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NOTE ON LOCAL ESTIMATES FOR WEAK SOLUTION OF BOUNDARY VALUE PROBLEM FOR SECOND ORDER PARABOLIC EQUATION

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Abstract. The aim of this note is to provide detailed proofs for local estimates near the boundary for weak solutions of second order parabolic equations in divergence form with time-dependent measurable coefficients subject to Neumann boundary condition. The corresponding parabolic equations with Dirichlet boundary condition are also considered.

1. Introduction and main results

Local boundedness and local Hölder continuity for weak solutions of elliptic or parabolic equations with bounded measurable coefficients are very well known and usually referred to as De Giorgi-Moser-Nash theory. There are a large number of references regarding this theory. One of most popular reference for elliptic equations is a book by Gilbarg and Trudinger [4]. We also refer the reader to [3] for elliptic equations with Dirichlet boundary condition as well as Neumann boundary condition. Recently, in [5] the author provide a detailed proof for local boundedness estimate near the boundary for weak solutions of Neumann problem for elliptic equation. The corresponding result for parabolic equation with Neumann boundary condition is of course well known to expert. However, it is very hard to locate a specific reference in the existing literature. There is rich literature discussing conormal boundary conditions, for example [6, 7], but none of them contains the exact local boundedness estimate and local Hölder estimate.

In this note, we give detailed proofs for local boundedness estimates and local Hölder estimates near the boundary for weak solutions of second order parabolic equations in divergence form with bounded measurable coefficients subject to Neumann boundary condition. We also consider the local estimates for weak solutions of parabolic equations with Dirichlet boundary condition.

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Throughout the note (except Theorem 1.5), we use Ω to denote a Sobolev extension domain in $\mathbb{R}^d$ ($d \geq 2$); i.e., there exists a linear operator $E : W^1_2(\Omega) \to W^1_2(\mathbb{R}^d)$ such that for any $u \in W^1_2(\Omega)$ we have

$$\|Eu\|_{L_2(\mathbb{R}^d)} \leq \mathcal{E}_0 \|u\|_{L_2(\Omega)} \quad \text{and} \quad \|Eu\|_{W^1_2(\mathbb{R}^d)} \leq \mathcal{E}_0 \|u\|_{W^1_2(\Omega)}, \tag{1.1}$$

where $W^1_2(\Omega)$ is the usual Sobolev space. Such domains include bounded Lipschitz domains, Lipschitz graph domains, and locally uniform domains (see Rogers [8]). We let

$$Q = \Omega \times (a, b) \quad \text{and} \quad S = \partial\Omega \times (a, b),$$

where $-\infty < a < b < \infty$. For any $(x, t) \in \Omega \times (a, b)$ and $r > 0$, we write

$$\begin{aligned}
&\Omega_r(x) = \Omega \cap B_r(x),\\
&Q_r(x,t) = Q \cap (B_r(x) \times (t - r^2, t)),\\
&S_r(x,t) = S \cap Q_r(x,t),
\end{aligned}$$

where $B_r(x)$ is the usual Euclidean ball of radius r centered at x.

To avoid confusion, spaces of functions defined on $Q = \Omega \times (a, b) \subset \mathbb{R}^{d+1}$ will be always written in *script letters* throughout the note. We write $u \in \mathscr{C}^\infty_c(\overline{Q})$ if u is an infinitely differentiable function on $\mathbb{R}^{d+1}$ with a compact support in $\overline{Q}$. For $p, q \geq 1$, we let $\mathscr{L}_{p,q}(Q)$ is the Banach space consisting of all measurable functions on Q with a finite norm

$$\|u\|_{\mathscr{L}_{p,q}(Q)} = \left(\int_a^b \left(\int_\Omega |u(x,t)|^p \, dx \right)^{q/p} dt \right)^{1/q}.$$

$\mathscr{L}_{p,p}(Q)$ will be denoted by $\mathscr{L}_p(Q)$. By $\mathscr{C}^{\alpha,\alpha/2}(Q)$, $\alpha \in (0, 1]$, we denote the set of all bounded measurable functions u on Q for which $|u|_{\alpha,\alpha/2;Q}$ is finite, where we define the parabolic Hölder norm as follows:

$$\begin{aligned}
|u|_{\alpha,\alpha/2;Q} &= [u]_{\alpha,\alpha/2;Q} + |u|_{0;Q}\\
&= \sup_{\substack{(x,t),(y,s)\in Q\\ (x,t)\neq(y,s)}} \frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}} + \sup_{(x,t)\in Q} |u(x,t)|.
\end{aligned}$$

The space $\mathscr{W}^{1,0}_p(Q)$ denotes the Banach space with the norm

$$\|u\|_{\mathscr{W}^{1,0}_p(Q)} = \|u\|_{\mathscr{L}_p(Q)} + \|D_x u\|_{\mathscr{L}_p(Q)}$$

and $\mathscr{W}^{1,1}_p(Q)$ denotes the Banach space with the norm

$$\|u\|_{\mathscr{W}^{1,1}_p(Q)} = \|u\|_{\mathscr{L}_p(Q)} + \|D_x u\|_{\mathscr{L}_p(Q)} + \|u_t\|_{\mathscr{L}_p(Q)}.$$

The space $\mathscr{V}^{1,0}_2(Q)$ is obtained by completing the set $\mathscr{W}^{1,1}_2(Q)$ with the norm

$$\|u\|_{\mathscr{V}^{1,0}_2(Q)} = \max_{a\leq t\leq b} \|u(\cdot,t)\|_{L_2(\Omega)} + \|D_x u\|_{\mathscr{L}_2(Q)}.$$

We consider the parabolic operator

$$\mathcal{P}u = u_t - D_i(A^{ij} D_j u + A^i u) + B^i D_i u + Cu.$$

Here, the leading coefficients $A^{ij} = A^{ij}(x,t)$ are bounded measurable functions defined on $\mathbb{R}^{d+1}$ such that for any $(x,t) \in \mathbb{R}^{d+1}$ and ξ, $\eta \in \mathbb{R}^d$, we have

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} A^{ij}(x,t)\xi_i\xi_j \quad \text{and} \quad \sum_{i,j=1}^{d} |A^{ij}(x,t)\xi_i\eta_j| < \lambda^{-1}|\xi||\eta|, \tag{1.2}$$

where λ is a positive constant. We denote

$$\boldsymbol{A} = (A^1, \ldots, A^d) \quad \text{and} \quad \boldsymbol{B} = (B^1, \ldots, B^d),$$

and let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathscr{L}_1(Q)^d$ and $C \in \mathscr{L}_1(Q)$. For $\boldsymbol{F} = (F^1, \ldots, F^d) \in \mathscr{L}_1(Q)^d$ and $f \in \mathscr{L}_1(Q)$, we say that $u \in \mathscr{V}_2^{1,0}(Q)$ is a weak solution of the problem

$$\begin{cases} \mathcal{P}u = \operatorname{div} \boldsymbol{F} + f \quad \text{in } \ Q, \\ (A^{ij}D_j u + A^i u + F^i)n_i = 0 \quad \text{on } \ S, \end{cases}$$

if u satisfies for all $t_1 \in [a,b]$ that

(1.3)
$$\int_\Omega u(x,t_1)v(x,t_1)\,dx - \int_a^{t_1}\int_\Omega uv_t\,dx\,dt + \int_a^{t_1}\int_\Omega (A^{ij}D_j u + A^i u)D_i v\,dx\,dt$$
$$+ \int_a^{t_1}\int_\Omega (B^i D_i u + Cu)v\,dx\,dt = \int_a^{t_1}\int_\Omega -F^i D_i v + fv\,dx\,dt$$

for all $v \in \mathscr{C}_c^\infty(\overline{Q})$ that are equal to zero for $t = a$.

1.1. Boundedness of solutions

The first main result is about the local boundedness up to the boundary for weak solutions of

$$\begin{cases} \mathcal{P}u = \operatorname{div} \boldsymbol{F} + f \quad \text{in } \ Q, \\ (A^{ij}D_j u + A^i u + F^i)n_i = 0 \quad \text{on } \ S. \end{cases} \tag{1.4}$$

Theorem 1.1. *Let $Q = \Omega \times (a,b)$, where Ω is a Sobolev extension domain in $\mathbb{R}^d$. Assume that*

$$\begin{aligned} &\mathcal{D} := \big\| |\boldsymbol{A}| + |\boldsymbol{B}| + |C|^{1/2} \big\|_{\mathscr{L}_{p_0,q_0}(Q)} < \infty, \\ &\mathcal{M} := \|\boldsymbol{F}\|_{\mathscr{L}_{p_1,q_1}(Q)} + \|f\|_{\mathscr{L}_{p_2,q_2}(Q)} < \infty, \\ &\begin{cases} p_0 > d, \quad q_0 > 2, \\ p_1 > d, \quad q_1 > 2, \\ p_2 > \dfrac{d}{2}, \quad q_2 > 1, \end{cases} \quad \textit{and} \quad \frac{d}{p_{\min}} + \frac{2}{q_{\min}} < 1, \end{aligned} \tag{1.5}$$

where we use the notation

$$p_{\min} = \min(p_0, p_1, 2p_2) \quad \textit{and} \quad q_{\min} = \min(q_0, q_1, 2q_2)\,.$$

If $u \in \mathscr{V}_2^{1,0}(Q)$ is a weak solution of (1.4)*, then there exists a constant $0 < R_0 \le \min\left(1, \sqrt{b-a}\right)$, depending only on d, λ, $\mathcal{E}_0$, p_i, q_i, and $\mathcal{D}$, such that for any $x_0 \in \overline{\Omega}$ and $0 < r \le R_0$, we have*

$$\begin{aligned} \|u\|_{\mathscr{L}_\infty(Q_{r/2})} \le Nr^{-\frac{d+2}{2}}\|u\|_{\mathscr{L}_2(Q_r)} + Nr^{1-\frac{d}{p_1}-\frac{2}{q_1}}\|\boldsymbol{F}\|_{\mathscr{L}_{p_1,q_1}(Q_r)} \\ + Nr^{2-\frac{d}{p_2}-\frac{2}{q_2}}\|f\|_{\mathscr{L}_{p_2,q_2}(Q_r)}, \end{aligned} \tag{1.6}$$

where $Q_r = Q_r(x_0, b)$ and $N = N(d, \lambda, \mathcal{E}_0, p_i, q_i)$.

Remark 1.1. By using a standard covering argument, it is easy to see that the constant R_0 in Theorem 1.1 is interchangeable with $c \cdot R_0$ for any $c \in (0, \infty)$ satisfying $c \cdot R_0 < \sqrt{b-a}$, possibly at the cost of changing the constant N in the theorem by $K \cdot N$, where $K = (d, c) > 0$.

Remark 1.2. Let $x_0 \in \overline{\Omega}$ and $0 < r \le \sqrt{b-a}$. We say that $u \in \mathscr{V}_2^{1,0}(Q_r(x_0, b))$ is a weak solution of

$$\begin{cases} \mathcal{P}u = \operatorname{div} \boldsymbol{F} + f \quad \text{in } \ Q_r(x_0, b), \\ (A^{ij}D_j u + A^i + F^i)n_i = 0 \quad \text{on } \ S_r(x_0, b), \end{cases} \tag{1.7}$$

if u satisfies (1.3) for all $t_1 \in [b-r^2, b]$ and $v \in \mathscr{C}_c^\infty(Q_r(x_0, b) \cup S_r(x_0, b))$. We note that the estimate (1.6) is local in nature. In fact, $u \in \mathscr{V}_2^{1,0}(Q_r(x_0, b))$ is a weak solution of (1.7), then the same proof will show that the estimate (1.6) still holds. Therefore, we verify that condition (A3) of [1] holds.

Remark 1.3. We say that $u \in \mathscr{V}_2^{1,0}(Q)$ is a weak solution of the (backward) problem

$$\begin{cases} -u_t - D_i(A^{ij}D_j u + A^i u) + B^i D_i u + Cu = \operatorname{div} \boldsymbol{F} + f \quad \text{in } \ Q, \\ (A^{ij}D_j u + A^i + F_i)n_i = 0 \quad \text{on } \ S, \end{cases} \tag{1.8}$$

if u satisfies for all $t_1 \in [a, b]$ that

$$\begin{aligned} \int_\Omega u(x, t_1)v(x, t_1)\,dx + \int_{t_1}^b\int_\Omega uv_t\,dx\,dt + \int_{t_1}^b\int_\Omega (A^{ij}D_j u + A^i u)D_i v\,dx\,dt \\ + \int_{t_1}^b\int_\Omega (B^i D_i u + Cu)v\,dx\,dt = \int_{t_1}^b\int_\Omega -F^i D_i v + fv\,dx\,dt \end{aligned}$$

for all $v \in \mathscr{C}_c^\infty(\overline{Q})$ that are equal to zero for $t = b$. By repeating essentially the same argument as in the proof of Theorem 1.1, if $u \in \mathscr{V}_2^{1,2}(Q)$ is a weak solution of (1.8), then the estimate (1.6) holds, provided that $Q_r = \Omega_r(x_0) \times (b - r^2, b)$ is replaced by $\Omega_r(x_0) \times (a, a + r^2)$.

Remark 1.4. Similar to [1, Remark 3.19], by setting $u = 1$, $A^{ij} = \delta_{ij}$, $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{F} = 0$, and $C = f = 0$ in (1.4), we get from (1.6) that

$$|\Omega_r(x_0)| \ge \theta r^d, \quad \forall x_0 \in \overline{\Omega}, \quad \forall r \in (0, R_0], \tag{1.9}$$

where $\theta = \theta(d, \mathcal{E}_0)$.

Remark 1.5. In [2], the author claims that if the (Neumann) Green function of the parabolic operator $\mathcal{P}$ satisfies the Gaussian upper bound

$$G(x,t;y,s) \le c^{-1}(t-s)^{-n/2}\exp\{-c|x-y|^2/4(t-s)\},$$

then the following local boundedness property holds: if u is a weak solution of

$$\begin{cases} \mathcal{P}u = 0 \quad \text{in } \ Q, \\ (A^{ij}D_j u + A^i u)n_i = 0 \quad \text{on } \ S, \end{cases}$$

then u satisfies

$$\|u\|_{\mathscr{L}_\infty(Q_r)} \le Nr^{-\frac{d+2}{2}}\left(\|u\|_{\mathscr{L}_2(Q_{2r})} + r^{-1}\|u\|_{\mathscr{L}_2(S_{3r/2})}\right). \tag{1.10}$$

To show (1.10), the author claims that the boundary integral term in [2, Eq. (11)] is bounded by the second term in the right hand side of (1.10). However, by Theorem 1.1, the weak solution u satisfies

$$\|u\|_{\mathscr{L}_\infty(Q_r)} \le Nr^{-\frac{d+2}{2}}\|u\|_{\mathscr{L}_2(Q_{2r})}. \tag{1.11}$$

Indeed, the boundary integral term in [2, Eq. (11)] is cancelled out in the weak formulation of the problem, and by using the Gaussian upper bound, it is not hard to see that u satisfies (1.11); see the proof of [1, Theorem 3.24]. Therefore, the local boundedness property (1.10)

$$\left(\text{or} \quad \|u\|_{\mathscr{L}_\infty(Q_r)} \le Nr^{-\frac{d+6}{2}}\|u\|_{\mathscr{L}_2(Q_{2r})}\right)$$

is not optimal.

Next we consider the local boundedness estimates for weak solutions of

$$\begin{cases} \mathcal{P}u = \operatorname{div}\boldsymbol{F} + f \quad \text{in } \ Q, \\ (A^{ij}D_j u + A^i u + F^i)n_i = g \quad \text{on } \ S. \end{cases} \tag{1.12}$$

Theorem 1.2. *Let $Q = \Omega \times (a,b)$, where Ω is a Sobolev extension domain in $\mathbb{R}^d$ such that the trace embedding is available*; *i.e., for any $p \in [1,d)$, there exists a positive constant $\mathcal{T}_0$ such that*

$$\|u\|_{L_{p(d-1)/(d-p)}(\partial\Omega)} \le \mathcal{T}_0\|u\|_{W^1_p(\Omega)}, \quad \forall u \in W^1_p(\Omega). \tag{1.13}$$

Assume that

$$\mathcal{D} := \big\||\boldsymbol{A}| + |\boldsymbol{B}| + |C|^{1/2}\big\|_{\mathscr{L}_{p_0,q_0}(Q)} < \infty,$$
$$\mathcal{M} := \|\boldsymbol{F}\|_{\mathscr{L}_{p_1,q_1}(Q)} + \|f\|_{\mathscr{L}_{p_2,q_2}(Q)} + \|g\|_{\mathscr{L}_{p_3,q_3}(S)} < \infty,$$
$$\begin{cases} p_0 > d, \quad q_0 > 2, \\ p_1 > d, \quad q_1 > 2, \\ p_2 > \dfrac{d}{2}, \quad q_2 > 1, \\ p_3 > d-1, \quad q_3 > 2, \end{cases} \quad \textit{and} \quad \frac{d}{p_{\min}} + \frac{2}{q_{\min}} < 1,$$

where we use the notation

$$p_{\min} = \min\left(p_0, p_1, 2p_2, \frac{dp_3}{d-1}\right) \quad \textit{and} \quad q_{\min} = \min\left(q_0, q_1, 2q_2, q_3\right).$$

If $u \in \mathscr{V}_2^{1,0}(Q)$ *is a weak solution of* (1.12), *then there exists a constant* $0 < R_0 \le \min\left(1, \sqrt{b-a}\right)$, *depending only on* d, λ, $\mathcal{E}_0$, $\mathcal{T}_0$, p_i, q_i, *and* $\mathcal{D}$, *such that for any* $x_0 \in \overline{\Omega}$ *and* $0 < r \le R_0$, *we have*

$$\begin{aligned} \|u\|_{\mathscr{L}_\infty(Q_{r/2})} &\le N r^{-\frac{d+2}{2}} \|u\|_{\mathscr{L}_2(Q_r)} + N r^{1-\frac{d}{p_1}-\frac{2}{q_1}} \|\boldsymbol{F}\|_{\mathscr{L}_{p_1,q_1}(Q_r)} \\ &\quad + N r^{2-\frac{d}{p_2}-\frac{2}{q_2}} \|f\|_{\mathscr{L}_{p_2,q_2}(Q_r)} + N r^{1-\frac{d-1}{p_3}-\frac{2}{q_3}} \|g\|_{\mathscr{L}_{p_3,q_3}(S_r)}, \end{aligned} \tag{1.14}$$

where $Q_r = Q_r(x_0, b)$, $S_r = S_r(x_0, b)$, *and* $N = N(d, \lambda, \mathcal{E}_0, \mathcal{T}_0, p_i, q_i)$.

1.2. Hölder continuity of solutions

In this subsection, we state main results concerning the local Hölder continuity up to the boundary for weak solutions of the Neumann problem. For this, we impose the following assumption that holds for the case when Ω is a convex domain.

Assumption 1.1 (R)**.** Denote

$$A_{k,\rho}(x_0) = \{x \in \Omega_\rho(x_0) : u(x) > k\}.$$

There exists a constant $\mathcal{E}_1 > 0$ such that for any $x_0 \in \overline{\Omega}$, $\rho \in (0, R]$, and $0 < k < l$, we have

$$\begin{gathered} (l-k)|A_{l,\rho}(x_0)| \le \mathcal{E}_1 \frac{\rho^{d+1}}{|\Omega_\rho(x_0) \setminus A_{k,\rho}(x_0)|} \int_{A_{k,\rho}(x_0)\setminus A_{l,\rho}(x_0)} |Du|\, dx, \\ \forall u \in W_1^1(\Omega_\rho(x_0)). \end{gathered}$$

Theorem 1.3. *Assume the same hypothesis of Theorem 1.1 holds. Denote*

$$\begin{gathered} M := \|u\|_{\mathscr{L}_\infty(Q)}, \\ 0 < \alpha < \beta := \min\left\{1 - \frac{d}{p_0} - \frac{2}{q_0}, 1 - \frac{d}{p_1} - \frac{2}{q_1}, 2 - \frac{d}{p_2} - \frac{2}{q_2}\right\}. \end{gathered} \tag{1.15}$$

Then there exists a constant $0 < R_1 \le \min\left(1, \sqrt{b-a}\right)$, *where*

$$R_1 = R_1(d, \lambda, p_i, q_i, \mathcal{D}, M\mathcal{D}, \mathcal{M}, \alpha),$$

such that, under Assumption 1.1 (R_1), *for any* $x_0 \in \overline{\Omega}$ *and* $0 < r \le R_1$, *we have*

$$[u]_{\alpha_0, \alpha_0/2; Q_{r/2}(x_0,b)} \le \frac{N}{r^{\alpha_0}} \max\left(r^\alpha, \|u\|_{\mathscr{L}_\infty(Q_r(x_0,b))}\right), \tag{1.16}$$

where $\alpha_0 = \alpha_0(d, \lambda, \mathcal{E}_0, \mathcal{E}_1, p_i, q_i, \alpha) \in (0, \alpha]$ *and* $N = N(d, \lambda, \mathcal{E}_0, \mathcal{E}_1, p_i, q_i) > 0$.

Remark 1.6. We point out that if $\mathcal{D} = 0$, then the constant R_1 in Theorem 1.3 is independent of M. Moreover, in the case when $\mathcal{D} = \mathcal{M} = 0$, the constant R_1 depends only on d, λ, and α. Therefore, by applying the estimate (1.16) to $u/\|u\|_{\mathscr{L}_\infty(Q_r(x_0,b))}$, we have

$$[u]_{\alpha_0,\alpha_0/2;Q_{r/2}(x_0,b)} \le \frac{N}{r^{\alpha_0}} \|u\|_{\mathscr{L}_\infty(Q_r(x_0,b))}.$$

Theorem 1.4. *Assume the same hypothesis of Theorem 1.2 holds. Denote* $M := \|u\|_{\mathscr{L}_\infty(Q)}$ *and*

$$0 < \alpha < \beta := \min\left\{1 - \frac{d}{p_0} - \frac{2}{q_0}, 1 - \frac{d}{p_1} - \frac{2}{q_1}, 2 - \frac{d}{p_2} - \frac{2}{q_2}, 1 - \frac{d-1}{p_3} - \frac{2}{q_3}\right\}.$$

Then there exists a constant $0 < R_1 \le \min\left(1, \sqrt{b-a}\right)$, *where*

$$R_1 = R_1(d, \lambda, p_i, q_i, \mathcal{D}, M\mathcal{D}, \mathcal{M}, \alpha),$$

such that, under Assumption 1.1 (R_1), *for any* $x_0 \in \overline{\Omega}$ *and* $0 < r \le R_1$ *we have the estimate* (1.16).

1.3. Estimates for weak solutions of Dirichlet problem

In this subsection, we consider the parabolic equations with Dirichlet boundary condition. We define $\mathring{\mathscr{V}}_2^{1,0}(Q) = \mathscr{V}_2^{1,0}(Q) \cap \mathring{\mathscr{W}}_2^{1,0}(Q)$, where $\mathring{\mathscr{W}}_2^{1,0}(Q)$ is the closure of $\mathscr{C}_c^\infty(\Omega \times [a,b])$ in the Hilbert space $\mathscr{W}_2^{1,0}(Q)$.

Theorem 1.5. *Let* $Q = \Omega \times (a,b)$, *where* Ω *is a domain in* $\mathbb{R}^d$. *Assume that* (1.5) *holds. If* $u \in \mathring{\mathscr{V}}_2^{1,0}(Q)$ *is a weak solution of*

$$\mathcal{P}u = \operatorname{div} \boldsymbol{F} + f \quad \text{in } Q, \tag{1.17}$$

then there exists a constant $0 < R_0 \le \min\left(1, \sqrt{b-a}\right)$, *depending only on* d, λ, p_i, q_i, *and* $\mathcal{D}$, *such that for any* $x_0 \in \overline{\Omega}$ *and* $0 < r \le R_0$, *we have the estimate* (1.6).

Proof. By following the proof of Theorem 1.1, and using [6, Eq. (3.4), p. 75] instead of Lemma 2.3, it is not hard to see that the conclusion of the theorem holds. We note that counterparts of Remarks 1.1–1.3 are also valid. □

Theorem 1.6. *Let* $Q = \Omega \times (a,b)$, *where* Ω *is a Sobolev extension domain in* $\mathbb{R}^d$. *Assume that* (1.5) *holds, and recall* (1.15). *If* $u \in \mathring{\mathscr{V}}_2^{1,0}(Q)$ *is a weak solution of* (1.17), *then there exists a constant* $0 < R_1 \le \min\left(1, \sqrt{b-a}\right)$, *where*

$$R_1 = R_1(d, \lambda, p_i, q_i, \mathcal{D}, M\mathcal{D}, \mathcal{M}, \alpha),$$

such that for any $x_0 \in \overline{\Omega}$ *and* $0 < r \le R_1$, *we have the estimate* (1.16) *with* $\alpha_0 = \alpha_0(d, \lambda, \mathcal{E}_0, p_i, q_i, \alpha) \in (0, \alpha]$ *and* $N = N(d, \lambda, \mathcal{E}_0, p_i, q_i) > 0$.

Proof. We point out that there exists a constant $\mathcal{E}_2 > 0$ such that for any $x_0 \in \overline{\Omega}$, $\rho \in (0, R_0]$, and $0 < k < l$, we have

$$(l-k)|A_{l,\rho}(x_0)| \leq \mathcal{E}_2 \frac{\rho^{d+1}}{|\Omega_\rho(x_0) \setminus A_{k,\rho}(x_0)|} \int_{A_{k,\rho}(x_0)\setminus A_{l,\rho}(x_0)} |Du|\, dx, \quad \forall u \in C_c^\infty(\Omega), \tag{1.18}$$

where

$$A_{k,\rho}(x_0) = \{x \in \Omega_\rho(x_0) : u(x) > k\}.$$

Indeed, by setting $u = 0$ on Ω^c, and then, applying [6, Eq. (5.5), p. 91], we get the above inequality. By following the proof of Theorem 1.3, and using (1.18) instead of Assumption 1.1 (R), it is not hard to see that the conclusion of the theorem holds. □

2. Proofs of main theorems

2.1. Auxiliary results

In this subsection, we provide some lemmas used to prove the main theorems. The following two lemmas are taken from [6, pp. 95–96]; see also [3, Lemma 15.1, p. 319].

Lemma 2.1. *Let $\{Y_n\}$ be a sequence of nonnegative numbers linked by the recursive inequalities*

$$Y_{n+1} \leq b^n K Y_n^{1+\sigma}$$

for some $b > 1$, $K > 0$, and $\sigma > 0$. If

$$Y_1 \leq b^{-1/\sigma^2} K^{-1/\sigma},$$

then $\{Y_n\} \to 0$ as $n \to \infty$.

Lemma 2.2. *Let $\{Y_n\}$ and $\{Z_n\}$ be sequences of nonnegative numbers linked by the system of recursive inequalities*

$$Y_{n+1} \leq b^n K \left(Y_n^{1+\sigma} + Z_n^{1+\kappa} Y_n^\sigma\right),$$
$$Z_{n+1} \leq b^n K \left(Y_n + Z_n^{1+\kappa}\right),$$

for some $b > 1$, $K > 0$, $\sigma > 0$, and $\kappa > 0$. If

$$Y_1 \leq G \quad \text{and} \quad Z_1 \leq G^{\frac{1}{1+\kappa}},$$

where

$$G = \min\left\{(2K)^{-\frac{1}{\sigma}} b^{-\frac{1}{\sigma\epsilon}}, (2K)^{-\frac{1+\kappa}{\kappa}} b^{-\frac{1}{\kappa\epsilon}}\right\}, \quad \epsilon = \min\left\{\sigma, \frac{\kappa}{1+\kappa}\right\},$$

then $\{Y_n + Z_n\} \to 0$ as $n \to \infty$.

We will use the following embedding.

Lemma 2.3. *Let $Q = \Omega \times (a, b)$, where Ω is a Sobolev extension domain in $\mathbb{R}^d$ $(d \geq 2)$. Assume that η is a smooth cut-off function in $\mathbb{R}^{d+1}$ satisfying*

$$\operatorname{supp} \eta \subset B_r(x) \times (b - r^2, b + r^2),$$

where $x \in \overline{\Omega}$ and $0 < r \leq \min\left(1, \sqrt{b-a}\right)$. If ηu belongs to $\mathscr{V}_2^{1,0}(Q)$, then we have

$$\|\eta u\|_{\mathscr{L}_{p,q}(Q)} \leq N \|\eta u\|_{\mathscr{V}_2^{1,0}(Q)}, \tag{2.1}$$

where $N = N(d, \mathcal{E}_0, p, q)$. Here, p and q satisfy

$$\frac{d}{p} + \frac{2}{q} = \frac{d}{2}, \quad \text{where} \quad \begin{cases} p \in \left[2, \dfrac{2d}{d-2}\right], & q \in [2, \infty] \quad \text{for } d \geq 3, \\ p \in [2, \infty), & q \in (2, \infty] \quad \text{for } d = 2. \end{cases} \tag{2.2}$$

Proof. Let us fix p and q satisfying (2.2), and denote $v = \eta u$ and $\alpha = 2/q$. Notice from [6, Theorem 2.2, p. 62] that there exists a constant $N_0 = N_0(d, q)$ such that

$$\|Ev(\cdot, t)\|_{L_p(\mathbb{R}^d)} \leq N_0 \|D(Ev)(\cdot, t)\|^{\alpha}_{L_2(\mathbb{R}^d)} \|Ev(\cdot, t)\|^{1-\alpha}_{L_2(\mathbb{R}^d)}, \quad \forall t \in (a, b).$$

From this together with (1.1) it follows that

$$\begin{aligned} \|v(\cdot, t)\|_{L_p(\Omega)} &\leq \|Ev(\cdot, t)\|_{L_p(\mathbb{R}^d)} \leq N_0 \|D(Ev)(\cdot, t)\|^{\alpha}_{L_2(\mathbb{R}^d)} \|Ev(\cdot, t)\|^{1-\alpha}_{L_2(\mathbb{R}^d)} \\ &\leq N_0 \mathcal{E}_0 \|v(\cdot, t)\|^{\alpha}_{W_2^1(\Omega)} \|v(\cdot, t)\|^{1-\alpha}_{L_2(\Omega)} \\ &\leq N \|v(\cdot, t)\|_{L_2(\Omega)} + N \|Dv(\cdot, t)\|^{\alpha}_{L_2(\Omega)} \|v(\cdot, t)\|^{1-\alpha}_{L_2(\Omega)}, \end{aligned}$$

where $N = N(d, \mathcal{E}_0, q)$. Therefore, we obtain that (use $r \leq 1$)

$$\begin{aligned} \|v\|^q_{\mathscr{L}_{p,q}(Q)} &\leq N \int_{b-r^2}^b \|v(\cdot, t)\|^q_{L_2(\Omega)}\, dt + N \int_a^b \|Dv(\cdot, t)\|^2_{L_2(\Omega)} \|v(\cdot, t)\|^{q-2}_{L_2(\Omega)}\, dt \\ &\leq N \max_{a \leq t \leq b} \|v(\cdot, t)\|^q_{L_2(\Omega)} + N \max_{a \leq t \leq b} \|v(\cdot, t)\|^{q-2}_{L_2(\Omega)} \int_a^b \|Dv(\cdot, t)\|^2_{L_2(\Omega)} dt, \end{aligned}$$

and thus, by Cauchy's inequality, we get (2.1). The lemma is proved. □

2.2. Proof of Theorem 1.1

We prove the theorem by adapting the idea of De Giorgi. Let $x_0 \in \overline{\Omega}$ and $0 < r \leq R_0 \leq \min\left(1, \sqrt{b-a}\right)$, where R_0 will be chosen later. For $n = 1, 2, \ldots$, we denote

$$r_n = \frac{r}{2} + \frac{r}{2^n}, \quad k_n = k\left(2 - \frac{1}{2^{n-1}}\right), \quad E_n(t) = \{x \in \Omega_{r_n}(x_0) : u(x, t) > k_n\}, \tag{2.3}$$

where $k > 0$ is a constant to be chosen later. Let us set

$$v_n = (u - k_n)_+,$$

and let $\eta = \eta_n$ be a smooth cut-off function in $\mathbb{R}^{d+1}$ satisfying

$$
\begin{aligned}
&0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } B_{r_{n+1}}(x_0) \times (b - r_{n+1}^2, b + r_{n+1}^2),\\
&\operatorname{supp} \eta \subset B_{r_n}(x_0) \times (b - r_n^2, b + r_n^2), \quad |\eta_t| + |D\eta|^2 \le 4^{n+3} r^{-2}.
\end{aligned}
\tag{2.4}
$$

By applying $\eta^2 v_n$ as a test function to the equation (1.4), we get for all $t_1 \in [a, b]$ that

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \eta^2(x,t_1) v_n^2(x,t_1)\,dx + \int_a^{t_1}\int_\Omega \eta^2 A^{ij} D_j v_n D_i v_n\,dx\,dt\\
= &\int_a^{t_1}\int_\Omega \big(\eta\eta_t v_n^2 - A^{ij} D_j v_n 2\eta D_i \eta v_n\big)\,dx\,dt\\
&- \int_a^{t_1}\int_\Omega A^i u(\eta^2 D_i v_n + 2\eta D_i \eta v_n)\,dx\,dt\\
&- \int_a^{t_1}\int_\Omega \big(B^i D_i v_n \eta^2 v_n + C u \eta^2 v_n\big)\,dx\,dt\\
&- \int_a^{t_1}\int_\Omega \big(F^i \eta^2 D_i v_n + F^i 2\eta D_i \eta v_n\big)\,dx\,dt + \int_a^{t_1}\int_\Omega f\eta^2 v_n\,dx\,dt.
\end{aligned}
$$

Then by using (1.2), Cauchy's inequality, and the properties of η, we have

$$
\begin{aligned}
&\max_{a\le t\le b}\int_\Omega |\eta v_n|^2\,dx + \int_Q \eta^2 |Dv_n|^2\,dx\,dt\\
&\le N(\lambda)\frac{4^n}{r^2}\int_{b-r_n^2}^b\int_{E_n} v_n^2\,dx\,dt + N(\lambda)\sum_{i=1}^4 I_i,
\end{aligned}
$$

where we set

$$
\begin{aligned}
I_1 &= \int_Q (|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2 + |C|)\eta^2 v_n^2\,dx\,dt,\\
I_2 &= \int_a^b\int_{E_n} (|\boldsymbol{A}|^2 + |C|)\eta^2 k_n^2\,dx\,dt,\\
I_3 &= \int_Q \big(|\boldsymbol{F}|\eta^2|Dv_n| + |\boldsymbol{F}|\eta|D\eta|v_n\big)\,dx\,dt,\\
I_4 &= \int_Q |f|\eta^2 v_n\,dx\,dt.
\end{aligned}
\tag{2.5}
$$

Therefore, from the following inequality

$$
\int_Q |D(\eta v_n)|^2\,dx\,dt \le 2\int_Q \eta^2|Dv_n|^2\,dx\,dt + 2\int_Q |D\eta|^2 v_n^2\,dx\,dt,
$$

we obtain

$$
\|\eta v_n\|^2_{\mathscr{V}_2^{1,0}(Q)} + \|\eta D v_n\|^2_{\mathscr{L}_2(Q)} \le N(\lambda)\frac{4^n}{r^2}\int_{b-r_n^2}^b\int_{E_n} v_n^2\,dx\,dt + N(\lambda)\sum_{i=1}^4 I_i.
\tag{2.6}
$$

Hereafter in the proof, we fix $p \in (d, p_{\min})$ and $q \in (2, q_{\min})$ satisfying

$$0 < \delta := 1 - \frac{d}{p} - \frac{2}{q} < 1 - \frac{d}{p_{\min}} - \frac{2}{q_{\min}}.$$

Estimate of I_1. To estimate I_1, we first note that Hölder's inequality implies

$$I_1 \leq \mathcal{D}^2 \|\eta v_n\|^2_{\mathscr{L}_{\chi_1,\zeta_1}(Q_r)}, \tag{2.7}$$

where

$$\chi_1 = \frac{2p_0}{p_0 - 2} \quad \text{and} \quad \zeta_1 = \frac{2q_0}{q_0 - 2}.$$

We also note that

$$\begin{cases} \chi_1 \in \left(2, \dfrac{2d}{d-2}\right), & \zeta_1 \in (2, \infty) \quad \text{for } d \geq 3, \\ \chi_1 \in (2, \infty), & \zeta_1 \in (2, \infty) \quad \text{for } d = 2, \end{cases}$$

and

$$\frac{d}{\chi_1} + \frac{2}{\zeta_1} = \frac{d}{2} + 1 - \left(\frac{d}{p_0} + \frac{2}{q_0}\right) > \frac{d}{2}.$$

Therefore, by choosing $\zeta_2 \in (\zeta_1, \infty)$ such that

$$\frac{d}{\chi_1} + \frac{2}{\zeta_2} = \frac{d}{2},$$

and then, applying Hölder's inequality and Lemma 2.3 to (2.7), we get

$$I_1 \leq r^{4\left(\frac{1}{\zeta_1} - \frac{1}{\zeta_2}\right)} \mathcal{D}^2 \|\eta v_n\|^2_{\mathscr{L}_{\chi_1,\zeta_2}(\Omega \times (b - r_n^2, b))} \leq N r^{2\mu} \mathcal{D}^2 \|\eta v_n\|^2_{\mathscr{V}_2^{1,0}(Q)},$$

where $N = N(d, \mathcal{E}_0, p_0, q_0) > 0$ and

$$\mu = 2\left(\frac{1}{\zeta_1} - \frac{1}{\zeta_2}\right) = 1 - \frac{d}{p_0} - \frac{2}{q_0} > 0.$$

Estimate of I_2. By using Hölder's inequality, we obtain

$$\begin{aligned} I_2 &\leq 4k^2 \left\| |\boldsymbol{A}| + |C|^{1/2} \right\|^2_{\mathscr{L}_{p,q}(Q_r)} \left(\int_{b - r_n^2}^{b} |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}} \\ &\leq N k^2 r^{2 - 2\delta - \frac{2d}{p_0} - \frac{4}{q_0}} \left\| |\boldsymbol{A}| + |C|^{1/2} \right\|^2_{\mathscr{L}_{p_0,q_0}(Q)} \left(\int_{b - r_n^2}^{b} |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}}, \\ &\leq N k^2 r^{2\mu - 2\delta} \mathcal{D}^2 \left(\int_{b - r_n^2}^{b} |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}}, \end{aligned}$$

where $N = N(d, p_0, p)$.

Estimate of I_3. By Cauchy's inequality and the properties of η, we get for any $\epsilon > 0$ that

$$I_3 \le \epsilon \int_Q \eta^2 |Dv_n|^2 \, dx \, dt + \frac{4^n}{r^2} \int_{b-r_n^2}^b \int_\Omega v_n^2 \, dx \, dt + N(\epsilon) \int_{b-r_n^2}^b \int_{E_n} |\boldsymbol{F}|^2 \, dx \, dt. \tag{2.8}$$

From Hölder's inequality it follows that

$$\begin{aligned}\int_{b-r_n^2}^b \int_{E_n} |\boldsymbol{F}|^2 \, dx \, dt &\le \|\boldsymbol{F}\|^2_{\mathscr{L}_{p,q}(Q_r)} \left(\int_{b-r_n^2}^b |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}} \\ &\le N r^{2-2\delta - \frac{2d}{p_1} - \frac{4}{q_1}} \|\boldsymbol{F}\|^2_{\mathscr{L}_{p_1,q_1}(Q_r)} \left(\int_{b-r_n^2}^b |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}},\end{aligned} \tag{2.9}$$

where $N = N(d, p_1, p)$. Therefore by combining (2.8) and (2.9), we have

$$\begin{aligned} I_3 \le \epsilon \int_Q \eta^2 |Dv_n|^2 \, dx \, dt &+ \frac{4^n}{r^2} \int_{b-r_n^2}^b \int_\Omega v_n^2 \, dx \, dt \\ &+ N r^{2-2\delta - \frac{2d}{p_1} - \frac{4}{q_1}} \|\boldsymbol{F}\|^2_{\mathscr{L}_{p_1,q_1}(Q_r)} \left(\int_{b-r_n^2}^b |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}}, \end{aligned}$$

where $N = N(d, p_1, p, \epsilon)$.

Estimate of I_4. To estimate I_4, let us set

$$\chi_0 = \frac{2dq}{2q + dq - 4}, \quad \zeta_0 = \frac{2q}{q+2}, \quad \chi_1 = \frac{\chi_0}{\chi_0 - 1}, \quad \zeta_1 = \frac{\zeta_0}{\zeta_0 - 1}.$$

We then find that

$$\chi_0 \in \left(1, \min\left(\frac{p}{2}, 2\right)\right), \quad \zeta_0 \in \left(1, \min\left(\frac{q}{2}, 2\right)\right),$$

and

$$\frac{d}{\chi_1} + \frac{2}{\zeta_1} = \frac{d}{2}.$$

By Hölder's inequality, Cauchy's inequality, and Lemma 2.3, we obtain for $\epsilon > 0$ that

$$\begin{aligned} I_4 &\le \|\eta v_n\|_{\mathscr{L}_{\chi_1,\zeta_1}(Q)} \|f\|_{\mathscr{L}_{\chi_0,\zeta_0}(E_n \times (b-r_n^2, b))} \\ &\le \epsilon \|\eta v_n\|^2_{\mathscr{V}_2^{1,0}(Q)} + N \|f\|^2_{\mathscr{L}_{\chi_0,\zeta_0}(E_n \times (b-r_n^2, b))}, \end{aligned} \tag{2.10}$$

where $N = N(d, \mathcal{E}_0, p, q, \epsilon)$. Notice from Hölder's inequality that (use $|E_n| \le N(d) r^d$)

$$\|f\|^2_{\mathscr{L}_{\chi_0,\zeta_0}(E_n \times (b-r_n^2, b))}$$

$$
\begin{aligned}
&\leq \|f\|^2_{\mathscr{L}_{p/2,q/2}(Q_r)} \left(\int_{b-r_n^2}^{b} |E_n|^{\frac{p-2\chi_0}{p\chi_0} \frac{q\zeta_0}{q-2\zeta_0}} \, dt \right)^{\frac{2(q-2\zeta_0)}{q\zeta_0}} \\
&= \|f\|^2_{\mathscr{L}_{p/2,q/2}(Q_r)} \left(\int_{b-r_n^2}^{b} |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} |E_n|^{\frac{2\delta}{d} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}} \\
&\leq N r^{4-2\delta-\frac{2d}{p_2}-\frac{4}{q_2}} \|f\|^2_{\mathscr{L}_{p_2,q_2}(Q_r)} \left(\int_{b-r_n^2}^{b} |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}}, \qquad (2.11)
\end{aligned}
$$

where $N = N(d, p_2, p, q)$. Therefore, we get from (2.10) and (2.11) that

$$
I_4 \leq \epsilon \|\eta v_n\|^2_{\mathscr{V}_2^{1,0}(Q)} + N r^{4-2\delta-\frac{2d}{p_2}-\frac{4}{q_2}} \|f\|^2_{\mathscr{L}_{p_2,q_2}(Q_r)} \left(\int_{b-r_n^2}^{b} |E_n|^{\frac{p-2}{p} \frac{q}{q-2}} \, dt \right)^{\frac{q-2}{q}},
$$

where $N = N(d, \mathcal{E}_0, p_2, p, q, \epsilon)$.

We are now ready to prove the theorem. By (2.6) and the estimates of I_i, we have

$$
\begin{aligned}
\|\eta v_n\|_{\mathscr{V}_2^{1,0}(Q)} \leq{}& N_0 \frac{2^n}{r} \left(\int_{b-r_n^2}^{b} \int_{E_n} v_n^2 \, dx \, dt \right)^{1/2} \\
&+ N_1 r^{\mu} \mathcal{D} \|\eta v_n\|_{\mathscr{V}_2^{1,0}(Q)} + N_2 k r^{\mu} r^{-\delta} \mathcal{D} \boldsymbol{E}_n + N_3 r^{-\delta} \mathcal{M}_0 \boldsymbol{E}_n,
\end{aligned}
$$

where

$$
\begin{aligned}
&N_0 = N_0(\lambda), \quad N_1 = N_1(d, \mathcal{E}_0, \lambda, p_0, q_0) \geq 1, \\
&N_2 = N_2(d, \lambda, p_0, p), \quad N_3 = N_3(d, \mathcal{E}_0, \lambda, p_1, p_2, p, q).
\end{aligned}
$$

Here, we use the notation

$$
\begin{aligned}
&\boldsymbol{E}_n = \left(\int_{b-r_n^2}^{b} |E_n|^{\frac{p-2}{2p} \frac{2q}{q-2}} \, dt \right)^{\frac{q-2}{2q}}, \\
&\mathcal{M}_0 = r^{1-\frac{d}{p_1}-\frac{2}{q_1}} \|\boldsymbol{F}\|_{\mathscr{L}_{p_1,q_1}(Q_r)} + r^{2-\frac{d}{p_2}-\frac{2}{q_2}} \|f\|_{\mathscr{L}_{p_2,q_2}(Q_r)}.
\end{aligned}
$$

Then by taking $R_1 \in (0, 1]$ so that

$$
R_1^{\mu} \mathcal{D} \leq \frac{1}{2N_1} \leq \frac{1}{2}, \qquad (2.12)
$$

we obtain for $0 < r \leq R_0 \leq R_1$ that

(2.13)

$$
\|\eta v_n\|_{\mathscr{V}_2^{1,0}(Q)} \leq 2N_0 \frac{2^n}{r} \left(\int_{b-r_n^2}^{b} \int_{E_n} v_n^2 dx dt \right)^{1/2} + (2N_2 + 2N_3) r^{-\delta} (k r^{\mu} \mathcal{D} + \mathcal{M}_0) \boldsymbol{E}_n.
$$

Next, denote

$$Y_n := \left(\int_{b-r_n^2}^{b} \left(\int_{E_n} |\eta_n v_n|^{\frac{2p}{p-2}} \, dx \right)^{\frac{p-2}{2p} \frac{2q}{q-2}} \, dt \right)^{\frac{q-2}{2q}},$$

and observe that

$$Y_n \geq \left(\int_{b-r_{n+1}^2}^{b} \left(\int_{E_{n+1}} |v_n|^{\frac{2p}{p-2}} \, dx \right)^{\frac{p-2}{2p} \frac{2q}{q-2}} \, dt \right)^{\frac{q-2}{2q}} \geq \frac{k}{2^n} \boldsymbol{E}_{n+1}. \tag{2.14}$$

Since p and q satisfy

$$d \left(\frac{p-2}{2p} \right) + 2 \left(\frac{q-2}{2q} \right) = \frac{d}{2} + \delta,$$

the constants

$$\chi := \frac{2p}{p-2} \left(1 + \frac{2}{d} \delta \right) \quad \text{and} \quad \zeta := \frac{2q}{q-2} \left(1 + \frac{2}{d} \delta \right)$$

satisfy

$$\frac{d}{\chi} + \frac{2}{\zeta} = \frac{d}{2}.$$

Therefore, by using Hölder's inequality and Lemma 2.3, we have

$$Y_n \leq \|\eta_n v_n\|_{\mathscr{L}_{\chi,\zeta}(\Omega \times (b-r_n^2, b))} \boldsymbol{E}_n^{\frac{2\delta}{d+2\delta}} \leq N_4 \|\eta_n v_n\|_{\mathscr{V}_2^{1,0}(Q)} \boldsymbol{E}_n^{\frac{2\delta}{d+2\delta}},$$

where $N_4 = N_4(d, \mathcal{E}_0, p, q)$, and thus, we get from (2.13) that

$$Y_n \leq N_5 \left(\frac{2^n}{r} \left(\int_{b-r_n^2}^{b} \int_{E_n} v_n^2 \, dx \, dt \right)^{1/2} + r^{-\delta} \left(kR_0^\mu \mathcal{D} + \mathcal{M}_0 \right) \boldsymbol{E}_n \right) \boldsymbol{E}_n^{\frac{2\delta}{d+2\delta}}, \tag{2.15}$$

where $N_5 = N_5(d, \mathcal{E}_0, \lambda, p_i, q_i)$. In particular, if $n = 1$, then by using the fact that

$$\boldsymbol{E}_1 \leq N(d,p) r^{\frac{d+2\delta}{2}},$$

we get

$$Y_1 \leq N_6 \left(r^{-1+\delta} \|\boldsymbol{u}\|_{\mathscr{L}_2(Q_r)} + \left(kR_0^\mu \mathcal{D} + \mathcal{M}_0 \right) r^{\frac{d+2\delta}{2}} \right), \tag{2.16}$$

where $N_6 = N_6(d, \mathcal{E}_0, \lambda, p_i, q_i) \geq 1$. Notice from Hölder's inequality and (2.14) that (use $v_n \geq v_{n+1}$)

$$\begin{aligned} &\left(\int_{b-r_{n+1}^2}^{b} \int_{E_{n+1}} v_{n+1}^2 \, dx \, dt \right)^{1/2} \\ &\leq N(d,p) r^{1-\delta} \left(\int_{b-r_{n+1}^2}^{b} \left(\int_{E_{n+1}} |v_{n+1}|^{\frac{2p}{p-2}} \, dx \right)^{\frac{p-2}{2p} \frac{2q}{q-2}} \, dt \right)^{\frac{q-2}{2q}} \\ &\leq N(d,p) r^{1-\delta} Y_n. \end{aligned} \tag{2.17}$$

Therefore, by using (2.14), (2.15), and (2.17), we obtain that

$$Y_{n+1} \le N_7 \frac{2^n}{r^\delta} \left(\frac{2^n}{k}\right)^{\frac{2\delta}{d+2\delta}} \left(1 + R_0^\mu \mathcal{D} + \frac{\mathcal{M}_0}{k}\right) Y_n^{1+\frac{2\delta}{d+2\delta}},$$

where $N_7 = N_7(d, \mathcal{E}_0, \lambda, p_i, q_i) \ge 1$. Since $R_0^\mu \mathcal{D} \le 1$ (see (2.12)), we have

$$Y_{n+1} \le 4^n \frac{N_7}{r^\delta k^{\frac{2\delta}{d+2\delta}}} \left(2 + \frac{\mathcal{M}_0}{k}\right) Y_n^{1+\frac{2\delta}{d+2\delta}} \le 4^n K Y^{1+\frac{2\delta}{d+2\delta}},$$

where we set

$$K = \frac{3N_7}{r^\delta k^{\frac{2\delta}{d+2\delta}}} \text{ and } k = 2N_6 (3N_7)^{\frac{d+2\delta}{2\delta}} 4^{\left(\frac{d+2\delta}{2\delta}\right)^2} \left(r^{-\frac{d+2}{2}} \|u\|_{\mathscr{L}_2(Q_r)} + \mathcal{M}_0\right) \ge \mathcal{M}_0.$$

We choose $R_0 \in (0, R_1]$ such that

$$R_0^\mu \mathcal{D} \le \frac{1}{2N_6} 4^{-\left(\frac{d+2\delta}{2\delta}\right)^2} (3N_7)^{-\frac{d+2\delta}{2\delta}}.$$

Then for $r \in (0, R_0]$, we obtain by (2.16) that

$$Y_1 \le 4^{-\left(\frac{d+2\delta}{2\delta}\right)^2} K^{-\frac{d+2\delta}{2\delta}}.$$

Therefore by Lemma 2.1, we have $Y_n \to 0$ as $n \to \infty$, and thus, we get

$$u \le 2k \quad \text{on } Q_{r/2}.$$

By applying the same argument to $-u$, we obtain the estimate (1.6) from the definition of $\mathcal{M}_0$ and k.

2.3. Proof of Theorem 1.2

We follow the proof of Theorem 1.1 with a few adjustments. By the same argument used in deriving (2.6), we obtain

$$\|\eta v_n\|^2_{\mathscr{V}_2^{1,0}(Q)} + \|\eta D v_n\|^2_{\mathscr{L}_2(Q)} \le N(\lambda) \frac{4^n}{r^2} \int_{b-r_n^2}^b \int_{E_n} v_n^2 \, dx\, dt + N(\lambda) \sum_{i=1}^5 I_i,$$

where we use the notation (2.3)–(2.5) and

$$I_5 = \int_S |g| \eta^2 v_n \, d\sigma \, dt.$$

Let us fix $p \in (d, p_{\min})$ and $q \in (2, q_{\min})$ so that

$$0 < \delta := 1 - \frac{d}{p} - \frac{2}{q} < 1 - \frac{d}{p_{\min}} - \frac{2}{q_{\min}}.$$

We write

$$\chi = \frac{dp_3}{dp_3 - d + 1} \in (1, 2)$$

and observe that

$$\chi < \frac{p}{p-1}, \qquad \frac{p_3}{p_3 - 1} = \frac{(d-1)\chi}{d - \chi}.$$

Then, by using Hölder's inequality and (1.13), we have

(2.18) $$I_5 \le \|\eta v_n\|_{\mathscr{L}_{p_3/(p_3-1),q_3/(q_3-1)}(S)} \|g\|_{\mathscr{L}_{p_3,q_3}(S_r)} \le \mathcal{T}_0 H \|g\|_{\mathscr{L}_{p_3,q_3}(S_r)},$$

where

$$H = \|\eta v_n\|_{\mathscr{L}_{\chi,q_3/(q_3-1)}(Q)} + \|D(\eta v_n)\|_{\mathscr{L}_{\chi,q_3/(q_3-1)}(Q)}.$$

Notice from Hölder's inequality that

(2.19)
$$\begin{aligned} H &\le N r^{1-\delta-\frac{d-1}{p_3}-\frac{2}{q_3}} \left(\|\eta v_n\|_{\mathscr{L}_{p/(p-1),q/(q-1)}(Q)} + \|D(\eta v_n)\|_{\mathscr{L}_{p/(p-1),q/(q-1)}(Q)} \right) \\ &\le N r^{1-\delta-\frac{d-1}{p_3}-\frac{2}{q_3}} \|\eta v_n\|_{\mathscr{W}_2^{1,0}(Q)} \boldsymbol{E}_n, \end{aligned}$$

where $N = N(d, p_3, p)$ and

$$\boldsymbol{E}_n = \left(\int_{b-r_n^2}^{b} |E_n|^{\frac{p-2}{2p}\frac{2q}{q-2}} \, dt \right)^{\frac{q-2}{2q}}.$$

By combining (2.18) and (2.19), and then, applying Cauchy's inequality, we obtain for $\epsilon > 0$ that

$$I_5 \le \epsilon \|\eta v_n\|^2_{\mathscr{W}_2^{1,0}(Q)} + N r^{2-2\delta-\frac{2(d-1)}{p_3}-\frac{4}{q_3}} \|g\|^2_{\mathscr{L}_{p_3,q_3}(S_r)} \boldsymbol{E}_n^2,$$

where $N = N(d, \mathcal{T}_0, p_3, p, \epsilon)$. Then by following the same steps as in the proof of Theorem 1.1, there exists a constant $R_1 \in (0, 1]$ so that for $0 < r \le R_0 \le R_1$, we have (see (2.13))

$$\|\eta v_n\|_{\mathscr{V}_2^{1,0}(Q)} \le N \frac{2^n}{r} \left(\int_{b-r_n^2}^{b} \int_{E_n} v_n^2 \, dx \, dt \right)^{1/2} + N r^{-\delta} (k r^{\mu} \mathcal{D} + \mathcal{M}_0) \boldsymbol{E}_n,$$

where $\mu = 1 - d/p_0 - 2/q_0$ and

$$\begin{aligned} \mathcal{M}_0 = {} & r^{1-\frac{d}{p_1}-\frac{2}{q_1}} \|\boldsymbol{F}\|_{\mathscr{L}_{p_1,q_1}(Q_r)} + r^{2-\frac{d}{p_2}-\frac{2}{q_2}} \|f\|_{\mathscr{L}_{p_2,q_2}(Q_r)} \\ & + r^{1-\frac{d-1}{p_3}-\frac{2}{q_3}} \|g\|_{\mathscr{L}_{p_3,q_3}(S_r)}. \end{aligned}$$

This implies the estimate (1.14) in the same way as (2.13) implies (1.6). The theorem is proved.

2.4. Proof of Theorem 1.3

To prove the Hölder continuity of u, we need to obtain the oscillation estimates. For this, we use the following four lemmas whose proofs will be given in Appendix. Hereafter in the proof, we let $(y, s) \in Q_{R_1/2}(x_0, b)$, where $R_1 \in (0, R_0]$ will be chosen later. We use the notations

$$\begin{aligned} &\Omega_\rho = \Omega_\rho(y), \quad Q_\rho^\gamma = \Omega_\rho \times (s - \gamma\rho^2, s), \\ &E_{k,\rho}(t) = \{x \in \Omega_\rho : u(x,t) > k\}, \\ &p \in (d, p_{\min}), \quad q \in (2, q_{\min}), \quad \delta = 1 - \frac{d}{p} - \frac{2}{q} \in (0, 1). \end{aligned}$$

We also denote by R_0 the constant in Theorem 1.1.

Lemma 2.4. *There exist constants* γ, $c_0 \in (0,1)$ *and* $R_1 \in (0, R_0]$, *where*

$$\gamma = \gamma(d, \lambda, \mathcal{E}_0, p_i, q_i),$$
$$c_0 = c_0(d, \lambda, \mathcal{E}_0, p_i, q_i),$$
$$R_1 = R_1(d, \lambda, \mathcal{E}_0, p_i, q_i, \mathcal{D}, M\mathcal{D}, \mathcal{M}, \alpha, \beta),$$

such that for any $\rho \in (0, R_1/2]$, *the following holds*: *If*

$$|E_{k,\rho}(s - \gamma\rho^2)| \le \frac{1}{2}|\Omega_\rho| \tag{2.20}$$

and

$$H := \operatorname*{ess\,sup}_{Q_\rho^\gamma} u - k > \rho^\alpha \tag{2.21}$$

for some $k \in (-\infty, M]$, *then we have*

$$\left|E_{k+\frac{3}{4}H,\rho}(t)\right| \le c_0|\Omega_\rho|, \quad \forall t \in [s - \gamma\rho^2, s]. \tag{2.22}$$

Proof. See Section 3.1. □

In the rest of the proof, γ and R_1 denote the constants in Lemma 2.4.

Lemma 2.5. *There exist constants* $c_1 \in (0,1]$, *depending only on* d, λ, $\mathcal{E}_0$, p_i, *and* q_i, *such that for any* $\rho \in (0, R_1/2]$, *the following holds: If*

$$\left|\left\{(x,t) \in Q_\rho^\gamma : u(x,t) > k\right\}\right| \le c_1\rho^{d+2} \tag{2.23}$$

and

$$H := \operatorname*{ess\,sup}_{Q_\rho^\gamma} u - k > \rho^\alpha \tag{2.24}$$

for some $k \in (-\infty, M]$, *then we have*

$$\operatorname*{ess\,sup}_{Q_{\rho/2}^\gamma} u \le k + \frac{1}{2}H. \tag{2.25}$$

Proof. See Section 3.2. □

Lemma 2.6. *Under Assumption 1.1* ($R_1/2$), *there exists a positive integer* $c_2 \ge 2$, *depending only on* d, λ, p_i, q_i, $\mathcal{E}_0$, *and* $\mathcal{E}_1$, *such that for any* $\rho \in (0, R_1/2]$, *we have either*

$$\omega \le 2^{c_2}\rho^\alpha,$$

or

$$\left|\left\{(x,t) \in Q_{\rho/2}^\gamma : u(x,t) > \Psi - \frac{\omega}{2^{c_2}}\right\}\right| \le c_1\left(\frac{\rho}{2}\right)^{d+2} \tag{2.26}$$

or

$$\left|\left\{(x,t) \in Q_{\rho/2}^\gamma : u(x,t) < \psi + \frac{\omega}{2^{c_2}}\right\}\right| \le c_1\left(\frac{\rho}{2}\right)^{d+2}, \tag{2.27}$$

where c_1 is the constant in Lemma 2.5. Here, we denote

$$\Psi = \operatorname*{ess\,sup}_{Q^\gamma_\rho} u, \quad \psi = \operatorname*{ess\,inf}_{Q^\gamma_\rho} u, \quad \omega = \operatorname*{osc}_{Q^\gamma_\rho} u = \Psi - \psi.$$

Proof. See Section 3.3. □

As a consequence of Lemmas 2.5 and 2.6, we get:

Lemma 2.7. *Under Assumption 1.1 ($R_1/2$), for any $\rho \in (0, R_1/2]$, we have either*

$$\operatorname*{osc}_{Q^\gamma_{\rho/4}} u \leq 2^{c_2+1}\rho^\alpha$$

or

$$\operatorname*{osc}_{Q^\gamma_{\rho/4}} u \leq \left(1 - \frac{1}{2^{c_2+1}}\right) \operatorname*{osc}_{Q^\gamma_\rho} u, \tag{2.28}$$

where c_2 is the constant in Lemma 2.6.

Proof. See Section 3.3. □

Let $0 < r \leq R_1/2$, and choose $\alpha_0 \in (0, \alpha]$ such that

$$4^{\alpha_0}\left(1 - \frac{1}{2^{c_2+1}}\right) \leq 1,$$

where c_2 is the constant in Lemma 2.6. For $k = 0, 1, 2, \dots$, we define

$$r_k = 4^{-k} r, \quad \omega_k = \operatorname*{osc}_{Q^\gamma_{r_k}} u, \quad y_k = 4^{k\alpha_0}\omega_k.$$

Then by Lemma 2.7, we obtain for $k = 1, 2, \dots$, that

$$y_k \leq 4^{k\alpha_0} \max\left(2^{c_2+1} r_k^\alpha, \left(1 - \frac{1}{2^{c_2+1}}\right)\omega_{k-1}\right) \leq \max\left(2^{c_2+1} r^\alpha, y_{k-1}\right),$$

and thus, by using $y_0 = \omega_0$, we have

$$y_k \leq N_0 := \max\left(2^{c_2+1} r^\alpha, \omega_0\right).$$

Therefore, we conclude that

$$\omega_k \leq N_0 4^{-k\alpha_0} \leq N_0 \left(\frac{r_k}{r}\right)^{\alpha_0}. \tag{2.29}$$

Assume that $\rho \in (0, r]$ and $r_k \leq \rho \leq r_{k-1}$ for some positive integer k. Then we get from (2.29) that

$$\begin{aligned}\operatorname*{osc}_{Q^\gamma_\rho(y,s)} u \leq \omega_{k-1} \leq N_0\left(\frac{r_{k-1}}{r}\right)^{\alpha_0} &= N_0 4^{\alpha_0}\left(\frac{\rho}{r}\right)^{\alpha_0} \\ &= N\left(\frac{\rho}{r}\right)^{\alpha_0} \max\left(r^\alpha, \|u\|_{\mathscr{L}_\infty(Q_r(y,s))}\right) \end{aligned} \tag{2.30}$$

for any $(y, s) \in Q_{R_1/2}(x_0, b)$ and $0 < \rho \leq r \leq R_1/2$, where $N = N(c_2)$.

Now we are ready to prove the theorem. Let us fix $r \in (0, R_1]$, and let (y,s), $(z,\tau) \in Q_{r/2}(x_0,b)$ satisfy

$$(y,s) \neq (z,\tau), \quad \gamma\rho := \max\left(|y-z|, |s-\tau|^{1/2}\right) \leq \gamma r/2.$$

Then by (2.30), we have

$$\begin{aligned} \frac{|u(y,s)-u(z,\tau)|}{|y-z|^{\alpha_0}+|s-\tau|^{\alpha_0/2}} &\leq \frac{1}{(\gamma\rho)^{\alpha_0}} \operatorname*{osc}_{Q_{\gamma\rho}(y,s)} u \\ &\leq \frac{N(c_2,\gamma)}{r^{\alpha_0}} \max\left(r^{\alpha}, \|u\|_{\mathscr{L}_\infty(Q_r(x_0,b))}\right). \end{aligned}$$

Therefore, by using a standard covering argument, we get the estimate (1.16). The theorem is proved.

3. Appendix

3.1. Proof of Lemma 2.4

Let $\gamma \in (0,1)$ and $R_1 \in (0, R_0]$ be constants to be chosen later. Fix $\rho \in (0, R_1/2]$. For $k \in (-\infty, M]$ satisfying (2.20) and (2.21), we define

$$v_k = (u-k)_+.$$

Let $\eta = \eta(x)$ be a smooth cut-off function in $\mathbb{R}^d$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{(1-\epsilon)\rho}(y), \quad \operatorname{supp}\eta \subset B_\rho(y), \quad |D\eta| \leq 4(\epsilon\rho)^{-1},$$

where $0 < \epsilon < 1$. Then by following the same argument used in deriving (2.13), there exists a constant $R_1' \in (0, R_0]$, depending only on d, λ, $\mathcal{E}_0$, p_i, q_i, and $\mathcal{D}$, such that for $\rho \in (0, R_1'/2]$, we have

$$\begin{aligned} &\max_{s-\gamma\rho^2 \leq t \leq s} \int_{E_{k,(1-\epsilon)\rho}} v_k^2\,dx \\ &\leq \frac{N}{(\epsilon\rho)^2} \int_{s-\gamma\rho^2}^s \int_{E_{k,\rho}} v_k^2\,dx\,dt \\ &\quad + \frac{10}{9} \int_{E_{k,\rho}(s-\gamma\rho^2)} v_k^2\,dx + N\rho^{2\beta-2\delta}(M\mathcal{D}+\mathcal{M})^2\boldsymbol{E}^2, \end{aligned} \tag{3.1}$$

where $N = N(d,\lambda,\mathcal{E}_0,p_i,q_i)$ and

$$\boldsymbol{E} = \left(\int_{s-\gamma\rho^2}^s |E_{k,\rho}|^{\frac{p-2}{2p}\frac{2q}{q-2}}\,dt\right)^{\frac{q-2}{2q}}.$$

Notice from (2.20) that

$$\frac{N}{(\epsilon\rho)^2}\int_{s-\gamma\rho^2}^s \int_{E_{k,\rho}} v_k^2\,dx\,dt + \frac{10}{9}\int_{E_{k,\rho}(s-\gamma\rho^2)} v_k^2\,dx \leq \left(\frac{N\gamma}{\epsilon^2} + \frac{5}{9}\right) H^2|\Omega_\rho|.$$

We also note that (1.9) implies

$$\boldsymbol{E}^2 \leq (\gamma\rho^2)^{\frac{q-2}{q}} |\Omega_\rho|^{\frac{p-2}{p}} \leq N\gamma^{\frac{q-2}{q}}\rho^{2\delta}|\Omega_\rho|.$$

By the above two inequalities, we get from (3.1) that
(3.2)
$$\max_{s-\gamma\rho^2\le t\le s}\int_{E_{k,(1-\epsilon)\rho}} v_k^2\,dx \le \left(\frac{N\gamma}{\epsilon^2}+\frac{5}{9}\right)H^2|\Omega_\rho| + N\rho^{2\beta}(M\mathcal{D}+\mathcal{M})^2\gamma^{\frac{q-2}{q}}|\Omega_\rho|.$$
We note that for $t\in[s-\gamma\rho^2,s]$, we have
$$\begin{aligned}\left|E_{k+\frac{3}{4}H,(1-\epsilon)\rho}(t)\right| &\le \frac{16}{9H^2}\int_{E_{k+\frac{3}{4}H,(1-\epsilon)\rho}(t)}|v_k(x,t)|^2\,dx\\ &\le \frac{16}{9H^2}\int_{E_{k,(1-\epsilon)\rho}(t)}|v_k(x,t)|^2\,dx.\end{aligned}$$
From this together with (2.21) and (3.2), it follows that
$$\left|E_{k+\frac{3}{4}H,(1-\epsilon)\rho}(t)\right| \le \frac{16}{9}\left(\frac{N\gamma}{\epsilon^2}+\frac{5}{9}+N\rho^{2(\beta-\alpha)}(M\mathcal{D}+\mathcal{M})^2\gamma^{\frac{q-2}{q}}\right)|\Omega_\rho|.$$
Then by taking $R_1\in(0,R_1']$ so that
$$R_1^{2(\beta-\alpha)}(M\mathcal{D}+\mathcal{M})^2\le 1, \tag{3.3}$$
we have
$$\left|E_{k+\frac{3}{4}H,(1-\epsilon)\rho}(t)\right| \le \frac{16}{9}\left(\frac{N\gamma}{\epsilon^2}+\frac{5}{9}+N\gamma^{\frac{q-2}{q}}\right)|\Omega_\rho|.$$
Therefore we obtain by (1.9) that
$$\left|E_{k+\frac{3}{4}H,\rho}(t)\right| \le \left|E_{k+\frac{3}{4}H,(1-\epsilon)\rho}(t)\right| + |B_\rho(y)\setminus B_{(1-\epsilon)\rho}(y)| \le c_0|\Omega_\rho|,$$
where
$$c_0=\frac{16}{9}\left(\frac{N\gamma}{\epsilon^2}+\frac{5}{9}+N\gamma^{\frac{q-2}{q}}\right)+N_0(d,\theta)(1-(1-\epsilon)^d).$$
Then by taking $\epsilon=\epsilon(d,\theta)$ sufficiently small, and then $\gamma=\gamma(N,q,\epsilon)$ sufficiently small, we have $0<c_0<1$, which implies (2.22).

3.2. Proof of Lemma 2.5

Let us fix $0<\rho\le R_1/2$, and let $c_1\in(0,1]$ be a constant to be chosen later. Assume $k\in(-\infty,M]$ satisfies (2.23) and (2.24). For $n=1,2,\dots$, we denote
$$\rho_n=\frac{\rho}{2}+\frac{\rho}{2^n},\quad k_n=k+\frac{H}{4}\left(2-\frac{1}{2^{n-1}}\right),\quad E_n(t)=E_{k_n,\rho_n}(t).$$
Let us set
$$v_n=(u-k_n)_+,$$
and let $\eta=\eta_n$ be a smooth cut-off function in $\mathbb{R}^{d+1}$ satisfying
$$\begin{aligned}&0\le\eta\le 1,\quad \eta\equiv 1 \text{ on } B_{\rho_{n+1}}(y)\times(s-\gamma\rho_{n+1}^2,s+\gamma\rho_{n+1}^2),\\ &\operatorname{supp}\eta\subset B_{\rho_n}(y)\times(s-\gamma\rho_n^2,s+\gamma\rho_n^2),\quad |\eta_t|+|D\eta|^2\le N_\gamma 4^n\rho^{-2},\end{aligned}$$

where N_γ is a constant depending only on γ. Then by following the same argument used in deriving (2.13), we find that (use $\gamma < 1$)

$$\text{(3.4)}\quad \|\eta v_n\|_{\mathscr{V}_2^{1,0}(Q_\rho^\gamma)} \le N\frac{2^n}{\rho}\left(\int_{s-\gamma\rho_n^2}^{s}\int_{E_n} v_n^2\,dx\,dt\right)^{1/2} + N\rho^{\beta-\delta}(M\mathcal{D}+\mathcal{M})\boldsymbol{E}_n,$$

where $N = N(d,\lambda,\mathcal{E}_0,p_i,q_i)$ and

$$\boldsymbol{E}_n = \left(\int_{s-\gamma\rho_n^2}^{s} |E_n|^{\frac{p-2}{2p}\frac{2q}{q-2}}\,dt\right)^{\frac{q-2}{2q}}.$$

Let us fix $\kappa = \kappa(d,p) > 0$ so that

$$\frac{2p}{p-2} < (1+\kappa)\frac{2p}{p-2} < \frac{2d}{d-2},$$

and choose $\tilde{p} \in (d,p)$ and $\tilde{q} \in (2,q)$ satisfying

$$\frac{2\tilde{p}}{\tilde{p}-2} = (1+\kappa)\frac{2p}{p-2} \quad \text{and} \quad \frac{2\tilde{q}}{\tilde{q}-2} = (1+\kappa)\frac{2q}{q-2}.$$

Then it follows from (3.4) that

$$\text{(3.5)}\qquad \|\eta v_n\|_{\mathscr{V}_2^{1,0}(Q_\rho^\gamma)} \le N2^n\rho^{d/2}HY_n + N\rho^{\beta+d/2}(M\mathcal{D}+\mathcal{M})Z_n^{1+\kappa},$$

where

$$Y_n := \frac{1}{\rho^{\frac{d+2}{2}}}\left(\int_{s-\gamma\rho_n^2}^{s}\int_{E_n}\left(\frac{v_n}{H}\right)^2 dx\,dt\right)^{1/2},$$

$$Z_n = \frac{1}{\rho^{\frac{1}{1+\kappa}\left(\frac{d}{2}+\delta\right)}}\left(\int_{s-\gamma\rho_n^2}^{s}|E_n|^{\frac{\tilde{p}-2}{2\tilde{p}}\frac{2\tilde{q}}{\tilde{q}-2}}\,dt\right)^{\frac{\tilde{q}-2}{2\tilde{q}}}.$$

Now, we claim that

$$\text{(3.6)}\qquad Y_1 \le c_1^{1/2} \quad \text{and} \quad Y_{n+1} \le N4^n\big(Y_n^{1+\sigma} + Z_n^{1+\kappa}Y_n^\sigma\big), \quad \sigma = \frac{2}{d+2},$$

where $N = N(d,\lambda,\mathcal{E}_0,p_i,q_i)$. From (2.23), it is not hard to see that the first inequality in (3.6) holds. By using Hölder's inequality and the fact that

$$\int_{s-\gamma\rho_{n+1}^2}^{s}|E_{n+1}(t)|\,dt \le \frac{1}{|k_{n+1}-k_n|^2}\int_{s-\gamma\rho_{n+1}^2}^{s}\int_{E_{n+1}}|v_n|^2\,dx\,dt \le 4^{n+2}\rho^{d+2}Y_n^2,$$

we have

$$\begin{aligned} &Y_{n+1} \\ &\le \frac{1}{H\rho^{\frac{d+2}{2}}}\left(\int_{s-\gamma\rho_{n+1}^2}^{s}|E_{n+1}(t)|\,dt\right)^{\frac{1}{d+2}}\left(\int_{s-\gamma\rho_{n+1}^2}^{s}\int_{E_{n+1}}|v_{n+1}|^{\frac{2(d+2)}{d}}\,dx\,dt\right)^{\frac{d}{2(d+2)}} \\ &\le 2^{n+2}Y_n^{\frac{2}{d+2}}\frac{1}{H\rho^{d/2}}\|\eta_n v_n\|_{\mathscr{L}_{\frac{2(d+2)}{d}}(Q_\rho^\gamma)}, \end{aligned}$$

and thus, we get from Lemma 2.3, (3.5), and (2.24) that

$$\begin{aligned} Y_{n+1} &\le N2^{n+2}Y_n^{\frac{2}{d+2}}\frac{1}{H\rho^{d/2}}\|\eta_n v_n\|_{\mathscr{V}_2^{1,0}(Q_\rho^\gamma)} \\ &\le N4^nY_n^{1+\frac{2}{d+2}}+N2^n\rho^\beta\frac{M\mathcal{D}+\mathcal{M}}{H}Y_n^{\frac{2}{d+2}}Z_n^{1+\kappa}, \\ &\le N4^nY_n^{1+\frac{2}{d+2}}+N2^n\rho^{\beta-\alpha}(M\mathcal{D}+\mathcal{M})Y_n^{\frac{2}{d+2}}Z_n^{1+\kappa}. \end{aligned}$$

This together with (3.3), we get the second inequality in (3.6).

Next, we claim that

$$Z_1 \le N\big(c_1^{\frac{p-2}{2p}}+c_1^{\frac{q-2}{2q}}\big)^{\frac{1}{1+\kappa}} \quad \text{and} \quad Z_{n+1}\le N4^n\big(Y_n+Z_n^{1+\kappa}\big), \tag{3.7}$$

where $N=N(d,\lambda,\mathcal{E}_0,p_i,q_i)$. Note that since

$$d\frac{\tilde p-2}{2\tilde p}+2\frac{\tilde q-2}{2\tilde q}=\frac{1}{1+\kappa}\left(\frac{d}{2}+\delta\right)\ge\frac{d}{2},$$

we get from Lemma 2.3 that

$$\begin{aligned} Z_{n+1} &\le \frac{1}{\rho^{\frac{1}{1+\kappa}\left(\frac{d}{2}+\delta\right)}|k_{n+1}-k_n|}\|\eta_n v_n\|_{\mathscr{L}_{2\tilde p/(\tilde p-2),2\tilde q/(\tilde q-2)}(Q_\rho^\gamma)} \\ &\le N\frac{2^{n+1}}{H\rho^{d/2}}\|\eta_n v_n\|_{\mathscr{V}_2^{1,0}(Q_\rho^\gamma)}. \end{aligned}$$

Then by using this together with (3.5), we have

$$Z_{n+1}\le N4^nY_n+N2^n\rho^{\beta-\alpha}(M\mathcal{D}+\mathcal{M})Z_n^{1+\kappa},$$

which gives the second inequality in (3.7). Assume that $\tilde p\ge\tilde q$. Then we obtain by (2.23) that

$$Z_1\le\frac{N}{\rho^{(d+2)\frac{\tilde q-2}{2\tilde q}}}\left(\int_{s-\gamma\rho^2}^s|E_1|\,dt\right)^{\frac{\tilde q-2}{2\tilde q}}\le Nc_1^{\frac{\tilde q-2}{2\tilde q}}=N\big(c_1^{\frac{q-2}{2q}}\big)^{\frac{1}{1+\kappa}},$$

where $N=N(d,p_i,q_i)$. On the other hand, if $\tilde p\le\tilde q$, then by Hölder's inequality and (2.23), we get

$$Z_1\le\frac{1}{\rho^{(d+2)\frac{\tilde p-2}{2\tilde p}}}\left(\int_{s-\gamma\rho^2}^s|E_1|\,dt\right)^{\frac{\tilde p-2}{2\tilde p}}\le c_1^{\frac{\tilde p-2}{2\tilde p}}=\big(c_1^{\frac{p-2}{2p}}\big)^{\frac{1}{1+\kappa}}.$$

By combining the above two inequalities, we get (3.7).

Finally, by taking $c_1=c_1(d,\lambda,\mathcal{E}_0,p_i,q_i)$ sufficiently small, and then, by using Lemma 2.2, (3.6), and (3.7), we have $Y_n\to0$ as $n\to\infty$, which implies (2.25).

3.3. Proof of Lemma 2.6

Assume that $\omega > 2^{c_2}\rho^\alpha$, where $c_2 \ge 2$ is a positive integer to be chosen later. Obviously, we have at least one of the inequalities

$$\left| E_{\Psi-\frac{\omega}{2},\frac{\rho}{2}}\left(s - \frac{\gamma\rho^2}{4}\right)\right| \le \frac{1}{2}|\Omega_{\rho/2}| \tag{3.8}$$

or

$$\left| \Omega_{\rho/2} \setminus E_{\Psi-\frac{\omega}{2},\frac{\rho}{2}}\left(s - \frac{\gamma\rho^2}{4}\right)\right| \le \frac{1}{2}|\Omega_{\rho/2}|. \tag{3.9}$$

Now, we claim that (3.8) implies (2.26). To see this, we only need to consider the case that

$$\operatorname*{ess\,sup}_{Q^\gamma_{\rho/2}} u > \Psi - \frac{\omega}{2^{c_2}}.$$

Let us fix i in $\{1, 2, \dots, c_2 - 1\}$. Notice from the above inequality that

$$H := \operatorname*{ess\,sup}_{Q^\gamma_{\rho/2}} u - \left(\Psi - \frac{\omega}{2^i}\right) \ge \frac{\omega}{2^i} - \frac{\omega}{2^{c_2}} \ge \frac{\omega}{2^{c_2}} > \rho^\alpha.$$

We also note that (3.8) yields

$$\left| E_{\Psi-\frac{\omega}{2^i},\frac{\rho}{2}}\left(s - \frac{\gamma\rho^2}{4}\right)\right| \le \frac{1}{2}|\Omega_{\rho/2}|.$$

Then by Lemma 2.4, we have

$$\left| \Omega_{\rho/2} \setminus E_{\Psi-\frac{\omega}{2^i}+\frac{3}{4}H,\frac{\rho}{2}}(t)\right| \ge (1-c_0)|\Omega_{\rho/2}|, \quad \forall t \in \left[s - \frac{\gamma\rho^2}{4}, s\right],$$

where c_0 is the constant in Lemma 2.4. Since $H \le \omega/2^i$, we have

$$\Psi - \frac{\omega}{2^i} + \frac{3}{4}H \le \Psi - \frac{\omega}{2^{i+2}}.$$

Therefore, we obtain

$$\left| \Omega_{\rho/2} \setminus E_{\Psi-\frac{\omega}{2^{i+2}},\frac{\rho}{2}}(t)\right| \ge (1-c_0)|\Omega_{\rho/2}|, \quad \forall t \in \left[s - \frac{\gamma\rho^2}{4}, s\right]. \tag{3.10}$$

Let us set

$$k = \Psi - \frac{\omega}{2^{i+2}}, \quad l = \Psi - \frac{\omega}{2^{i+3}}, \quad B(t) = E_{k,\rho/2}(t) \setminus E_{l,\rho/2}(t).$$

Then by Assumption 1.1 ($R_1/2$), (3.10), and (1.9), we obtain for $t \in \left[s - \frac{\gamma\rho^2}{4}, s\right]$ that

$$\frac{\omega}{2^{i+3}}\left|E_{l,\frac{\rho}{2}}(t)\right| \le \mathcal{E}_1 \frac{\rho^{d+1}}{|\Omega_{\rho/2} \setminus E_{k,\rho/2}(t)|}\int_{B(t)} |Du|\, dx \le N\rho \int_{B(t)} |Du|\, dx,$$

where $N = N(c_0, \mathcal{E}_1, \theta) = N(d, \lambda, \mathcal{E}_0, \mathcal{E}_1, p_i, q_i)$. Therefore, by integrating the above inequality over $\left[s - \frac{\gamma\rho^2}{4}, s\right]$, and then, using Hölder's inequality, we get

$$\begin{aligned}
(3.11)\qquad & \frac{\omega}{2^{i+3}} \left|\left\{(x,t) \in Q^{\gamma}_{\rho/2} : u(x,t) > l\right\}\right| \\
&\leq N\rho \int_{s-\gamma\rho^2/4}^{s} \int_B |Du| \, dx \, dt \\
&\leq N\rho \left(\int_{s-\gamma\rho^2/4}^{s} \int_{E_{k,\rho/2}} |Du|^2 \, dx \, dt\right)^{1/2} \left(\int_{s-\gamma\rho^2/4}^{s} |B| \, dt\right)^{1/2}.
\end{aligned}$$

We remark that by following the same argument used in deriving (2.13), we have

$$\|(u-k)_+\|_{\mathscr{V}_2^{1,0}(Q^{\gamma}_{\rho/2})} \leq \frac{N}{\rho} \|(u-k)_+\|_{\mathscr{L}_2(Q^{\gamma}_{\rho})} + N\rho^{\beta-\delta}(M\mathcal{D} + \mathcal{M})\boldsymbol{E},$$

where

$$\boldsymbol{E} = \left(\int_{s-\gamma\rho^2}^{s} |E_{k,\rho}|^{\frac{p-2}{2p}\frac{2q}{q-2}} \, dt\right)^{\frac{q-2}{2q}}.$$

Then it is easy to see that (use (3.3))

$$\begin{aligned}
\int_{s-\gamma\rho^2/4}^{s} \int_{E_{k,\rho/2}} |Du|^2 \, dx \, dt &\leq N \frac{\omega^2}{4^i} \rho^d + N(M\mathcal{D} + \mathcal{M})^2 \rho^{d+2\beta} \\
(3.12)\qquad &\leq N \frac{\omega^2}{4^i} \rho^d + N\rho^{d+2\alpha}.
\end{aligned}$$

From (3.11) and (3.12), it follows that

$$\left|\left\{(x,t) \in Q^{\gamma}_{\rho/2} : u(x,t) > l\right\}\right|^2 \leq N\left(1 + \frac{4^i}{\omega^2}\rho^{2\alpha}\right) \rho^{d+2} \int_{s-\gamma\rho^2/4}^{s} |B| \, dt,$$

and thus, by using the fact that

$$\frac{4^i}{\omega^2}\rho^{2\alpha} \leq \frac{4^i}{4^{c_2}} \leq 1,$$

we conclude

$$\left|\left\{(x,t) \in Q^{\gamma}_{\rho/2} : u(x,t) > \Psi - \frac{\omega}{2^{c_2}}\right\}\right|^2 \leq N\rho^{d+2} \int_{s-\gamma\rho^2/4}^{s} |B| \, dt,$$

where $N = N(d, \lambda, \mathcal{E}_0, \mathcal{E}_1, p_i, q_i)$. We sum the above inequalities over i to obtain

$$\begin{aligned}
(c_2 - 1)\left|\left\{(x,t) \in Q^{\gamma}_{\rho/2} : u(x,t) > \Psi - \frac{\omega}{2^{c_2}}\right\}\right|^2 &\leq N\rho^{d+2} \int_{s-\gamma\rho^2/4}^{s} |\Omega_{\rho/2}| \, dt \\
&\leq N\left(\frac{\rho}{2}\right)^{2d+4}.
\end{aligned}$$

By taking $c_2 \geq 2$ so that $\frac{N}{c_2 - 1} \leq c_1$, we find that (2.26) holds. Moreover, by applying the same argument to $-u$, it is not hard to see that (3.9) implies (2.27). The lemma is proved.

3.4. Proof of Lemma 2.7

Suppose that

$$\operatorname*{osc}_{Q^{\gamma}_{\rho/4}} u > 2^{c_2+1}\rho^{\alpha}.$$

We then have

$$\omega = \operatorname*{osc}_{Q^{\gamma}_{\rho}} u > 2^{c_2+1}\rho^{\alpha} > 2^{c_2}\rho^{\alpha}, \tag{3.13}$$

and thus, by Lemma 2.6, we get either (2.26) or (2.27). Assume the inequality (2.26) holds. We denote

$$H := \operatorname*{ess\,sup}_{Q^{\gamma}_{\rho/2}} u - \left(\Psi - \frac{\omega}{2^{c_2}}\right).$$

If $H > \rho^{\alpha}$, then we obtain by Lemma 2.5 that

$$\operatorname*{ess\,sup}_{Q^{\gamma}_{\rho/4}} u \le \Psi - \frac{\omega}{2^{c_2}} + \frac{1}{2}H \le \Psi - \frac{\omega}{2^{c_2}} + \frac{\omega}{2^{c_2+1}} \le \Psi - \frac{\omega}{2^{c_2+1}}.$$

From this, we get

$$\operatorname*{osc}_{Q^{\gamma}_{\rho/4}} u \le \Psi - \operatorname*{ess\,inf}_{Q^{\gamma}_{\rho/4}} u - \frac{\omega}{2^{c_2+1}} \le \left(1 - \frac{1}{2^{c_2+1}}\right)\omega.$$

On the other hand, if $H \le \rho^{\alpha}$, then we obtain by (3.13) that

$$\operatorname*{ess\,sup}_{Q^{\gamma}_{\rho/2}} u \le \Psi - \frac{\omega}{2^{c_2}} + \rho^{\alpha} \le \Psi - \frac{\omega}{2^{c_2+1}},$$

and thus, we have

$$\operatorname*{osc}_{Q^{\gamma}_{\rho/2}} u \le \Psi - \operatorname*{ess\,inf}_{Q^{\gamma}_{\rho/2}} u - \frac{\omega}{2^{c_2+1}} \le \left(1 - \frac{1}{2^{c_2+1}}\right)\omega.$$

By applying the same argument to $-u$, it is not hard to see that (2.27) implies (2.28). The lemma is proved.

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