking $X = \xi$ to this and using the fact that $m\beta = 0$, we have

$$A_{N}\xi = F(A^{*}_{\xi}U) + m\theta(U)\xi - m\beta U - [\tau(U) - \ell\theta(U)]V. \tag{5.20}$$

Taking the scalar product with N to this equation, we have

$$m\theta(U) = 0. \tag{5.21}$$

Comparing (5.15) and (5.20), we obtain

$$F(A_{N}V) = F(A^{*}_{\xi}U) - \{\tau(U) - \ell\theta(U)\}V.$$

Taking the scalar product with U to this equation, we have $\tau(U) - \ell\theta(U) = 0$. Applying F to the last equation and using (3.12), we have

$$A^{*}_{\xi}U = A_{N}V + m\theta(V)U. \tag{5.22}$$

Taking the scalar product with X to (5.22) and using (3.5) and (3.12), we have

$$C(V, X) - C(X, V) = m\{\theta(X) - \theta(V)v(X)\}.$$

Replacing X by U to this and using (5.18) and (5.21), we get

$$C(U, V) = 0. \tag{5.23}$$

Taking the scalar product with V to (5.16) and using (5.23), we have

$$B(Y, U) = -\tau(FY) - \ell\theta(FY) - m\theta(Y). \tag{5.24}$$

Taking $X = V$ to (5.7) and using (3.4), (3.7), (3.12) and $FV = \xi$, we obtain

$$\begin{aligned} &u(Y)A_{N}V - F(A^{*}_{\xi}FY) - A^{*}_{\xi}Y + m\theta(V)u(Y)U \\ &- \{\tau(Y) - \ell\theta(Y) + m\theta(FY)\}\xi + \{\tau(FY) - \ell\theta(FY) - m\theta(Y)\}V = 0. \end{aligned}$$

Taking the scalar product with U and using $C(V, U) = 0$, we have

$$B(Y, U) = \tau(FY) - \ell\theta(FY) - m\theta(Y).$$

Comparing this equation with (5.24), we see that $\tau(FX) = 0$. Replacing X by FY to this and using (3.4), we have

$$\tau(Y) = \ell\theta(U)u(Y).$$

If $\ell = 0$, then $\tau = 0$. Also if $m = 1$, then $\theta(U) = 0$ by (5.21). Thus $\tau = 0$.

(3) Put $f = C(U, U)$. Taking the scalar product with U to (5.16), we have

$$C(Y, U) = f\,u(Y).$$

Applying ∇_X to this and using (3.6) and (3.9), we have

$$(\nabla_X C)(Y,U) = (Xf)u(Y) - 2fu(Y)\tau(X) - fB(X,FY) \\ - g(A_{\scriptscriptstyle N}Y, F(A_{\scriptscriptstyle N}X)).$$

Substituting the last two equations into (4.4), we have

$$\begin{aligned} &(Xf)u(Y) - (Yf)u(X) + 2g(A_{\scriptscriptstyle N}X, F(A_{\scriptscriptstyle N}Y)) \\ &+f\{3u(X)\tau(Y) - 3u(Y)\tau(X) - B(X,FY) + B(Y,FX)\} \\ &- f\ell\{\theta(X)u(Y) - \theta(Y)u(X)\} \\ = \ &\frac{c}{2}\{\eta(X)v(Y) - \eta(Y)v(X)\}. \end{aligned}$$

Taking $X = \xi$ and $Y = V$ to this equation, we have

$$-2g(A_{\scriptscriptstyle N}V, F(A_{\scriptscriptstyle N}\xi)) - f\{B(\xi,\xi) + B(V,V)\} = \frac{c}{2}.$$

Using (3.13), (5.14), (5.15), (5.17) and (5.22), we obtain

$$g(A^*_\xi U, A^*_\xi U) = g(A_{\scriptscriptstyle N}V, A_{\scriptscriptstyle N}V) = g(A_{\scriptscriptstyle N}\xi, A_{\scriptscriptstyle N}\xi) = \frac{c}{4}.$$

Thus if one of the lengths of $A_{\scriptscriptstyle N}\xi$, $A_{\scriptscriptstyle N}V$ and $A^*_\xi U$ is zero, then $c = 0$. □

Definition 5. The Jacobi operator on M with respect to the vector field X is defined by $R(\cdot\,, X)X$. In case $X = U$, the Jacobi operator is called *structure Jacobi operator* and it is denoted by $\phi = R(\cdot\,, U)U$.

Theorem 5.3. *Let M be a Lie recurrent lightlike hypersurfaces of an indefinite complex space form $\bar{M}(c)$ with a symmetric metric connection of type (ℓ, m). If the structure Jacobi operator ϕ is satisfied $\phi = 0$, then $c = 0$.*

Proof. Taking $Y = Z = U$ to (4.2) and using (5.24), we have

$$\phi(X) = -\frac{c}{4}v(X)U - B(X,U)A_{\scriptscriptstyle N}U.$$

As $\phi = 0$, taking the scalar product with V to the last equation and using (5.23), we obtain $\frac{c}{4}v(X) = 0$. Replacing X by V to this, we get $c = 0$. □

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