

FIRST EIGENVALUES OF GEOMETRIC OPERATORS UNDER THE YAMABE FLOW

SHOUWEN FANG AND FEI YANG

ABSTRACT. Let $(M, g(t))$ be a compact Riemannian manifold and the metric $g(t)$ evolve by the Yamabe flow. In the paper we derive the evolution for the first eigenvalue of geometric operator $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow, where Δ_ϕ is the Witten-Laplacian operator, $\phi \in C^2(M)$, and R is the scalar curvature with respect to the metric $g(t)$. As a consequence, we construct some monotonic quantities under the Yamabe flow.

1. Introduction

Recently, there have been many results on the eigenvalue problems under various geometric flows, especially the Ricci flow. In a seminal preprint [13], Perelman introduced the so-called F -entropy functional and proved that it is nondecreasing along the Ricci flow coupled to a backward heat-type equation. The nondecreasing of the functional F implies the monotonicity of the first eigenvalue of $-4\Delta + R$ along the Ricci flow. With his entropy and the monotonicity of the first eigenvalue, Perelman was able to rule out nontrivial steady or expanding breathers on compact manifolds. In [11] Ma obtained the monotonicity of the first eigenvalue of the Laplacian operator on a domain with Dirichlet boundary condition along the Ricci flow. Cao [1] considered the eigenvalues of $-\Delta + \frac{R}{2}$ and showed that they are nondecreasing under the Ricci flow for manifolds with nonnegative curvature operator. Li got the monotonicity of eigenvalues of the operator $-4\Delta + kR$ and ruled out compact steady Ricci breathers by using their monotonicity [7]. Later, Cao [2] also improved his own previous results and proved that the first eigenvalues of $-\Delta + cR$ ($c \geq \frac{1}{4}$) are nondecreasing under the Ricci flow on the manifolds without curvature assumption. Ling studied the first nonzero eigenvalue under the normalized Ricci flow, gave a Faber-Krahn type of comparison theorem and a sharp bound [9], and constructed a class of monotonic quantities on closed n -dimensional manifolds

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[10]. Moreover, Zhao obtained the evolution equation for the first eigenvalue of the Laplacian operator along the Yamabe flow, gave some monotonic quantities under the Yamabe flow [15], and proved that the first eigenvalue of the p -Laplace operator is increasing and the differentiable almost everywhere along the unnormalized powers of the m th mean curvature flow [16] and the unnormalized H^k -flow [17]. Guo, Philipowski and Thalmaier [5] derived an explicit formula for the evolution of the lowest eigenvalue of the Laplace-Beltrami operator with potential in abstract geometric flows. The first author and his collaborators proved that the eigenvalues of some geometric operators related to the Witten-Laplacian are nondecreasing under the Ricci flow in [3] and [4].

In this paper, we consider an n -dimensional compact Riemannian manifold M with a time-dependent Riemannian metric $g(t)$. $(M, g(t))$ is a smooth solution to the Yamabe flow equation

$$(1.1) \quad \frac{\partial}{\partial t} g_{ij}(t) = -Rg_{ij}(t).$$

Let ∇ be the Levi-Civita connection on (M, g) , Δ the Laplace-Beltrami operator, $d\nu$ the Riemannian volume measure, and $d\mu$ the weight volume measure on (M, g) , i.e.,

$$d\mu = e^{-\phi(x)} d\nu,$$

where $\phi \in C^2(M)$. Then the Witten-Laplacian (also called symmetric diffusion operator) is defined by

$$\Delta_\phi = \Delta - \nabla\phi \cdot \nabla,$$

which is a symmetric operator on $L^2(M, \mu)$.

When ϕ is a constant function, the Witten-Laplacian operator is just the Laplace-Beltrami operator. As an extension of the Laplace-Beltrami operator, many classical results in Riemannian geometry asserted in terms of the Laplace-Beltrami operator have been extended to the analogous ones on the Witten-Laplacian operator. For example, we can see these results ([3], [4], [8], and [14]). Inspired by Cao [1] and Zhao [15], we study the first eigenvalue of the geometric operator $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow. The purpose of this paper is to get the evolution equations for the first eigenvalue of the operator along the Yamabe flow and the normalized Yamabe flow on compact Riemannian manifolds. As an application, we can get some corollaries and some monotonic quantities depending on eigenvalues.

The rest of this paper is organized as follows. In Section 2, we will derive the evolution equation of the first eigenvalue under the Yamabe flow. As applications some corollaries will be obtained. In Section 3, we will construct some monotonic quantities along the Yamabe flow using the evolution equation of the first eigenvalue. In Section 4, we will study the evolution equation of the first eigenvalue under the normalized Yamabe flow and give a monotonic quantity under the normalized Yamabe flow on a compact surface.

2. Evolution equation of the first eigenvalue

In this section, we establish the evolution equation for the first eigenvalue of the geometry operator $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow.

Let $(M, g(t))$ be a compact Riemannian manifold, and $(M, g(t)), t \in [0, T)$ be a smooth solution to the Yamabe flow equation (1.1). Let λ be an eigenvalue of the operator $-\Delta_\phi + \frac{R}{2}$ at time t_0 where $0 \leq t_0 < T$, and f the corresponding eigenfunction, i.e.,

$$(2.1) \quad -\Delta_\phi f + \frac{R}{2} f = \lambda f,$$

with the normalization

$$\int_M f^2 d\mu = 1.$$

We assume that $f(x, t)$ is a C^1 -family of smooth functions on M , and satisfies the following condition

$$\frac{d}{dt} \left[\int_M f^2 d\mu \right] = 0.$$

Hence, we have

$$(2.2) \quad \int_M f [f_t d\mu + (f d\mu)_t] = 0,$$

where $f_t = \frac{\partial f}{\partial t}$.

We also need to define a functional

$$\lambda(f, t) = \int_M \left(-f \Delta_\phi f + \frac{R}{2} f^2 \right) d\mu = \int_M \left(-\Delta_\phi f + \frac{R}{2} f \right) f d\mu,$$

where f satisfies the equality (2.2). At time t , if f is the eigenfunction of λ , then

$$\lambda(f, t) = \lambda(t).$$

Let us first give the evolution equation of the above functional under the general geometric flow.

Lemma 2.1. *Suppose that λ is an eigenvalue of the operator $-\Delta_\phi + \frac{R}{2}$, f is the eigenfunction of λ at time t_0 , and the metric $g(t)$ evolves by*

$$\frac{\partial}{\partial t} g_{ij} = v_{ij},$$

where v_{ij} is a symmetric two-tensor. Then we have

$$(2.3) \quad \begin{aligned} \frac{d}{dt} \lambda(f, t)|_{t=t_0} &= \int_M \left(v_{ij} f_{ij} - v_{ij} \phi_i f_j + \frac{1}{2} \frac{\partial R}{\partial t} f \right) f d\mu \\ &+ \int_M \left(v_{ij,i} - \frac{1}{2} V_j \right) f_j f d\mu, \end{aligned}$$

where $V = \text{Tr}(v)$.

The proof of Lemma 2.1 can be found in [4]. In fact, Lemma 2.1 also give us the evolution of eigenvalues. It is easy to see that the evolution equation (2.3) does not depend on the evolution equation of f , as long as f satisfies (2.2). Hence we have

$$(2.4) \quad \frac{d}{dt}\lambda(t) = \frac{d}{dt}\lambda(f, t)$$

for any time t , when f is the eigenfunction of λ at time t .

Remark 2.1. The above identity (2.4) holds due to the assumption that $f(x, t)$ is a C^1 -family of smooth functions on M . Note that the Witten-Laplacian operator is also self-adjoint similarly as the Laplacian operator. Therefore, it is generally accepted that the eigenvalues λ_k and corresponding eigenfunctions are no longer differentiable in time t . But for the first eigenvalue and first eigenfunction, one can always assume that they are smooth in time along the Yamabe flow (cf. [12, §6.6], [6, §7] and the references therein).

Now we can calculate the evolution equation for the first eigenvalue of the geometric operator under the Yamabe flow.

Theorem 2.1. *Let $g(t), t \in [0, T)$, be a solution to the Yamabe flow (1.1) on a compact manifold M^n . Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\phi + \frac{R}{2}$, $f(x, t) > 0$ satisfies*

$$-\Delta_\phi f(x, t) + \frac{R}{2}f(x, t) = \lambda(t)f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\mu = 1.$$

Then under the Yamabe flow the eigenvalue $\lambda(t)$ evolves by

$$(2.5) \quad \begin{aligned} \frac{d}{dt}\lambda(t) &= \frac{1}{2} \int_M R|\nabla f|^2 d\mu + \frac{n-1}{2} \int_M R|\nabla f - f\nabla\phi|^2 d\mu + \frac{n}{4} \int_M R^2 f^2 d\mu \\ &\quad - \frac{n-1}{2} \int_M Rf^2 \Delta\phi d\mu - \left(\frac{n}{2} - 1\right)\lambda \int_M Rf^2 d\mu. \end{aligned}$$

Proof. The proof follows from a direct computation. In Lemma 2.1, we take the symmetric two-tensor $v_{ij} = -Rg_{ij}$. Note that the evolution of scalar curvature is

$$\frac{\partial R}{\partial t} = (n-1)\Delta R + R^2.$$

Using (2.4) and substituting $v_{ij} = -Rg_{ij}$ into the equality (2.3), we have

$$(2.6) \quad \begin{aligned} \frac{d}{dt}\lambda(t) &= \int_M \left(-Rf\Delta_\phi f + \frac{n-1}{2}\Delta Rf^2 + \frac{1}{2}R^2f^2\right) d\mu \\ &\quad + \left(\frac{n}{2} - 1\right) \int_M \nabla_i Rf_i f d\mu. \end{aligned}$$

Using integration by parts, we get the following three formulas

$$(2.7) \quad \int_M \Delta Rf^2 d\mu = -2 \int_M \nabla_i Rf_i f d\mu + \int_M \nabla_i R\phi_i f^2 d\mu,$$

$$(2.8) \quad \int_M \nabla_i Rf_i f d\mu = - \int_M Rf \Delta_\phi f d\mu - \int_M R|\nabla f|^2 d\mu,$$

$$(2.9) \quad \int_M \nabla_i R\phi_i f^2 d\mu = - \int_M Rf^2 \Delta\phi d\mu - 2 \int_M Rf_i \phi_i f d\mu + \int_M R|\nabla\phi|^2 f^2 d\mu.$$

Combining (2.6), (2.7), (2.8), and (2.9), we arrive at

$$\begin{aligned} \frac{d}{dt}\lambda(t) &= \int_M (-Rf \Delta_\phi f + \frac{n-1}{2} \nabla_i R\phi_i f^2 + \frac{1}{2} R^2 f^2) d\mu - \frac{n}{2} \int_M \nabla_i Rf_i f d\mu \\ &= \frac{1}{2} \int_M R^2 f^2 d\mu + (\frac{n}{2} - 1) \int_M Rf \Delta_\phi f d\mu + \frac{n}{2} \int_M R|\nabla f|^2 d\mu \\ &\quad - \frac{n-1}{2} \int_M Rf^2 \Delta\phi d\mu - (n-1) \int_M Rf_i \phi_i f d\mu \\ &\quad + \frac{n-1}{2} \int_M R|\nabla\phi|^2 f^2 d\mu \\ &= \frac{1}{2} \int_M R^2 f^2 d\mu + \frac{1}{2} \int_M R|\nabla f|^2 d\mu + \frac{n-1}{2} \int_M R|\nabla f - f\nabla\phi|^2 d\mu \\ &\quad - \frac{n-1}{2} \int_M Rf^2 \Delta\phi d\mu + (\frac{n}{2} - 1) \int_M Rf \Delta_\phi f d\mu \\ &= \frac{1}{2} \int_M R|\nabla f|^2 d\mu + \frac{n-1}{2} \int_M R|\nabla f - f\nabla\phi|^2 d\mu + \frac{n}{4} \int_M R^2 f^2 d\mu \\ &\quad - \frac{n-1}{2} \int_M Rf^2 \Delta\phi d\mu - (\frac{n}{2} - 1)\lambda \int_M Rf^2 d\mu. \end{aligned}$$

Here in the last equality we have used (2.1). □

In Theorem 2.1 if ϕ is a constant function, we can get the evolution for the first eigenvalue of the geometric operator $-\Delta + \frac{R}{2}$ under the Yamabe flow.

Corollary 2.1. *Let $g(t), t \in [0, T)$, be a solution to the Yamabe flow (1.1) on a compact manifold M^n . Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta + \frac{R}{2}$, $f(x, t) > 0$ satisfies*

$$-\Delta f(x, t) + \frac{R}{2} f(x, t) = \lambda(t) f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\nu = 1.$$

Then under the Yamabe flow the eigenvalue $\lambda(t)$ evolves by

$$(2.10) \quad \frac{d}{dt}\lambda(t) = \frac{n}{2} \int_M R|\nabla f|^2 d\nu + \frac{n}{4} \int_M R^2 f^2 d\nu - (\frac{n}{2} - 1)\lambda \int_M Rf^2 d\nu.$$

When M is a two-dimensional surface, we have the monotonicity of the first eigenvalue from the above theorem and corollary.

Corollary 2.2. *In dimension two, if a compact Riemannian manifold has nonnegative scalar curvature, the first eigenvalue of the operator*

$$-\Delta + \frac{R}{2}$$

is nondecreasing under the Yamabe flow. Moreover, if the scalar curvature also satisfies $R \geq \Delta\phi$, the first eigenvalue of the operator

$$-\Delta_\phi + \frac{R}{2}$$

is also nondecreasing under the Yamabe flow.

Remark 2.2. In two-dimension case the Yamabe flow is equivalent to the Ricci flow. The same result under the Ricci flow was obtained by Fang, Yang and Zhu [4] and Cao [1].

3. Some monotonic quantities

In this section, we obtain some monotonic quantities using the evolution equation of the first eigenvalue under the Yamabe flow.

Note that the scalar curvature under the Yamabe flow evolves by

$$\frac{\partial R}{\partial t} = (n-1)\Delta R + R^2.$$

Let $\rho(t)$ and $\sigma(t)$ be two solutions to the ODE $y' = y^2$ with initial value respectively

$$\rho(0) = \max_{x \in M} R(0) \quad \text{and} \quad \sigma(0) = \min_{x \in M} R(0).$$

By the maximum principle, we have

$$(3.1) \quad R(t) \leq \rho(t) = \left(\frac{1}{\max_{x \in M} R(0)} - t \right)^{-1}$$

and

$$(3.2) \quad R(t) \geq \sigma(t) = \left(\frac{1}{\min_{x \in M} R(0)} - t \right)^{-1}.$$

Now it is easy to get the following monotonic quantity from Theorem 2.1.

Theorem 3.1. *Suppose that $g(t), t \in [0, T)$, is a solution to the Yamabe flow (1.1) on a compact manifold M^n with nonnegative scalar curvature and the scalar curvature satisfies*

$$R \geq \frac{2(n-1)}{n} \Delta\phi, \forall t \in [0, T).$$

Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\phi + \frac{R}{2}$, $f(x, t) > 0$ satisfies

$$-\Delta_\phi f(x, t) + \frac{R}{2} f(x, t) = \lambda(t) f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\mu = 1.$$

Then the quantity $e^{-\int_0^t [\sigma(\tau) - \frac{n}{2} \rho(\tau)] d\tau} \lambda(t)$ is nondecreasing under the Yamabe flow.

Proof. According to (2.5), (3.1) and (3.2), we have

$$\begin{aligned} \frac{d}{dt} \lambda(t) &= \frac{1}{2} \int_M R |\nabla f|^2 d\mu + \frac{n-1}{2} \int_M R |\nabla f - f \nabla \phi|^2 d\mu + \frac{n}{4} \int_M R^2 f^2 d\mu \\ &\quad - \frac{n-1}{2} \int_M R f^2 \Delta \phi d\mu - \left(\frac{n}{2} - 1\right) \lambda \int_M R f^2 d\mu \\ &\geq -\left(\frac{n}{2} - 1\right) \lambda \int_M R f^2 d\mu \geq \lambda \left[\sigma(t) - \frac{n}{2} \rho(t)\right]. \end{aligned}$$

Hence the theorem follows from the last inequality. □

As a corollary, we can get other monotonic quantity when ϕ is a constant function.

Corollary 3.1. *Let $g(t), t \in [0, T)$, be a solution to the Yamabe flow (1.1) on a compact manifold M^n with nonnegative scalar curvature. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta + \frac{R}{2}$, $f(x, t) > 0$ satisfies*

$$-\Delta f(x, t) + \frac{R}{2} f(x, t) = \lambda(t) f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\nu = 1.$$

Then the quantity $e^{-\int_0^t [\sigma(\tau) - \frac{n}{2} \rho(\tau)] d\tau} \lambda(t)$ is nondecreasing under the Yamabe flow.

4. Eigenvalues under the normalized Yamabe flow

In the last section, we come to consider the normalized Yamabe flow, i.e.,

$$(4.1) \quad \frac{\partial}{\partial t} g_{ij} = -(R - r) g_{ij},$$

where $r = \frac{\int_M R d\nu}{\int_M d\nu}$ is the average scalar curvature. In Lemma 2.1, if we evolve the metric by the normalized Yamabe flow, we can get the evolution for the first eigenvalue of the geometric operator $-\Delta_\phi + \frac{R}{2}$ under the normalized Yamabe flow.

Theorem 4.1. *Let $g(t), t \in [0, T)$, be a solution to the normalized Yamabe flow (4.1) on a compact manifold M^n . Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\phi + \frac{R}{2}$, $f(x, t) > 0$ satisfies*

$$-\Delta_\phi f(x, t) + \frac{R}{2} f(x, t) = \lambda(t) f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\mu = 1.$$

Then under the Yamabe flow the eigenvalue $\lambda(t)$ evolves by

$$\begin{aligned} \frac{d}{dt} \lambda(t) &= \frac{1}{2} \int_M R |\nabla f|^2 d\mu + \frac{n-1}{2} \int_M R |\nabla f - f \nabla \phi|^2 d\mu + \frac{n}{4} \int_M R^2 f^2 d\mu \\ &\quad - \frac{n-1}{2} \int_M R f^2 \Delta \phi d\mu - \left(\frac{n}{2} - 1\right) \lambda \int_M R f^2 d\mu - r \lambda. \end{aligned}$$

Proof. We note that the evolution of scalar curvature is

$$\frac{\partial R}{\partial t} = (n-1) \Delta R + R(R-r),$$

and

$$v_{ij} = -(R-r)g_{ij}.$$

The proof can be obtained from the same calculation with Theorem 2.1. So it is easy to get the extra term $-r\lambda$. \square

If ϕ is a constant function, we obtain the evolution for the first eigenvalue of the geometric operator $-\Delta + \frac{R}{2}$ under the normalized Yamabe flow.

Corollary 4.1. *Let $g(t), t \in [0, T)$, be a solution to the normalized Yamabe flow (4.1) on a compact manifold M^n . Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta + \frac{R}{2}$, $f(x, t) > 0$ satisfies*

$$-\Delta f(x, t) + \frac{R}{2} f(x, t) = \lambda(t) f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\nu = 1.$$

Then under the Yamabe flow the eigenvalue $\lambda(t)$ evolves by

$$\frac{d}{dt} \lambda(t) = \frac{n}{2} \int_M R |\nabla f|^2 d\nu + \frac{n}{4} \int_M R^2 f^2 d\nu - \left(\frac{n}{2} - 1\right) \lambda \int_M R f^2 d\nu - r \lambda.$$

When M is a two-dimensional surface, r is a constant. We have the following corollary.

Corollary 4.2. *Assume that a two-dimensional compact Riemannian manifold has nonnegative scalar curvature and $R \geq \Delta \phi$, if $\lambda(t)$ is the first eigenvalue of the geometric operator $-\Delta_\phi + \frac{R}{2}$, then $e^{rt} \lambda(t)$ is nondecreasing under the normalized Yamabe flow.*

Remark 4.1. The same result was obtained by Fang, Yang and Zhu [4]. When ϕ is a constant function, the similar result without the scalar curvature assumption was given by Cao [2].

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SHOUWEN FANG
 SCHOOL OF MATHEMATICAL SCIENCE
 YANGZHOU UNIVERSITY
 YANGZHOU 225002, P. R. CHINA
E-mail address: shwfang@163.com

FEI YANG
SCHOOL OF MATHEMATICS AND PHYSICS
CHINA UNIVERSITY OF GEOSCIENCES
WUHAN 430074, P. R. CHINA
E-mail address: yangfei810712@163.com