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A CHARACTERIZATION OF THE UNIT GROUP IN $\mathbb{Z}[T \times C_2]$

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Abstract. Describing the group of units $U(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$, for a finite group G, is a classical and open problem. In this note, we show that $U_1(\mathbb{Z}[T \times C_2]) \cong [F_{97} \rtimes F_5] \rtimes [T \times C_2]$, where $T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5 b\rangle$ and F_{97}, F_5 are free groups of ranks 97 and 5, respectively.

1. Introduction

Given a finite group G and the ring of integers $\mathbb{Z}$, we denote the integral group ring as $\mathbb{Z}G$. Its elements are all finite formal sums

$$\sum_{g \in G} r_g g, \text{ where } r_g \in \mathbb{Z}.$$

There is a surjective ring homomorphism ϵ: $\mathbb{Z}G \to \mathbb{Z}$, defined by

$$\sum_{g \in G} r_g g \longmapsto \sum_{g \in G} r_g.$$

The ring homomorphism ϵ is called the *augmentation map* and its kernel $\Delta_{\mathbb{Z}}(G) = \langle g - 1 : g \in G\rangle$ is the *augmentation ideal*. We will denote the group of units of $\mathbb{Z}G$ by $U(\mathbb{Z}G)$. $U_1(\mathbb{Z}G)$ will denote the units of augmentation one in $U(\mathbb{Z}G)$. Thus, $U_1(\mathbb{Z}G)$ is a normal subgroup of $U(\mathbb{Z}G)$ and $\pm U_1(\mathbb{Z}G) = U(\mathbb{Z}G)$. Observe that $\pm G \leq U(\mathbb{Z}G)$. The elements $\pm G$ are called the *trivial units* of $\mathbb{Z}G$.

Describing the units of the integral group ring is a classical and difficult problem. Over the years, it has drawn the attention of those working in the areas of algebra, number theory, and algebraic topology. Most descriptions of $U(\mathbb{Z}G)$ in the mathematical literature either give an explicit description of the units, the general structure of $U(\mathbb{Z}G)$, or a subgroup of finite index of the unit group $U(\mathbb{Z}G)$. These results were often obtained by using techniques from representation theory and algebraic number theory.

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In 1940, substantial work on the unit problem was done by Graham Higman [5, 6]. He first showed that if $U(\mathbb{Z}G) = \pm G$, then $U(\mathbb{Z}[G \times C_2]) = \pm(G \times C_2)$. Using this, he showed that $U(\mathbb{Z}G) = \pm G \iff$ G is abelian of exponent 2, 3, 4, or 6 or $G = E \times K_8$ where K_8 is the quaternion group of order 8 and E is an elementary abelian 2–group. Furthermore, Higman gave a general structure theorem for $U(\mathbb{Z}A)$, where A is a finite abelian group. Other results include: A_4 and S_4 by Allen-Hobby [1, 2], D_{2p} by Passman-Smith [21], $G = C_p \rtimes C_q$, where q is a prime dividing $p-1$ by Galovitch-Reiner-Ullom [4], $|G| = p^3$ by Ritter-Sehgal [23], and $U(\mathbb{Z}S_3)$ by Hughes-Pearson [7]. Jespers and Parmenter [10] gave a more explicit description of $U(\mathbb{Z}S_3)$. In 1993, Jespers and Parmenter [11] completed the description of $U(\mathbb{Z}G)$ for all groups of order 16. Jespers [9], in 1995, gave a description of $U(\mathbb{Z}G)$, for the dihedral group of order 12 and for $G = D_8 \times C_2$. More recently, Bilgin [3] gave a characterization of $U_1(\mathbb{Z}C_{12})$. Kusmus and Denizler [15] gave a construction of $U(\mathbb{Z}C_{24})$. Kelebek and Bilgin [14] described the structure of $U_1(\mathbb{Z}[C_n \times K_4])$. The interested reader is directed to Sehgal's [24] comprehensive survey on the unit problem in integral group rings.

In [17, 18], a general algebraic framework was developed to study $U(\mathbb{Z}G^*)$, where $G^* = G \times C_p$ with p prime. In the following sections of this note, we focus on the case where $p = 2$ and then resolve a conjecture found in [17].

2. $U(\mathbb{Z}[G \times C_2])$

Here, we obtain a result which helps us to answer the following question: Assuming that we have a good description of $U(\mathbb{Z}G)$, can we obtain a description of $U(\mathbb{Z}G^*)$, where $G^* = G \times C_2$?

Let $G^* = G \times \langle x \rangle$, $x^2 = 1$, with $|G| = n$. Decomposing G^* into two cosets, we have that $G^* = G \cup xG = \{g_1, g_2, \ldots, g_n, xg_1, \ldots, xg_n\}$. Thus, $\mathbb{Z}G^* = \mathbb{Z}G \oplus x\mathbb{Z}G$, a direct sum of abelian groups. Here, the equal sign denotes equality as sets. Now, consider the surjective group homomorphism $\pi : G^* \to G$ defined by $g \mapsto g, x \mapsto 1$. This induces a ring homomorphism $\pi : \mathbb{Z}G^* \to \mathbb{Z}G$; where $\pi(P_1 + xP_2) = P_1 + P_2$, and P_1, $P_2 \in \mathbb{Z}G$. At the ring level, $\mathrm{Ker}(\pi) = K^* = (x-1)\mathbb{Z}G$. So, we have the sequence of maps

$$K^* \xrightarrow{\iota} \mathbb{Z}G^* \xrightarrow{\pi} \mathbb{Z}G.$$

Restricting π to the group of units, we obtain the split exact sequence of groups:

$$K \xrightarrow{\iota} U(\mathbb{Z}G^*) \xrightarrow{\pi} U(\mathbb{Z}G),$$

where $K = \mathrm{Ker}(\pi)$. Hence, $U(\mathbb{Z}G^*) = K \rtimes U(\mathbb{Z}G)$. Note that $K = U(\mathbb{Z}G^*) \cap (1 + K^*)$. Thus, a unit in K has the form $1 + (x-1)P$, where $P \in \mathbb{Z}G$, and has an inverse $1 + (x-1)Q$, where $Q \in \mathbb{Z}G$.

Also, let us consider the surjective ring homomorphism $\rho : \mathbb{Z}G \twoheadrightarrow \mathbb{Z}_2G$, where ρ reduces the coefficients modulo 2. The kernel of ρ, say M^* (as an ideal), is $M^* = 2\mathbb{Z}G$. Thus, we have the following sequence of maps:

$$M^* \xrightarrow{\iota} \mathbb{Z}G \xrightarrow{\rho} \mathbb{Z}_2G.$$

Furthermore, ρ induces the following exact sequence of groups, which does not necessarily split:

$$M \xrightarrow{\ \iota\ } U(\mathbb{Z}G) \xrightarrow{\ \rho\ } U(\mathbb{Z}_2G),$$

where M is the kernel of the group homomorphism ρ. Observe that $M = U(\mathbb{Z}G) \cap (1 + M^*)$. Thus, a unit in M has the form $1 + 2P$, where $P \in \mathbb{Z}G$ and has an inverse $1 + 2Q$, where $Q \in \mathbb{Z}G$. Notice that here at the group level, ρ is not necessarily surjective.

Since $G^* = G \times \langle x \rangle$ and $x^2 = 1$, we have the group homomorphism $\sigma : G^* \to U(\mathbb{Z}G)$, where $\sigma(g) = g$ and $\sigma(x) = -1$. This extends to a ring homomorphism $\sigma : \mathbb{Z}G^* \to \mathbb{Z}G$. So, we have the following diagram of rings:

$$\begin{array}{ccccc} K^* & \xrightarrow{\ \iota\ } & \mathbb{Z}G^* & \xrightarrow{\ \pi\ } & \mathbb{Z}G \\ \sigma\downarrow & & \sigma\downarrow & & \rho\downarrow \\ M^* & \xrightarrow{\ \iota\ } & \mathbb{Z}G & \xrightarrow{\ \rho\ } & \mathbb{Z}_2G. \end{array}$$

Observe that $\rho \circ \pi = \rho \circ \sigma$. Hence, $\sigma(K^*) \subseteq M^*$. Note that σ maps the element $1+(x-1)P \in K$ to the element $1-2P \in M$, where $P \in \mathbb{Z}G$. Thus, $\sigma(K) \subseteq M$.

Lemma 2.1. *Let $G^* = G \times \langle x \rangle$, where x has order 2, $u = 1 + (x - 1)P$, $v = 1 + (x - 1)Q$, where P, $Q \in \mathbb{Z}G$. Then u and v are multiplicative inverses of each other in $K \iff 1 - 2P$ and $1 - 2Q$ are multiplicative inverses of each other in $U(\mathbb{Z}G)$.*

Proof. Let u, $v \in K$; with $uv = 1$. It is straightforward to see that $uv = 1 + (x - 1)(P + Q - 2PQ)$.

$$\begin{aligned} \text{Hence, } uv = 1 &\iff (x-1)(P+Q-2PQ) = 0 \\ &\iff (2PQ - P - Q) + (P + Q - 2PQ)x = 0 \\ &\iff 2PQ - P - Q = 0 \\ &\iff 4PQ - 2P - 2Q = 0 \\ &\iff 1 - 2P - 2Q + 4PQ = 1 \\ &\iff (1 - 2P)(1 - 2Q) = 1. \end{aligned}$$

□

Lemma 2.2. *The map $\sigma : K \to M$ is an isomorphism of groups.*

Proof. Note that σ maps the element $1 + (x - 1)P$ of K to the element $1 - 2P$ of M. It is then easy to show that σ is injective. It follows from Lemma 2.1 that σ is surjective. □

Summarizing, we have the following diagram of groups:

$$\begin{array}{ccccc} K & \xrightarrow{\ \iota\ } & U(\mathbb{Z}G^*) & \xrightarrow{\ \pi\ } & U(\mathbb{Z}G) \\ \cong\downarrow & & \sigma\downarrow & & \rho\downarrow \\ M & \xrightarrow{\ \iota\ } & U(\mathbb{Z}G) & \xrightarrow{\ \rho\ } & U(\mathbb{Z}_2G). \end{array}$$

Theorem 2.3. *$U(\mathbb{Z}G^*) = K \rtimes U(\mathbb{Z}G) \cong M \rtimes U(\mathbb{Z}G)$.*

Proof. The elements of the semi-direct product $M \rtimes U(\mathbb{Z}G)$ should be viewed as ordered pairs (u, w), where $u \in M$ and $w \in U(\mathbb{Z}G)$. If $k \in K$ and $w \in U(\mathbb{Z}G)$, then the isomorphism maps kw to $(\sigma(k), w)$ with the action of $U(\mathbb{Z}G)$ on M induced by conjugation in $U(\mathbb{Z}G)$. □

The problem of describing $U(\mathbb{Z}G^*)$ has been reduced to the problem of describing M. In the next section, we apply Theorem 2.3 and resolve a conjecture involving $U(\mathbb{Z}G^*)$, where G^* is a particular non-abelian group of order 24.

3. Resolution of a conjecture

It was shown by Jespers [8] that there are only four finite groups G with the property that G has a non-abelian free normal complement in $U_1(\mathbb{Z}G)$, namely $G = S_3$, D_4 (the dihedral group of order 8), $P = \langle a, b : a^4 = 1 = b^4, bab^{-1}a^{-1} = a^2\rangle$, and the non-abelian group T (of order 12) described by the presentation

$$T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5 b\rangle.$$

In [9, 11, 17], the structure of $U(\mathbb{Z}[G \times C_2])$ is determined for $G = S_3, D_4$ and P. In this section, we disprove the following conjecture, first posed in [17]:

Conjecture. *Let $T^* = T \times C_2$, where $T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5 b\rangle$. Then, $U_1(\mathbb{Z}T^*) \cong [F_{33} \rtimes F_5] \rtimes T^*$, where F_i is a free group of rank i.*

This is certainly a plausible conjecture. Later, it was shown in [12] that $U(\mathbb{Z}[T \times C_2])$ is commensurable with a free-by-free group. We will show that if F_{33} is replaced with F_{97}, then a correct result is obtained.

In 1993, Parmenter [20] showed that $U_1(\mathbb{Z}T) = V \rtimes T$, where $V = \langle v_1, v_2, v_3, v_4, v_5\rangle$ is a free group of rank five. He also gave the generators of V to be:

$$\begin{aligned}
v_1 &= 1 + (1 + a^3)(-a^2 + ba^2)(1 - a^2),\\
v_2 &= 1 + (1 + a^3)(-a^2 + ba)(1 - a^2),\\
v_3 &= 1 + (1 + a^3)(-a^2 + b)(1 - a^2),\\
v_4 &= 1 + [-1 + (1 + a^3)a^2(a^2 + ba^2)](1 - a^2),\\
v_5 &= 1 + [-1 - a^2 + (1 + a^3)a(1 - a - 2ba^2)](1 - a^2).
\end{aligned}$$

Let us determine $\rho(V)$. It is straight-forward to verify the following facts. First, $\rho(v_i)\rho(v_j) = \rho(v_j)\rho(v_i)$, where $1 \le i, j \le 3$. Also, $\rho(v_1)^2 = \rho(v_2)^2 = \rho(v_3)^2 = 1$ and thus, $E = \langle \rho(v_1), \rho(v_2), \rho(v_3)\rangle \cong C_2 \times C_2 \times C_2$. Now, calculations show that $a^2\rho(v_1)a^4 = \rho(v_2), a^2\rho(v_2)a^4 = \rho(v_3), a^2\rho(v_3)a^4 = \rho(v_1), a^2\rho(v_1) = \rho(v_4)$, and $[\rho(v_4)]^3 a^4 = \rho(v_5)$. Thus, $\langle a^2, \rho(v_1)\rangle = \langle a^2, \rho(v_2)\rangle = \langle a^2, \rho(v_3)\rangle = \rho(V)$.

Lemma 3.1. *$\rho(V) = E \rtimes \langle a^2\rangle$, a group of order 24.*

Proof. Since $E = \langle \rho(v_1), \rho(v_2), \rho(v_3)\rangle$ is normalized by $\rho(v_1), \rho(v_2), \rho(v_3)$, and a^2, we have that $E \trianglelefteq \rho(V)$. So, $E \cdot \langle a^2\rangle \le \rho(V)$. In fact, $E \cdot \langle a^2\rangle = \rho(V)$ and

$E \cap \langle a^2 \rangle = 1$. Thus, $\rho(V) = E \rtimes \langle a^2 \rangle \cong [C_2 \times C_2 \times C_2] \rtimes C_3$, a group of order 24. $\square$

Lemma 3.2. $\rho[U_1(\mathbb{Z}T)] = E \rtimes T$.

Proof. Clearly, $\rho[U_1(\mathbb{Z}T)] = \rho(V \rtimes T) = \rho(V) \cdot T$. Since $E \trianglelefteq \rho(V)$, we have that $\rho[U_1(\mathbb{Z}T)] = E \cdot T$. Since $a\rho(v_1)a^5 = \rho(v_3), a\rho(v_3)a^5 = \rho(v_2), a\rho(v_2)a^5 = \rho(v_1), b\rho(v_1)b^3 = \rho(v_1), b\rho(v_2)b^3 = \rho(v_3), b\rho(v_3)b^3 = \rho(v_2)$, we see that E is normalized by T. Note that $E \cap T = 1$. Hence, the lemma is established. $\square$

A remark should be made at this point. Since $\rho[U_1(\mathbb{Z}T)]$ has order 96, $|\rho(V)| = 24$, and $|T| = 12$, this implies that $|\rho(V) \cap T| = 3$. But $\langle a^2 \rangle \leq \rho(V) \cap T$, where the order of a^2 is 3. Hence, $\rho(V) \cap T = \langle a^2 \rangle$. Now, we have the diagram:

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & U(\mathbb{Z}T^*) & \xrightarrow{\pi} & U(\mathbb{Z}T) \\
\cong\big\downarrow & & \sigma\big\downarrow & & \rho\big\downarrow \\
M & \xrightarrow{\iota} & U(\mathbb{Z}T) & \xrightarrow{\rho} & U(\mathbb{Z}_2T) \\
\iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow \\
M^+ & \xrightarrow{\iota} & U_1(\mathbb{Z}T) & \xrightarrow{\text{onto}} & E \rtimes T \\
\iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow \\
M^+ \cap V & \xrightarrow{\iota} & V & \xrightarrow{\text{onto}} & E \rtimes \langle a^2 \rangle.
\end{array}$$

Lemma 3.3. $\rho(V) = \langle \rho(v_1), \rho(v_2), \rho(v_3), \rho(v_4) \rangle = \{[\rho(v_1)]^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4} : 0 \leq i_1, i_2, i_3 \leq 1; 0 \leq i_4 \leq 2\}$. *Furthermore, this canonical representation is unique.*

Proof. Note that $\rho(V) = \langle a^2, \rho(v_1) \rangle = \langle \rho(v_1), \rho(v_2), \rho(v_3), \rho(v_4) \rangle$. Also, calculations show the following:

$\rho(v_4)\rho(v_1) = \rho(v_2)a^2\rho(v_1) = \rho(v_2)^2 a^2 = a^2 = \rho(v_2)\rho(v_4)$,

$\rho(v_4)\rho(v_2) = \rho(v_2)a^2\rho(v_2) = \rho(v_2)\rho(v_3)a^2 = \rho(v_2)\rho(v_3)\rho(v_2)\rho(v_4) = \rho(v_3)\rho(v_4)$,

$\rho(v_4)\rho(v_3) = \rho(v_2)a^2\rho(v_3) = \rho(v_2)\rho(v_1)a^2 = \rho(v_2)\rho(v_1)\rho(v_2)\rho(v_4) = \rho(v_1)\rho(v_4)$,

$\rho(v_4)^2\rho(v_1) = \rho(v_3)\rho(v_4)^2$,

$\rho(v_4)^2\rho(v_2) = \rho(v_1)\rho(v_4)^2$,

$\rho(v_4)^2\rho(v_3) = \rho(v_2)\rho(v_4)^2$,

$\rho(v_4)^3\rho(v_1) = \rho(v_2)\rho(v_3)$,

$\rho(v_4)^3\rho(v_2) = \rho(v_1)\rho(v_3)$,

$\rho(v_4)^3\rho(v_3) = \rho(v_1)\rho(v_2)$,

$\rho(v_4)^4\rho(v_1) = \rho(v_4)\rho(v_4)^3\rho(v_1) = \rho(v_1)\rho(v_3)\rho(v_4)$,

$\rho(v_4)^4\rho(v_2) = \rho(v_4)\rho(v_4)^3\rho(v_2) = \rho(v_1)\rho(v_2)\rho(v_4)$,

$\rho(v_4)^4\rho(v_3) = \rho(v_4)\rho(v_4)^3\rho(v_3) = \rho(v_2)\rho(v_3)\rho(v_4),$

$\rho(v_4)^5\rho(v_1) = \rho(v_1)\rho(v_2)\rho(v_3)\rho(v_4)^2\rho(v_1) = \rho(v_1)\rho(v_2)\rho(v_4)^2,$

$\rho(v_4)^5\rho(v_2) = \rho(v_1)\rho(v_2)\rho(v_3)\rho(v_4)^2\rho(v_2) = \rho(v_2)\rho(v_3)\rho(v_4)^2,$

$\rho(v_4)^5\rho(v_3) = \rho(v_1)\rho(v_2)\rho(v_3)\rho(v_4)^2\rho(v_3) = \rho(v_1)\rho(v_3)\rho(v_4)^2.$

Thus, every word in $\rho(V)$ can be put into the canonical form $[\rho(v_1)]^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4}$, where $0 \leq i_1, i_2, i_3 \leq 1$ and $0 \leq i_4 \leq 2$. This representation is unique, since $|\rho(V)| = 24$. □

Lemma 3.4. *Let $w[\rho(v_1), \rho(v_2), \rho(v_3)] \in E, t \in T$, with $w[\rho(v_1), \rho(v_2), \rho(v_3)] \cdot t = 1$. Then $t = 1_T$.*

Proof. Suppose that $w[\rho(v_1), \rho(v_2), \rho(v_3)] \cdot t = 1 = E \cap T$. Then, we have $w[\rho(v_1),\ \rho(v_2), \rho(v_3)] = t^{-1} \in T$ and $w[\rho(v_1), \rho(v_2), \rho(v_3)] \in E$. This implies that $w[\rho(v_1),\ \rho(v_2), \rho(v_3)] \in E \cap T = 1_T$. Thus, $w[\rho(v_1), \rho(v_2), \rho(v_3)] = 1$, which implies that $t^{-1} = 1_T$. Hence, $t = 1_T$. □

Lemma 3.5. *$M^+ \leq V \rtimes \langle a^2 \rangle$.*

Proof. Suppose that $w(v_1, v_2, v_3, v_4, v_5) \cdot t \in M^+$, where $t \in T$. This implies that $\rho[w(v_1, v_2, v_3, v_4, v_5) \cdot t] = 1$. By Lemma 3.3, we have that $([\rho(v_1)]^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4}) \cdot t = 1$, where $0 \leq i_1, i_2, i_3 \leq 1; 0 \leq i_4 \leq 2; t \in T$. Now, $[\rho(v_1)]^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4}$ has three possible forms:

$$\begin{cases} \rho(v_1)^{i_1} \cdot \rho(v_2)^{i_2} \cdot \rho(v_3)^{i_3} \cdot [\rho(v_2)a^2], & \text{if } i_4 = 1; \\ \rho(v_1)^{i_1} \cdot \rho(v_2)^{i_2} \cdot \rho(v_3)^{i_3} \cdot [\rho(v_2)\rho(v_3)a^4], & \text{if } i_4 = 2; \\ \rho(v_1)^{i_1} \cdot \rho(v_2)^{i_2} \cdot \rho(v_3)^{i_3}, & \text{if } i_4 = 0. \end{cases}$$

Using Lemma 3.4, we have $([\rho(v_1)]^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4}) \cdot t = 1$ implies that $t \in \langle a^2 \rangle$. □

Lemma 3.6. *M^+ is a free group of rank 97.*

Proof. Since $M^+ \leq \rho^{-1}[E \rtimes T]$, where $E = \langle \rho(v_1), \rho(v_2), \rho(v_3) \rangle \cong C_2 \times C_2 \times C_2$, we see that M^+ consists of the elements of the form

$$\rho^{-1}[\rho(v_1)^{j_1}\rho(v_2)^{j_2}\rho(v_3)^{j_3} \cdot t],$$

where $0 \leq j_1, j_2, j_3 \leq 1$ and $t \in T$. Since M^+ is an appropriate kernel of ρ, then $\rho(M^+) = 1$. If we consider an element in M^+ as

$$\alpha = \rho^{-1}[\rho(v_1)^{j_1}\rho(v_2)^{j_2}\rho(v_3)^{j_3} \cdot t],$$

we see that $\rho(\alpha) = \rho(v_1)^{j_1}\rho(v_2)^{j_2}\rho(v_3)^{j_3} \cdot t = 1$. By Lemma 3.4, $t = 1$. This implies that M^+ consists of elements of the form

$$\rho^{-1}[\rho(v_1)^{j_1}\rho(v_2)^{j_2}\rho(v_3)^{j_3}] \in V.$$

Thus, $M^+ \leq V$. Since V is a free group, the Nielson-Schreier Theorem states that M^+ is a free group. Note that $M^+ = M^+ \cap V$. Now, consider the induced isomorphism $\bar{\rho} : \frac{V}{M^+ \cap V} \twoheadrightarrow E \rtimes \langle a^2 \rangle$, which implies that $[V : M^+ \cap V] = [V :$

$M^+] = 24$. Since V is a free group of rank 5, this implies that M^+ is a free group of rank $(24)(5) - 24 + 1 = 97$. □

Theorem 3.7. *Let* $T^* = T \times C_2$*, where* $T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5 b\rangle$*. Then,* $U_1(\mathbb{Z}T^*) \cong [F_{97} \rtimes F_5] \rtimes T^*$*, where* F_i *is a free group of rank* i*.*

Proof. Invoking Theorem 2.3, we obtain $U_1(\mathbb{Z}[T \times C_2]) = K \rtimes (V \rtimes T) \cong M \rtimes (V \rtimes T) = [M^+ \times C_2] \rtimes (V \rtimes T) = [M^+ \rtimes V] \rtimes (T \times C_2) = [F_{97} \rtimes F_5] \rtimes (T \times C_2)$, where F_i is a free group of rank i. □

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