Bull. Korean Math. Soc. **53** (2016), No. 4, pp. 1087–1094
http://dx.doi.org/10.4134/BKMS.b150521
pISSN: 1015-8634 / eISSN: 2234-3016

WEAKLY EINSTEIN CRITICAL POINT EQUATION

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Abstract. On a compact n-dimensional manifold M, it has been conjectured that a critical point of the total scalar curvature, restricted to the space of metrics with constant scalar curvature of unit volume, is Einstein. In this paper, after derivng an interesting curvature identity, we show that the conjecture is true in dimension three and four when g is weakly Einstein. In higher dimensional case $n \geq 5$, we also show that the conjecture is true under an additional Ricci curvature bound. Moreover, we prove that the manifold is isometric to a standard n-sphere when it is n-dimensional weakly Einstein and the kernel of the linearized scalar curvature operator is nontrivial.

1. Introduction

Let M be a compact n-dimensional orientable manifold and $\mathcal{M}_1$ be the set of all smooth Riemannian metrics of unit volume on M. It is well known that the critical points of the total scalar curvature $\mathcal{S}$ on $\mathcal{M}_1$ given by

$$\mathcal{S}(g) = \int_M s_g \, dv_g$$

are Einstein metrics [2]. Here, s_g is the scalar curvature, and dv_g is the volume form determined by the metric and orientation.

Consider the space $\mathcal{C}$ of constant scalar curvature metrics with unit volume. Due to N. Koiso [4], it turns out that g is a critical point of $\mathcal{S}$ restricted to $\mathcal{C}$ if there exists a function f such that

$$z_g = s_g'^*(f). \tag{1}$$

We call (1) the critical point equation, or the CPE. Here, z_g is the traceless Ricci tensor of g, and $s_g'^*$ is given by

$$s_g'^*(f) = D_g df - (\Delta_g f)\, g - f r_g, \tag{2}$$

Received July 1, 2015; Revised August 9, 2015.

2010 *Mathematics Subject Classification.* Primary 53C25, 58E11.

Key words and phrases. total scalar curvature, critical point metric, weakly Einstein, Einstein metric, linearized scalar curvature.

The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2011-0007465).

where $D_g d$ and Δ_g denote the Hessian and (negative) Laplacian, respectively, and r_g is the Ricci curvature of g. For the remainder of this paper, we denote s_g by s, r_g by r, z_g by z, and the Riemann curvature tensor R_g and the Weyl curvature tensor $\mathcal{W}_g$ by R and $\mathcal{W}$, respectively, when there is no ambiguity.

When g is Einstein, (1) is equivalent to Obata's equation; hence, such a solution is isometric to a standard n-sphere [6]. It has been conjectured that this is the only possible case [2]. We refer to this conjecture as the Besse conjecture.

There are a few known results on Besse conjecture when the solution function f to the CPE is nontrivial. Lafontaine showed that the conjecture holds if a solution metric g is conformally flat and the kernel of $s_g'^*$ is nontrivial, or $\ker s_g'^* \neq 0$ [5]. Recently, Yun, Chang, and Hwang showed that the Besse conjecture is true for Riemannian manifolds with harmonic curvature [8].

On the other hand, we observe that there are non-Einstein metrics that are critical for the full Riemannian curvature functional on $\mathcal{M}_1$ defined by

$$g \mapsto \int_M |R_g|^2 \, dv_g. \tag{3}$$

It is well known that an Einstein metric is critical to the functional given in (3) if and only if the Riemann curvature tensor R satisfies the following equation [1]:

$$\check{R} = \frac{1}{n}|R|^2 g. \tag{4}$$

Here, $\check{R}$ is defined by

$$\check{R}(X,Y) = \sum_{i,j,k=1}^{n} R(X, e_i, e_j, e_k) R(Y, e_i, e_j, e_k)$$

for an orthonormal frame $\{e_i\}$, $i = 1, \dots, n$. In [3], metrics satisfying (4) in four dimensional space are referred to as *weakly Einstein metrics*. For simplicity, we call (4) the n-dimensional weakly Einstein metrics.

Little is known about the weakly Einstein condition. Clearly not every manifold is weakly Einstein; for example, $\mathbb{R} \times S^3$ is not weakly Einstein. When $\dim M = 4$, an Einstein metric is automatically weakly Einstein. However, the converse does not hold; a standard product metric on $\mathbb{S}^2 \times \mathbb{H}^2$ is weakly Einstein, but not Einstein. Moreover, when $\dim M > 4$, a generic Einstein metric does not necessarily satisfy the weakly Einstein condition (4).

In light of these facts, it is natural to ask whether an n-dimensional weakly Einstein metric that is a nontrivial solution to the CPE is Einstein. Our main result confirms that this is true when $n \leq 4$, and with additional Ricci curvature bound for $n \geq 5$.

Theorem 1.1. *Let (g, f) be a nontrivial solution to the CPE on a compact n-dimensional manifold M, $n \geq 3$. Suppose that g is n-dimensional weakly Einstein. For $n \geq 5$, we assume in addition that the Ricci curvature in ∇f*

direction is bounded below by $r_g(\nabla f, \nabla f) > -\frac{s_g}{n-4}|\nabla f|^2$. *Then the Besse conjecture holds.*

To prove Theorem 1.1, we need the following interesting curvature identity.

Theorem 1.2. *Let* (M^n, g) *be a Riemannian manifold. Then*

$$\check{R} - \frac{1}{n}|R|^2 g = \check{\mathcal{W}} - \frac{1}{n}|\mathcal{W}|^2 g + \frac{2n}{(n-2)^2}\left(z \circ z - \frac{1}{n}|z|^2 g\right) + \frac{4}{n-2}\left(\mathring{R}r - \frac{1}{n}|r|^2 g\right).$$

Moreover, we consider the condition $\ker s_g'^* \neq 0$ on a compact Riemannian manifold (M, g). Then we have the following result.

Theorem 1.3. *Assume that* $\varphi \in \ker s_g'^* \setminus \{0\}$ *on a compact Riemannian manifold* (M, g). *Suppose that* g *is* n*-dimensional weakly Einstein. For* $n \geq 5$, *we assume in addition that the Ricci curvature in* $\nabla\varphi$ *direction is bounded below by* $r_g(\nabla\varphi, \nabla\varphi) > -\frac{s_g}{n-4}|\nabla\varphi|^2$. *Then* (M, g) *is isometric to a standard* n*-sphere.*

We remark that if a function φ on M satisfies $\varphi \in \ker s_g'^* \setminus \{0\}$, by (2)

$$D_g d\varphi - (\Delta_g \varphi)\, g - \varphi\, r_g = 0.$$

For the detail about this equation, refer to [9].

The remainder of this paper is organized as follows. In Section 2, we investigate the relationship between the CPE and weakly Einstein metrics. In Section 3, we prove our main result. Finally in Section 4, we discuss compact Riemannian manifolds when the kernel of $s_g'^*$ is nontrivial.

2. Weakly Einstein metrics

In this section, we study the CPE and weakly Einstein metrics. To begin, we fix our convention. The Riemann curvature tensor R is defined by

$$R(X, Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z$$

and the Ricci curvature r is defined by

$$r(X, Y) = \sum_{i=1}^{n} \langle R(X, e_i)Y, e_i \rangle$$

for a local orthonormal frame $\{e_i\}$, $i = 1, \ldots, n$, as in [2]. Moreover, the Laplacian of the function f is defined by $\Delta f = -\delta df$; note that there is a sign difference between our Laplacian and the one found in [2].

From the definition of $\check{R}$ and the well know algebraic decomposition of the Riemann tensor R, we deduce that

$$\check{R} = \frac{2}{(n-2)^2}\left((n-4) z \circ z + |z|^2 g\right) \tag{5}$$

$$+\frac{2s^2}{n^2(n-1)}g+\check{\mathcal{W}}+\frac{4s}{n(n-1)}z+\frac{4}{n-2}\mathring{\mathcal{W}}z,$$

and

$$|R|^2=\frac{2s^2}{n(n-1)}+\frac{4}{n-2}|z|^2+|\mathcal{W}|^2. \tag{6}$$

Here, the composition $h\circ k$ of two symmetric 2-tensors h and k is defined by

$$h\circ k(X,Y)=\sum_{i=1}^{n}h(X,e_i)k(Y,e_i)$$

for any vectors X and Y and orthonormal frame $\{e_i\}$, $i=1,\dots,n$, and $\mathring{\mathcal{W}}z$ is defined by

$$(\mathring{\mathcal{W}}z)(X,Y)=\sum_{i=1}^{n}z(\mathcal{W}(X,e_i)Y,e_i).$$

Similar to $\check{R}$, $\check{\mathcal{W}}$ is defined by

$$\check{\mathcal{W}}(X,Y)=\sum_{i,j,k}\mathcal{W}(X,e_i,e_j,e_k)\mathcal{W}(Y,e_i,e_j,e_k).$$

Therefore, Theorem 1.2 follows from (5) and (6). Note that Theorem 1.2 holds even when M is not compact.

Next, we investigate some properties of the CPE required to prove our main result. Assume that (M,g,f) is a nontrivial solution to the CPE. The equation (2) can be rewritten as

$$(1+f)z=Ddf+\frac{sf}{n(n-1)}\,g. \tag{7}$$

The differential operator d^D of $C^\infty(S^2M)$ into $\Lambda^2M\otimes T^*M$ is defined by

$$d^D\eta(X,Y,Z)=(D_X\eta)(Y,Z)-(D_Y\eta)(X,Z)$$

for sections of symmetric 2-tensors $\eta\in C^\infty(S^2M)$. Note that the product of a 1-form β and symmetric 2-tensor η is defined by $\beta\wedge\eta(x,y,z)=\beta(x)\eta(y,z)-\beta(y)\eta(x,z)$. Applying d^D to (7) enables us to obtain the following.

Lemma 2.1 ([8])**.** *Let (g,f) be a solution to the* CPE. *Then*

$$(1+f)\,d^Dr=\tilde{i}_{\nabla f}\mathcal{W}-\frac{n-1}{n-2}\,df\wedge z-\frac{1}{n-2}\,i_{\nabla f}z\wedge g. \tag{8}$$

Here, we define the interior product $\tilde{i}$ of a 4-tensor $\mathcal{S}$ by

$$\tilde{i}_\xi\mathcal{S}(X,Y,Z)=\mathcal{S}(X,Y,Z,\xi)$$

for a vector ξ; i_X is the usual interior product with respect to X.

Refer to [8, Lemma 3.1] for the proof of (8). Consequently,

$$\begin{aligned} &\frac{(n-2)^2}{(n-1)^2}\sum_{i,j}|\mathcal{W}(e_i,N,e_j,\nabla f)-(1+f)\,d^D r(e_i,N,e_j)|^2 \\ &= |\nabla f|^2\left(|z|^2-\frac{n}{(n-1)^2}\,\alpha^2-\frac{n(n-2)}{(n-1)^2}\,z\circ z(N,N)\right), \end{aligned} \tag{9}$$

where $N=\nabla f/|\nabla f|$ and $\alpha=z(N,N)$. In particular, we :

Lemma 2.2. *At non-critical points of f in the set $B=f^{-1}(-1)$,*

$$|z|^2=\frac{n}{n-1}\,\alpha^2+\left(\frac{n-2}{n-1}\right)^2\sum_{i,j}|\mathcal{W}(e_i,N,e_j,N)|^2.$$

Proof. Substituting the triple $(X,\nabla f,\nabla f)$ with $X\perp\nabla f$ into (8), on the set $B=f^{-1}(-1)$, we obtain

$$0=\mathcal{W}(X,\nabla f,\nabla f,\nabla f)=df(X)z(\nabla f,\nabla f)-z(\nabla f,X)|\nabla f|^2,$$

implying that we have $z(X,\nabla f)=0$ on B. Thus, at non-critical points of f in B

$$z\circ z(N,N)=\alpha^2.$$

The proof follows immediately by combining this equation with (9). $\square$

It should be remarked that critical points of f in B are isolated (c.f. see Proposition 2.1 in [8]).

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For the proof, we need the following lemma.

Lemma 3.1. *Let (g,f) be a non-trivial solution to the CPE. Suppose that g is n-dimensional weakly Einstein. When $n=3$ or 4, $\alpha\geq 0$ on B. If $r(\nabla f,\nabla f)>-\frac{s}{n-4}|\nabla f|^2$ for $n\geq 5$, then α is again non-negative on B.*

Proof. When $n=3$, Theorem 1.2 reduces to

$$\check{R}-\frac{1}{3}|R|^2g=\frac{2}{3}\,|z|^2g-2\,z\circ z+\frac{2}{3}\,z.$$

If the metric is weakly Einstein, then

$$|z|^2g=3\,z\circ z-z. \tag{10}$$

By Lemma 2.2, at non-critical points of f in B,

$$|z|^2=\frac{3}{2}\alpha^2, \tag{11}$$

since $\mathcal{W}\equiv 0$ in a three-dimensional manifold. Therefore, by (10) and (11),

$$\frac{3}{2}\,\alpha^2=\alpha,$$

implying that $\alpha \geq 0$.

When $n = 4$, Theorem 1.2 reduces to

$$\check{R} - \frac{1}{4}|R|^2 g = \frac{s}{3}\, z + 2\mathring{\mathcal{W}}z, \tag{12}$$

which recovers identity (4.72) in [2]. Note that $\check{\mathcal{W}} = \frac{1}{n}|\mathcal{W}|^2 g$ for $n = 4$; this follows from the fact that the Hodge star operator induces a self-adjoint involution of $\Lambda^2 M$. Since g is weakly Einstein,

$$\mathring{\mathcal{W}}z = -\frac{s}{6}\, z. \tag{13}$$

By introducing normal coordinates $\{e_i\}$, for any vector ξ tangent to M,

$$\begin{aligned}-\frac{s}{6}\, z(\xi, \nabla f) &= \mathring{\mathcal{W}}z(\xi, \nabla f)\\ &= \mathcal{W}(e_i, \xi, e_j, \nabla f)\, z(e_i, e_j)\\ &= (1+f)\, d^D r(e_i, \xi, e_j) z(e_i, e_j) + 2\, z \circ z(\xi, \nabla f) - \frac{3}{2} df(\xi)|z|^2.\end{aligned}$$

In particular, on B we have

$$-\frac{s}{6} z(\xi, \nabla f) = 2\, z \circ z(\xi, \nabla f) - \frac{3}{2} df(\xi)|z|^2. \tag{14}$$

Recall that $z(X, \nabla f) = 0$ for X orthogonal to ∇f on B. Therefore, by (14), for $\xi = N$

$$-\frac{s}{6}\alpha|\nabla f| = 2\alpha^2|\nabla f| - \frac{3}{2}|\nabla f||z|^2,$$

which implies that

$$|z|^2 = \frac{4}{3}\, \alpha^2 + \frac{s}{9}\, \alpha. \tag{15}$$

On the other hand, by Lemma 2.2,

$$|z|^2 = \frac{4}{3}\, \alpha^2 + \frac{4}{9} \sum_{i,j} |\mathcal{W}(e_i, N, e_j, N)|^2$$

on B. Comparing this to (15) gives

$$\frac{s}{4}\, \alpha = \sum_{i,j} |\mathcal{W}(e_i, N, e_j, N)|^2 \geq 0. \tag{16}$$

For $n \geq 5$, on B, we have

$$\mathring{\mathcal{W}}z(N, N) = -\frac{n-2}{n-1}\, |\mathcal{W}_N|^2,$$

with $\check{\mathcal{W}}(N, N) = 2|\mathcal{W}_N|^2$. Hence, on B,

$$\frac{n-1}{2n}\, |\mathcal{W}|^2 = \frac{n-4}{n-2}\, \alpha^2 + \frac{2s}{n}\, \alpha + \frac{(n-2)(n^2-2n-2)}{n(n-1)}\, |\mathcal{W}_N|^2,$$

which implies that

$$\frac{n-4}{n-2}\, \alpha^2 + \frac{2s}{n}\, \alpha \geq 0.$$

Therefore, either $\alpha \geq 0$ or $\alpha \leq -\frac{2s}{n}\frac{n-2}{n-4}$ on B. From the assumption on the lower Ricci curvature bound, we may conclude that $\alpha \geq 0$ on B. $\square$

It should be remarked that, as a consequence of (12), if the metric g is Einstein, then it is easy to see that g is weakly Einstein since $z = \mathring{\mathcal{W}}z = 0$. Moreover, when the dimension is four, a result in [3] can be recovered; specifically,

$$\check{R} - \frac{1}{4}|R|^2 g = 2\, r \circ r + 2\mathring{R}r - sr - |z|^2 g.$$

Now, we are ready to prove our main result. Because of Obata's result [6], it suffices to prove that the metric g is Einstein, or equivalently $z = 0$ on M. Let M_0 be the subset of M defined by $M_0 = \{x \in M \mid f(x) < -1\}$. From the identity (cf. see (12) of [8])

$$\mathrm{div}(i_{\nabla f} z) = (1+f)|z|^2,$$

and the fact that $\alpha \geq 0$ on B by Lemma 3.1, we have

$$0 \geq \int_{M_0} (1+f)|z|^2 = \int_{\partial M_0} z(\nabla f, N) = \int_{\partial M_0} \alpha |\nabla f| \geq 0,$$

implying that

$$\int_{M_0} (1+f)|z|^2 = 0.$$

Therefore,

$$0 = \int_M (1+f)|z|^2 = \int_{M \setminus M_0} (1+f)|z|^2,$$

and thus, we conclude that

$$z \equiv 0,$$

or g is Einstein on all of M. This completes the proof of Theorem 1.1.

4. Manifolds with nontrivial kernels

In this section we prove Theorem 1.3. As mentioned in Introduction, if $\varphi \in \ker s_g'^* \setminus \{0\}$ for a compact Riemannian manifold (M, g), then we obtain

$$\varphi\, z = D_g d\varphi + \frac{s\,\varphi}{n(n-1)}\, g. \tag{17}$$

Applying d^D to both sides of (17) and using the Ricci identity yield

$$\varphi\, d^D r = \tilde{i}_{\nabla\varphi}\mathcal{W} - \frac{n-1}{n-2}\, d\varphi \wedge z - \frac{1}{n-2}\, i_{\nabla\varphi} z \wedge g \tag{18}$$

which appears similar to (8). An argument very similar to that used in the proof of Theorem 1.1 enables us to prove Theorem 1.3. For a proof of (18), and more details on the properties of $\varphi \in \ker s_g'^* \setminus \{0\}$, refer to [9].

Finally, we remark that (17) looks like an h-almost gradient Ricci soliton [7]. An h-almost gradient Ricci soliton is a complete Riemannian manifold (M, g)

with a potential function $u : M \to \mathbb{R}$, a solition function $\lambda : M \to \mathbb{R}$, and a function $h : M \to \mathbb{R}^+$, or $h : M \to \mathbb{R}^-$, satisfying the equation

$$r_g + h\, D_g du = \lambda\, g.$$

In particular, even though a non-trivial solution of CPE is not related to an h-almost gradient Ricci solition, $\varphi \in \ker s_g'^*$ looks like an h-almost gradient Ricci soliton; it satisfies

$$r_g - \frac{1}{\varphi}\, Dd\varphi = \frac{s}{n-1}\, g \tag{19}$$

with $h = -\frac{1}{\varphi}$ and $\lambda = \frac{s}{n-1}$. However, (g, φ) of (19) cannot be a non-trivial h-almost gradient Ricci soliton; the condition $h \geq 0$, or $h \leq 0$, implies that φ is identically zero due to the fact that φ is an eigenfunction of the Laplacian.

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