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GEOMETRIC INEQUALITIES FOR SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Abstract. B. Y. Chen introduced a series of curvature invariants, known as Chen invariants, and proved sharp estimates for these intrinsic invariants in terms of the main extrinsic invariant, the squared mean curvature, for submanifolds in Riemannian space forms.

Special classes of submanifolds in Sasakian manifolds play an important role in contact geometry. F. Defever, I. Mihai and L. Verstraelen [8] established Chen first inequality for C-totally real submanifolds in Sasakian space forms.

Also, the differential geometry of slant submanifolds has shown an increasing development since B. Y. Chen defined slant submanifolds in complex manifolds as a generalization of both holomorphic and totally real submanifolds. The slant submanifolds of an almost contact metric manifolds were defined and studied by A. Lotta, J. L. Cabrerizo et al.

A Chen first inequality for slant submanifolds in Sasakian space forms was established by A. Carriazo [4]. In this article, we improve this Chen first inequality for special contact slant submanifolds in Sasakian space forms.

1. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $\widetilde{M}$ it said to be a Sasakian manifold if it admits an endomorphism ϕ of its tangent bundle $T\widetilde{M}$, a vector field ξ and a 1-form η, satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi),$$
$$(\widetilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X,\ \widetilde{\nabla}_X \xi = -\phi X,$$

for any vector fields X, Y on $T\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T\widetilde{M}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a

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ϕ-section is called a ϕ-sectional curvature. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a Sasakian space form and is denoted by $\widetilde{M}^{2n+1}(c)$.

The curvature tensor $\widetilde{R}$ of a Sasakian space form $\widetilde{M}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) + \frac{c-1}{4}(\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z)\end{aligned} \tag{1.1}$$

for any tangent vector fields X, Y, Z to $\widetilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and S^{2m+1} with standard Sasakian structures (see [1], [12]).

Let M be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}(c)$ of constant ϕ-sectional curvature c. We denote by $K(\pi)$ the sectional curvature associated with a plane section $\pi \subset T_pM$, $p \in M$, and ∇ the Riemannian connection of M, respectively. Also, let h be the second fundamental form and R the Riemannian curvature tensor of M.

Then the equation of Gauss is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) \\ & - g(h(X,Z),h(Y,W))\end{aligned} \tag{1.2}$$

for any vector fields X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, e_2, \ldots, e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}$, such that e_1, e_2 and e_n are tangent to M at p. We denote by H the mean curvature vector, that is $H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$.

Also, we set

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)).$$

2. Special contact slant submanifolds

A submanifold normal to ξ in a Sasakian manifold is said to be a C-totally real submanifold. It is known that ϕ maps any tangent space T_pM to the normal space $T_p^{\perp}M$, $p \in M$.

B. Y. Chen [5] introduced the notion of a slant submanifold in a Hermitian manifold. In the geometry of submanifolds in contact manifolds we have the following corresponding notion.

Definition 2.1. A submanifold M tangent to ξ in a Sasakian manifold $\widetilde{M}$ is said to be a *contact slant submanifold* if for any $p \in M$ and any $X \in T_pM$ linearly independent on ξ, the angle between ϕX and T_pM is a constant θ, called the slant angle of M.

For any vector field X tangent to M, we put $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX.

A proper slant submanifold (see [3]) is a contact slant submanifold which is neither invariant nor anti-invariant.

A particular case of a contact slant submanifold, which is called special contact slant submanifold, was considered in [11].

Definition 2.2. A proper contact θ-slant submanifold is said to be *special contact slant* if

$$(\nabla_X P)(Y) = \cos^2(\theta)[g(X,Y)\xi - \eta(Y)X], \ \forall X, Y \in \Gamma(TM). \tag{2.1}$$

Remark. Any 3-dimensional proper contact slant submanifold of a Sasakian space form is a special contact slant submanifold.

Before giving some examples of special contact slant submanifolds, we remind the following result from [3].

Theorem 2.1. *Suppose that*

$$x(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v))$$

defines a slant surface S in complex space $\mathbb{C}^2$ with its usual Kaehlerian structure, such that $\partial/\partial u$ and $\partial/\partial v$ are non-zero and perpendicular. Then

$$y(u,v,t) = 2(f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v), t)$$

defines a 3-dimensional slant submanifold M in $(\mathbb{R}^5, \phi_0, \eta, g)$, such that if we put

$$e_1 = \frac{\partial}{\partial u} + (2f_3\frac{\partial f_1}{\partial u} + 2f_4\frac{\partial f_2}{\partial v})\frac{\partial}{\partial t}$$

and

$$e_2 = \frac{\partial}{\partial v} + (2f_3\frac{\partial f_1}{\partial v} + 2f_4\frac{\partial f_2}{\partial v})\frac{\partial}{\partial t},$$

then $\{e_1, e_2, \xi\}$ is an orthogonal basis of the tangent bundle of the submanifold.

Example 2.1. For any constant k,

$$x(u,v,t) = 2(u, k\cos v, v, k\sin v, t)$$

defines a special contact slant submanifold M with slant angle

$$\theta = \cos^{-1}(1/\sqrt{1+k^2}).$$

Proof. Let consider on the manifold $\mathbb{R}^5$ the standard Sasakian structure, as follows

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{2} y^i dx^i),$$

$$g = \eta \otimes \eta + \frac{1}{4}(\sum_{i=1}^{2} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi_0(\sum_{i=1}^{2}(X_i\frac{\partial}{\partial x^i}+Y_i\frac{\partial}{\partial y^i})+Z\frac{\partial}{\partial z})=\sum_{i=1}^{2}(Y_i\frac{\partial}{\partial x^i}-X_i\frac{\partial}{\partial y^i})+\sum_{i=1}^{2}Y_iy^i\frac{\partial}{\partial z}.$$

So, we have

$$\frac{\partial x}{\partial u}=2(1,0,0,0,0),\ \frac{\partial x}{\partial v}=(0,-2k\sin v,2,2k\cos v,0),\ \frac{\partial x}{\partial t}=(0,0,0,0,1),$$

and

$$\frac{\partial f_1}{\partial u}=1,\ \frac{\partial f_2}{\partial u}=0,\ \frac{\partial f_1}{\partial v}=0,\ \frac{\partial f_2}{\partial v}=-k\sin v.$$

It follows that

$$e_1=(2,0,0,0,2v),\ e_2=(0,-2k\sin v,2,2k\cos v,-2k^2\sin^2 v).$$

By orthonormalisation we find an orthonormal frame $\{e_1^*,e_2^*,\xi\}$.

Since the dimension is 3, x is a special contact slant immersion. □

As usual, we denote by $\tau(p)$ the scalar curvature at $p\in M$,

$$\tau(p)=\sum_{1\le i<j\le n}K(e_i\wedge e_j),$$

with $\{e_1,e_2,\dots,e_n\}$ an orthonormal basis of T_pM.

For a 2-plane section $\pi\subset T_pM$ orthogonal to ξ, we denote (see [4]) by

$$\psi^2(\pi)=g^2(Pe_1,e_2),$$

where $\{e_1,e_2\}$ is an orthonormal basis of π.

Then $\psi^2(\pi)$ is a real number in $[0,1]$ which is independent of the choice of the orthonormal basis $\{e_1,e_2\}$ of π.

3. An improved Chen first inequality for special slant submanifolds

B. Y. Chen has introduced several curvature invariants: the first Chen's invariant is

$$\delta_M(p)=\tau(p)-\inf K(p),$$

where τ is the scalar curvature and $\inf K(p)=\inf\{K(\pi);\pi\subset T_pM,\dim\pi=2\}$. Afterwards he defined more general curvature invariants by

$$\delta(n_1,\dots,n_k)(p)=\tau(p)-S(n_1,\dots,n_k)(p),$$

where $S(n_1,\dots,n_k)(p)=\inf\{\tau(L_1)+\cdots+\tau(L_k)\}$, with L_i subspaces of T_pM of dimension n_i, and $n_1+\cdots+n_k\le n$. Moreover B. Y. Chen established sharp estimations of all these intrinsic invariants in terms of the main extrinsic characteristic of an immersion, as can be the norm of the mean curvature $\|H\|$. These inequalities are known as Chen inequalities.

For the motivation of the introduction of Chen invariants and further developments in the theory of Chen invariants and Chen inequalities, we refer to [6].

A. Carriazo has proved the first Chen inequality for submanifolds tangent to the structure vector field ξ of a Sasakian space form. More precisely, he has proved the following theorem for proper slant submanifolds in Sasakian space forms.

Theorem 3.1. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(c)$ be an isometric immersion from a Riemannian $(n+1)$-manifold into a Sasakian space form $\widetilde{M}^m(c)$ such that $\xi \in TM$. Then, for any point $p \in M$ and any plane sections $\pi \subset D_p$, we have*

$$\begin{aligned}\tau - K(\pi) \leq & \frac{(n+1)^2(n-1)}{2n}\|H\|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} \\ & + n + \frac{3}{2}\|T\|^2\frac{c-1}{4} - 3\Phi^2(\pi)\frac{c-1}{4} - \|N\|^2.\end{aligned}$$

Then, D. Cioroboiu and A. Oiagă [7] have proved a Chen first inequality for slant submanifolds, with plane sections π orthogonal to ξ in Sasakian space forms. This result is given by the following theorem.

Theorem 3.2. *Let M be an $(n = 2k+1)$-dimensional θ-slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then we have*

$$\delta(M) \leq \frac{n-2}{2}\{\frac{n^2}{n-1}\|H\|^2 + \frac{(c+3)(n+1)}{4}\} + \frac{(c-1)}{8}[3(n-3)\cos^2\theta - 2(n-1)].$$

Moreover they proved general Chen inequalities involving the Chen invariants $\delta(n_1, \ldots, n_k)$ for such submanifolds.

Also, in [10] an improved Chen first inequality for purely real submanifolds M in complex space forms $\widetilde{M}(4c)$ of holomorphic sectional curvature $4c$ has been proven by A. Mihai. This inequality has the following form.

Theorem 3.3. *Let M be an n-dimensional $(n \geq 3)$ purely real submanifold of an m-complex space form $\widetilde{M}(4c)$, $p \in M$ and $\pi \subset T_pM$ a 2-plane section. Then*

$$\tau(p) - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + [(n+1)(n+2) + 3\|P\|^2 - 6\Phi^2(\pi)]\frac{c}{2},$$

where $\Phi^2(\pi) = g^2(Je_1, e_2)$ and e_1, e_2 is an orthonormal basis of π.

We establish a corresponding inequality for special contact slant submanifolds in Sasakian space forms $\widetilde{M}(c)$.

For $(n+1)$-dimensional special slant submanifolds in $(2n+1)$-dimensional Sasakian space forms $\widetilde{M}(c)$ we prove a Chen first inequality which improves the Chen first inequality of Theorem 3.1 from [4].

Theorem 3.4. *Let M be an $(n+1)$-dimensional special contact slant submanifold into a Sasakian space form $\widetilde{M}(c)$ and $p \in M$, $\pi \subset T_pM$ a 2-plane section orthogonal to ξ. Then*

$$\tau(p) - \inf K(p) \leq \frac{n^2(2n-3)}{2(2n+3)}\|H\|^2 + \frac{(n+1)(n-2)}{8}(c+3) \tag{3.1}$$

$$+ 3n\cos^2\theta\frac{(c-1)}{8} - \frac{3\psi^2(\pi)}{4}(c-1) + n\cos^2\theta,$$

where H is the mean curvature vector.

Moreover, the equality case of the inequality holds for some plane section π at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_0 = \xi, e_1, e_2, \dots, e_n\}$ at p such that $\pi = span\{e_1, e_2\}$ and with respect to this basis the second fundamental form takes the following form

$$h(e_1,e_1) = aFe_1 + 3bFe_3, \quad h(e_1,e_3) = 3bFe_1, \quad h(e_3,e_j) = 4bFe_j,$$
$$h(e_2,e_2) = -aFe_1 + 3bFe_3, \quad h(e_2,e_3) = 3bFe_2, \quad h(e_j,e_k) = 4bFe_3\delta_{jk},$$
$$h(e_1,e_2) = -aFe_2, \qquad h(e_3,e_3) = 12bFe_3, \quad h(e_1,e_j) = h(e_2,e_j) = 0$$

for some numbers a, b and $j, k = 4, \dots, n$.

Proof. We assume that M is an $(n+1)$-dimensional special contact submanifold of a $(2n+1)$-dimensional Sasakian space form$\widetilde{M}(c)$. Let $p \in M$, $\pi \subset T_pM$ a 2-plane section orthogonal to ξ and $\{e_0 = \xi, e_1, e_2, \dots, e_n\}$ an orthonormal basis of the tangent space T_pM such that $e_1, e_2 \in \pi$. An orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ of the normal space $T_p^\perp M$, is given by $e_{n+j} = \frac{Fe_j}{\sin\theta}$, $\forall j = 1, 2, \dots, n$.

If we choose $X = W = e_i$ and $Y = Z = e_j$, $i, j = 1, \dots, n$, then (1.1) implies

$$\widetilde{R}(e_i, e_j, e_i, e_j) = \frac{c+3}{4} + \frac{3(c-1)}{4}g^2(Pe_i, e_j). \tag{3.2}$$

By using the Gauss equation and summing after i, j in (3.2), it follows that

$$\begin{aligned}2\tau(p) = &\sum_{i,j=1}^{n}\frac{c+3}{4} + \sum_{i,j=1}^{n}\frac{3(c-1)}{4}g^2(Pe_i, e_j)\\ &+ \sum_{i,j=1}^{n} g(h(e_i,e_i), h(e_j,e_j)) - \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j))\\ &+ 2\sum_{j=1}^{n}\tilde{K}(\xi\wedge e_j) - 2\sum_{j=1}^{n} g(h(\xi,e_j), h(\xi,e_j)).\end{aligned}$$

Let A be the shape operator and h the second fundamental form of M. We can write

$$h(e_i, e_j) = \sum_{r=1}^{n} g(h(e_i,e_j), e_r^*)e_r^*,$$

with $e_r^* = e_{n+r}$, $\forall i, j, r = 1, 2, \dots, n$.

Using the fact that $\widetilde{M}$ is, in particular, a Sasakian manifold and the formula of Gauss,

$$\widetilde{\nabla}_X\xi = \nabla_X\xi + h(X,\xi),$$

we obtain

$$g(-\phi X, e_r^*) = g(\nabla_X\xi, e_r^*) + g(h(X,\xi), e_r^*),$$

which implies

$$g(-FX, e_r^*) = g(h(X,\xi), e_r^*),\ r \in \overline{1,n}.$$

We denote, as usual, by $h_{ij}^k = g(h(e_i,e_j), e_k^*)$ for all $i,j,k = 1,\dots,n$, the components of the second fundamental form.

Hence, we obtain

$$\begin{aligned} 2\tau(p) = {} & \frac{n(n-1)(c+3)}{4} + n\cos^2\theta \frac{3(c-1)}{4} \\ & + 2\sum_{r=1}^{n} \sum_{1\le i<j\le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + 2n - 2n\sin^2\theta, \end{aligned}$$

which implies

$$\tau(p) = \sum_{r=1}^{n} \sum_{1\le i<j\le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{n(n-1)(c+3)}{8} + n\cos^2\theta \frac{3(c-1)}{8} + n\cos^2\theta.$$

Similarly, by computing $K(\pi)$, we get

$$K(\pi) = \sum_{r=1}^{n} [h_{11}^r h_{22}^r - (h_{12}^r)^2] + \frac{c+3}{4} + \frac{3(c-1)}{4}\psi^2(\pi).$$

It follows that

$$\begin{aligned} \tau(p) - K(\pi) = {} & \sum_{r=1}^{n} \Big\{ \sum_{j=3}^{n} (h_{11}^r + h_{22}^r) h_{jj}^r + \sum_{3\le i<j\le n} h_{ii}^r h_{jj}^r \\ & - \sum_{j=3}^{n} [(h_{1j}^r)^2 + (h_{2j}^r)^2] - \sum_{2\le i\neq j\le n} (h_{ij}^r)^2 \Big\} \\ & + \frac{(n+1)(n-2)}{8}(c+3) + 3n\cos^2\theta \frac{c-1}{8} \\ & - \frac{3(c-1)}{4}\psi^2(\pi) + n\cos^2\theta. \end{aligned} \tag{3.3}$$

Because M is a special contact slant submanifold, from (2.1) it follows that the components of the second fundamental form are symmetric, i.e.,

$$h_{ij}^k = h_{ik}^j = h_{jk}^i,\ \forall i,j,k = 1,\dots,n.$$

In particular, $h_{1j}^1 = h_{11}^j$, $h_{1j}^j = h_{jj}^1$ for $3 \le j \le n$, and $h_{ij}^j = h_{jj}^i$ for $2 \le i \neq j \le n$. Then the above inequality implies

$$\begin{aligned} \tau(p) - K(\pi) \le {} & \sum_{r=1}^{n} \Big\{ \sum_{j=3}^{n} (h_{11}^r + h_{22}^r) h_{jj}^r + \sum_{3\le i<j\le n} h_{ii}^r h_{jj}^r \\ & - \sum_{j=3}^{n} [(h_{11}^j)^2 - \sum_{j=3}^{n} (h_{jj}^1)^2 - \sum_{2\le i\neq j\le n} (h_{jj}^i)^2 \Big\} \\ & + \frac{(n+1)(n-2)}{8}(c+3) + 3n\cos^2\theta \frac{c-1}{8} \end{aligned}$$

$$-\frac{3(c-1)}{4}\psi^2(\pi)+n\cos^2\theta.$$

In order to achieve the proof we will use some ideas from [2].

We use the following inequalities

$$\begin{aligned}\sum_{j=3}^{n}(h_{11}^r+h_{22}^r)h_{jj}^r+\sum_{3\le i<j\le n}h_{ii}^rh_{jj}^r-\sum_{j=3}^{n}(h_{jj}^r)^2\\ \le\frac{n-2}{2(n+1)}(h_{11}^r+h_{22}^r+\cdots+h_{nn}^r)^2\\ \le\frac{2n-3}{2(2n+3)}(h_{11}^r+\cdots+h_{nn}^r)^2\end{aligned}\tag{3.5}$$

for $r=1,2$. The first inequality in (3.5) is equivalent to

$$\sum_{j=3}^{n}(h_{11}^r+h_{22}^r-3h_{jj}^r)^2+3\sum_{3\le i<j\le n}(h_{ii}^r-h_{jj}^r)^2\ge 0.$$

The equality holds if and only if $3h_{jj}^r=h_{11}^r+h_{22}^r,\ \forall j=3,\dots,n$.

The equality also holds in the second inequality if and only if $h_{11}^r+h_{22}^r=0$ and $h_{jj}^r=0,\ \forall j=3,\dots,n$ and $r=1,2$. Also, we have

$$\begin{aligned}\sum_{j=3}^{n}(h_{11}^r+h_{22}^r)h_{jj}^r+\sum_{3\le i<j\le n}h_{ii}^rh_{jj}^r-\sum_{j=1,j\ne r}^{n}(h_{jj}^r)\\ \le\frac{2n-3}{2(2n+3)}(h_{11}^r+\cdots+h_{nn}^r)^2\end{aligned}\tag{3.6}$$

for $r=3,\dots,n$, which is equivalent to

$$\begin{aligned}&\sum_{3\le j\le n,j\ne r}[2(h_{11}^r+h_{22}^r)-3h_{jj}^r]^2+(2n+3)(h_{11}^r-h_{22}^r)^2\\&+6\sum_{3\le i<j\le n,i,j\ne r}(h_{ii}^r-h_{jj}^r)^2+2\sum_{j=3}^{n}(h_{rr}^r-h_{jj}^r)^2\\&+3[h_{rr}^r-2(h_{11}^r+h_{22}^r)]^2\ge 0.\end{aligned}$$

The equality holds if and only if

$$h_{11}^r=h_{22}^r=3\lambda^r, `h_{jj}^r=4\lambda^r;\forall j=3,\dots,n,\ j\ne r,\ r=3,\dots,n,$$

and $h_{rr}^r=12\lambda^r$, $\lambda^r\in\mathbb{R}$.

By summing the inequalities (3.5), respectively (3.6), we obtain the following inequality

$$\begin{aligned}\tau-K(\pi)\le\frac{n^2(2n-3)}{2(2n+3)}\|H\|^2+\frac{(n+1)(n-2)}{8}(c+3)+3n\cos^2\theta\frac{(c-1)}{8}\\-\frac{3\psi^2(\pi)}{4}(c-1)+n\cos^2\theta,\end{aligned}$$

which is the inequality (3.1).

Combining the above equality cases, we get the desired forms of the second fundamental form. □

Remark. The above inequality does not hold for arbitrary contact slant submanifolds in Sasakian space forms. In the proof we used the symmetry of the components of the second fundamental form, which is characteristic to special contact slant submanifolds.

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