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ON DEGENERATE q-BERNOULLI POLYNOMIALS

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Abstract. In this paper, we introduce the degenerate Carlitz q-Bernoulli numbers and polynomials and give some interesting identities and properties of these numbers and polynomials which are derived from the generating functions and p-adic integral equations.

1. Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p < 1$. We use the notation $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

In [2], L. Carlitz considered q-Bernoulli numbers as follows:

$$\beta_{0,q} = 1,\ q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.1}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$. The q-Bernoulli polynomials are defined by

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,q} q^l [x]_q^{n-l} \quad \text{(see [2, 8])}. \tag{1.2}$$

In [4, 3], L. Carlitz defined the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda)\frac{t^n}{n!} \quad \text{(see [2, 5])}. \tag{1.3}$$

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When $x=0$, $\beta_n(\lambda)=\beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers.* Note that $\lim_{\lambda\to 0}\beta_n(x|\lambda)=B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials (see [1-12]). Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in UD(\mathbb{Z}_p)$, the *p-adic q-integral on* $\mathbb{Z}_p$ is defined by

$$I_q(f)=\int_{\mathbb{Z}_p} f(x)d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x \quad \text{(see [8]).} \tag{1.4}$$

The Carlitz's q-Bernoulli polynomials can be represented by p-adic q-integrals on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}[x+y]_q^n d\mu_q(y)=\beta_{n,q}(x) \quad (n\geq 0). \tag{1.5}$$

Thus, by (1.4), we get

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t}d\mu_q(y)=\sum_{n=0}^{\infty}\beta_{n,q}(x)\frac{t^n}{n!} \quad \text{(see [8]).} \tag{1.6}$$

From (1.6), we can derive the following equation:

$$\beta_{m,q}(x)=\frac{1}{(1-q)^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^j q^{jx}\frac{j+1}{[j+1]_q} \quad (m\geq 0). \tag{1.7}$$

In this paper, we introduce the degenerate Carlitz q-Bernoulli numbers and polynomials and give some interesting identities and properties of these numbers and polynomials which are derived from the generating functions and p-adic integral equations on $\mathbb{Z}_p$.

2. Degenerate Carlitz q-Bernoulli numbers and polynomials

In this section, we assume that $\lambda, t\in\mathbb{C}_p$ with $0<|\lambda|_p\leq 1$, $|t|_p<p^{-\frac{1}{p-1}}$. Then, as $|\lambda t|_p<p^{-\frac{1}{p-1}}$, $|\log(1+\lambda t)|_p=|\lambda t|_p$ and hence $|\frac{1}{\lambda}\log(1+\lambda t)|_p=|t|_p<p^{-\frac{1}{p-1}}$ and now it makes sense to take the limit as $\lambda\to 0$.

In the viewpoint of (1.3), we consider the *degenerate Carlitz q-Bernoulli polynomials* which are given by the generating function to be

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_q}d\mu_q(y)=\sum_{n=0}^{\infty}\beta_{n,q}(x|\lambda)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$, $\beta_{n,q}(\lambda)=\beta_{n,q}(0|\lambda)$ are called the *degenerate Carlitz q-Bernoulli numbers.*

Now, we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x+y]_q}{\lambda}}d\mu_q(y)&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{[x+y]_q}{\lambda}}{n}d\mu_q(y)\lambda^n t^n\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_q(y)\lambda^n\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

where $\left(\frac{[x+y]_q}{\lambda}\right)_n = \frac{[x+y]_q}{\lambda} \times \left(\frac{[x+y]_q}{\lambda} - 1\right) \times \cdots \times \left(\frac{[x+y]_q}{\lambda} - n + 1\right)$.

Now, we define $[x+y]_{n,\lambda}$ as $[x+y]_{0,\lambda} = 1$,

$$[x+y]_{n,\lambda} = [x+y]_q \left([x+y]_q - \lambda\right) \cdots \left([x+y]_q - (n-1)\lambda\right) \quad (n \geq 1). \tag{2.3}$$

Therefore, by (2.1), (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} [x+y]_{n,\lambda} d\mu_q(y) = \beta_{n,q}(x|\lambda).$$

Let $S_1(n,m)$ be the *Stirling numbers of the first kind* which are defined by $(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l$, $(n \geq 0)$. Then, by (2.2), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_q(y) &= \sum_{l=0}^{n} S_1(n,l)\lambda^{-l} \int_{\mathbb{Z}_p} [x+y]_q^l d\mu_q(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{-l}\beta_{l,q}(x). \end{aligned} \tag{2.4}$$

Therefore, by (2.2) and (2.4), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\beta_{l,q}(x).$$

Note that $\lim_{\lambda \to 0} \beta_{n,q}(x|\lambda) = \beta_{n,q}(x)$.

Corollary 2.3. *For $n \geq 0$, we have*

$$\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} \sum_{j=0}^{l} \frac{S_1(n,l)}{(1-q)^l} \binom{l}{j} (-1)^j q^{jx} \frac{j+1}{[j+1]_q} \lambda^{n-l}.$$

We observe that

$$\begin{aligned} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} &= e^{\frac{[x+y]_q}{\lambda}\log(1+\lambda t)} = \sum_{n=0}^{\infty} \left(\frac{[x+y]_q}{\lambda}\right)^n \frac{1}{n!} \left(\log(1+\lambda t)\right)^n \\ &= \sum_{m=0}^{\infty} \left(\frac{[x+y]_q}{\lambda}\right)^m \frac{1}{m!} m! \sum_{n=m}^{\infty} S_1(n,m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^{n-m} S_1(n,m)[x+y]_q^m \right) \frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

Thus, by (2.5), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x+y]_q}{\lambda}}d\mu_q(y) &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}[x+y]_q^m d\mu_q(x)\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\beta_{m,q}(x)\right)\frac{t^n}{n!}.\end{aligned}$$

Replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (2.1), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}e^{[x+y]_q t}d\mu_q(y) &= \sum_{m=0}^{\infty}\beta_{m,q}(x|\lambda)\frac{1}{m!}\frac{1}{\lambda^m}\left(e^{\lambda t}-1\right)^m\\ &= \sum_{m=0}^{\infty}\beta_{m,q}(x|\lambda)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\beta_{m,q}(x|\lambda)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!},\end{aligned}\tag{2.6}$$

where $S_2(n,m)$ are the Stirling numbers of the second kind.

We note that the left hand side of (2.6) is given by

$$\begin{aligned}\int_{\mathbb{Z}_p}e^{[x+y]_q t}d\mu_q(y) &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}[x+y]_q^n d\mu_q(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\beta_{n,q}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\beta_{n,q}(x) = \sum_{m=0}^{n}\beta_{m,q}(x|\lambda)\lambda^{n-m}S_2(n,m).$$

Note that

$$\begin{aligned}(1+\lambda t)^{\frac{[x+y]_q}{\lambda}} &= (1+\lambda t)^{\frac{[x]_q}{\lambda}}(1+\lambda t)^{\frac{q^x[y]_q}{\lambda}}\\ &= \left(\sum_{m=0}^{\infty}[x]_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{q^{lx}}{\lambda^l}\frac{[y]_q^l\left(\log(1+\lambda t)\right)^l}{l!}\right)\\ &= \left(\sum_{m=0}^{\infty}[x]_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\lambda^{k-l}q^{lx}[y]_q^l S_1(k,l)\right)\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}[x]_{n-k,\lambda}\lambda^{k-l}q^{lx}[y]_q^l S_1(k,l)\binom{n}{k}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.8}$$

Thus, by (2.8), we get

(2.9)
$$\sum_{n=0}^{\infty}\beta_{n,q}(x|\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}[x]_{n-k,\lambda}\lambda^{k-l}q^{lx}\int_{\mathbb{Z}_p}[y]_q^l d\mu_q(y)S_1(k,l)\binom{n}{k}\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}[x]_{n-k,\lambda}\lambda^{k-l}q^{lx}\beta_{l,q}S_1(k,l)\right)\frac{t^n}{n!}.$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.5. *For $n\geq 0$, we have*

$$\beta_{n,q}(x|\lambda)=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}[x]_{n-k,\lambda}\lambda^{k-l}q^{lx}S_1(k,l)\beta_{l,q}.$$

For $r\in\mathbb{N}$, we define the *degenerate Carlitz q-Bernoulli polynomials of order r* as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x_1+\cdots+x_r+x]_q}{\lambda}}d\mu_q(x_1)\cdots d\mu_q(x_r)=\sum_{n=0}^{\infty}\beta_{n,q}^{(r)}(x|\lambda)\frac{t^n}{n!}. \tag{2.10}$$

We observe that

(2.11)
$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x_1+\cdots+x_r+x]}{\lambda}}d\mu_q(x_1)\cdots d\mu_q(x_r)$$
$$=\sum_{m=0}^{\infty}\lambda^{-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r+x]_q^m d\mu_q(x_1)\cdots d\mu_q(x_r)\frac{1}{m!}\left(\log(1+\lambda t)\right)^m$$
$$=\sum_{m=0}^{\infty}\beta_{m,q}^{(r)}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n}{n!}t^n$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}\beta_{m,q}^{(r)}(x)S_1(n,m)\right)\frac{t^n}{n!},$$

where $\beta_{m,q}^{(r)}(x)$ are the Carlitz q-Bernoulli polynomials of order r.

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.6. *For $n\geq 0$, we have*

$$\beta_{n,q}^{(r)}(x|\lambda)=\sum_{m=0}^{n}\lambda^{n-m}\beta_{m,q}^{(r)}(x)S_1(n,m).$$

Replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (2.10), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]_q t} d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{m=0}^{\infty}\beta_{m,q}^{(r)}(x|\lambda)\frac{1}{m!}\lambda^{-m}\left(e^{\lambda t}-1\right)^m\\
&=\sum_{m=0}^{\infty}\beta_{m,q}^{(r)}(x|\lambda)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}\beta_{m,q}^{(r)}(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.12}$$

The left hand side of (2.12) is given by

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_r+x]_q t} d\mu_q(x_1)\cdots d\mu_q(x_r)=\sum_{n=0}^{\infty}\beta_{n,q}^{(r)}(x)\frac{t^n}{n!}.\tag{2.13}$$

By comparing the coefficients on the both sides of (2.12) and (2.13), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$\beta_{m,q}^{(r)}(x)=\sum_{m=0}^{n}\lambda^{n-m}S_2(n,m)\beta_{m,q}^{(r)}(x|\lambda).$$

We recall that

$$\begin{aligned}
\int_{\mathbb{Z}_p} f(x)d\mu_q(x)&=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}f(x)q^x\\
&=\lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{x=0}^{dp^N-1}f(x)q^x,
\end{aligned}$$

where $d\in\mathbb{N}$ and $f\in UD(\mathbb{Z}_p)$.

Now, we observe that

$$\beta_{n,q}(x|\lambda)=\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\int_{\mathbb{Z}_p}[x+y]_q^l d\mu_q(y),\tag{2.14}$$

and

$$\begin{aligned}
\int_{\mathbb{Z}_p}[x+y]_q^l d\mu_q(y)&=\frac{1}{[m]_q}\sum_{i=0}^{m-1}q^i[m]_q^l\int_{\mathbb{Z}_p}\left[\frac{x+i}{m}+y\right]_{q^m}^l d\mu_{q^m}(y)\\
&=[m]_q^{l-1}\sum_{i=0}^{m-1}q^i\beta_{l,q^m}\left(\frac{x+i}{m}\right),
\end{aligned}\tag{2.15}$$

where $l\in\mathbb{Z}_{\geq 0}$ and $m\in\mathbb{N}$.

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.8. *For $n \geq \mathbb{N}_{\geq 0}$, $m \in \mathbb{N}$, we have*

$$\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} \sum_{i=0}^{m-1} S_1(n,l)\lambda^{n-l}[m]_q^{l-1} q^i \beta_{l,q^m}\left(\frac{x+i}{m}\right).$$

From (1.4), we note that

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f^{'}(0), \tag{2.16}$$

where $f^{'}(0) = \frac{df(x)}{dx}\Big|_{x=0}$.

By (2.16), we get

$$q\beta_{n,q}(x+1|\lambda) - \beta_{n,q}(x|\lambda) = (q-1)\lambda^n \left(\frac{[x]_q}{\lambda}\right)_n + \sum_{l=1}^{n} S_1(n,l)\lambda^{n-l} l [x]_q^{l-1} q^x, \tag{2.17}$$

where $n \in \mathbb{N}$.

Therefore, by (2.17), we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$q\beta_{n,q}(x+1|\lambda) - \beta_{n,q}(x|\lambda) = (q-1)\lambda^n \left(\frac{[x]_q}{\lambda}\right)_n + \sum_{l=1}^{n} S_1(n,l)\lambda^{n-l} l [x]_q^{l-1} q^x.$$

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