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CHARACTERIZATION OF FUNCTIONS VIA COMMUTATORS OF BILINEAR FRACTIONAL INTEGRALS ON MORREY SPACES

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Abstract. For $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let $\mathcal{I}_\alpha$ be the bilinear fractional integral operator, and $[b, \mathcal{I}_\alpha]_i$ be the commutator of $\mathcal{I}_\alpha$ with pointwise multiplication b ($i = 1, 2$). This paper shows that if the commutator $[b, \mathcal{I}_\alpha]_i$ for $i = 1$ or 2 is bounded from the product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to the Morrey space $L^{q,\lambda}(\mathbb{R}^n)$ for some suitable indexes λ, λ_1, λ_2 and p_1, p_2, q, then $b \in BMO(\mathbb{R}^n)$, as well as that the compactness of $[b, \mathcal{I}_\alpha]_i$ for $i = 1$ or 2 from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$ implies that $b \in CMO(\mathbb{R}^n)$ (the closure in $BMO(\mathbb{R}^n)$ of the space of $C^\infty(\mathbb{R}^n)$ functions with compact support). These results together with some previous ones give a new characterization of $BMO(\mathbb{R}^n)$ functions or $CMO(\mathbb{R}^n)$ functions in essential ways.

1. Introduction

Let $\mathbb{R}^n$ be the Euclidean space with $n \geq 2$, and $BMO(\mathbb{R}^n)$ denote the space of functions with bounded mean oscillation, which consists of all locally integrable functions b, such that

$$\|b\|_* := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx < \infty,$$

where Q is a cube with sides parallel to the axes, and b_Q is the average of b over Q. Also, let $CMO(\mathbb{R}^n)$ be the closure in the $BMO(\mathbb{R}^n)$ norm of $C_c^\infty(\mathbb{R}^n)$, which represents the space of infinitely differentiable functions with compact support.

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For $0 < \alpha < 2n$, let us consider the bilinear fractional integral operator $\mathcal{I}_\alpha$ defined originally for f, $g \in C_c^\infty(\mathbb{R}^n)$ by

$$\mathcal{I}_\alpha(f,g)(x) := \int_{\mathbb{R}^{2n}} \frac{1}{(|x-y|+|x-z|)^{2n-\alpha}} f(y)g(z)dydz, \tag{1}$$

and its commutators with symbol b given by

$$[b,\mathcal{I}_\alpha]_1\,(f,g) := \mathcal{I}_\alpha(bf,g) - b\,\mathcal{I}_\alpha(f,g), \tag{2}$$

and

$$[b,\mathcal{I}_\alpha]_2\,(f,g) := \mathcal{I}_\alpha(f,bg) - b\,\mathcal{I}_\alpha(f,g). \tag{3}$$

The boundedness and compactness of $[b,\,\mathcal{I}_\alpha]_i$ on variant function spaces have been the topic in many articles recently, see [1, 2, 3, 5, 6, 12, 13, 17, 18, 19, 26], among numerous references. One of the interesting questions on $[b,\,\mathcal{I}_\alpha]_i$ is whether it can be used to characterize $BMO(\mathbb{R}^n)$ by boundedness, or $CMO(\mathbb{R}^n)$ by compactness, as those in the linear setting (see [7, 9, 27] etc). Recently, Chaffee [5] established the following result.

Theorem A. *For $b \in L^1_{\text{loc}}\,(\mathbb{R}^n)$, $0 < \alpha < 2n$ and $1 < p_1,\, p_2$, and q satisfying*

$$0 < \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q} < 1,$$

we have, for $i = 1$ or 2,

$[b,\mathcal{I}_\alpha]_i : L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ *is a bounded operator* $\Longleftrightarrow b\in\mathrm{BMO}(\mathbb{R}^n)$.

Subsequently, Chaffee and Torres [6] obtained the following characterization by compactness in Lebesgue spaces.

Theorem B. *For $b \in L^1_{\text{loc}}\,(\mathbb{R}^n)$, $0 < \alpha < 2n$ and $1 < p_1,\, p_2$, and q satisfying*

$$0 < \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q} < 1,$$

we have, for $i = 1$ or 2,

$[b,\mathcal{I}_\alpha]_i : L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ *is a compact operator* $\Longleftrightarrow b\in\mathrm{CMO}(\mathbb{R}^n)$.

In this paper, we aim to extend the above results to Morrey spaces, which is defined as follows.

Definition. For $0 < \lambda < n$, $1 \le p < \infty$, the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{loc} : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{t\in\mathbb{R}^n, R>0} \left(\frac{1}{R^\lambda}\int_{B(t,R)} |f(x)|^p dx\right)^{1/p},$$

and $B(t,R)$ is the ball centered at t and with radius $R > 0$.

The space $L^{p,\lambda}(\mathbb{R}^n)$ becomes a Banach space with norm $\|\cdot\|_{L^{p,\lambda}(\mathbb{R}^n)}$. Moreover, for $1 \le p < \infty$, then $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ isometrically. If $\lambda > n$, then $L^{p,\lambda}(\mathbb{R}^n) = \{0\}$.

It is well known that the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ was originally introduced by Morrey [20] to study certain problems in elliptic equations and was subsequently found to have many important applications to partial differential equations, such as elliptic equations, Navier-Stokes equations and Scrödinger equations, see [4, 15, 21, 23, 24] *et al.* and references therein. Also, the boundedness for the classical operators in harmonic analysis and the compactness of the commutators for such classical operators in $L^{p,\lambda}(\mathbb{R}^n)$ were extensively studied, for examples see [8, 9, 10, 11, 14, 17, 28] and references therein. In particular, Ding and Mei [17] established the following boundedness and compactness of $[b, \mathcal{I}_\alpha]_i$ for $i = 1, 2$ in Morrey spaces.

Theorem C. *For $0 < \alpha < 2n$, $0 < \lambda$, λ_1, $\lambda_2 < n$. Suppose that $1/2 < p < \infty$, $1 < p_1$, $p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, $1 < q < \infty$ with $1/q = 1/p - \alpha/(n-\lambda)$. Then*

(i) *for $b \in BMO(\mathbb{R}^n)$, there exists a constant $C > 0$ such that for $i = 1, 2$,*

$$\|[b, \mathcal{I}_\alpha]_i(f, g)\|_{L^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)};$$

(ii) *for $b \in CMO(\mathbb{R}^n)$, $[b, \mathcal{I}_\alpha]_i$ is a compact operator from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, $i = 1, 2$.*

Compared Theorem C with Theorems A and B, it is natural to ask whether the boundedness or compactness of the commutator $[b, \mathcal{I}_\alpha]_i$ for $i = 1$ or 2 from the product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to the Morrey space $L^{q,\lambda}(\mathbb{R}^n)$ can imply that $b \in BMO(\mathbb{R}^n)$ or $b \in CMO(\mathbb{R}^n)$. The main purpose of this paper is to address the question above. Our results can be formulated as follows.

Theorem 1.1. *For $0 < \alpha < 2n$, $0 < \lambda$, λ_1, $\lambda_2 < n$, suppose that $1 < p_1, p_2 < \infty$, $1/2 < p < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, $1 < q < \infty$ with $1/q = 1/p - \alpha/(n-\lambda)$. If the commutator $[b, \mathcal{I}_\alpha]_i$ for $i = 1$ or 2 is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$.*

Theorem 1.2. *For $0 < \alpha < 2n$, $0 < \lambda_1, \lambda_2, \lambda < n$, suppose that $1 < p_1$, $p_2 < \infty$, $1/2 < p < \infty$ with $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, and $1 < q < \infty$ with $1/q = 1/p - \alpha/(n-\lambda)$. If the commutator $[b, \mathcal{I}_\alpha]_i$ for $i = 1$ or 2 is a compact operator from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then $b \in CMO(\mathbb{R}^n)$.*

Moreover, combining Theorems 1.1 and 1.2 with Theorem C, we have the following equivalent characterizations.

Theorem 1.3. *Under the assumptions of Theorem 1.1 or 1.2, for $i = 1$ or 2, we have*

(i) $[b, \mathcal{I}_\alpha]_i : L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n) \to L^{q,\lambda}(\mathbb{R}^n)$ *is bounded* $\Longleftrightarrow b \in BMO(\mathbb{R}^n)$;

(ii) $[b, \mathcal{I}_\alpha]_i : L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n) \to L^{q,\lambda}(\mathbb{R}^n)$ *is compact* $\Longleftrightarrow b \in CMO(\mathbb{R}^n)$.

Remark. Obviously, Theorems A and B can be regarded as the extreme case of Theorem 1.3 in $\lambda = \lambda_1 = \lambda_2 = 0$ since $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for any $1 \leq p < \infty$. Therefore, our results essentially extend the corresponding ones in [5, 6]. In addition, Theorem 1.3 can also be regarded as the generalization of the corresponding result in [9] from the linear setting to the multilinear setting.

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1.1 and the proof of Theorem 1.2 will be given in Section 3. We remark that our ideas are greatly motivated by [5, 6, 9, 16].

We shall use the following conventions:

- C always denotes a positive constant that is independent of main parameters involved but whose value may differ from line to line.
- For a set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function.
- For $p \in [1, \infty)$, we use p' to denote the dual exponent of p, namely $p' = p/(p-1)$.
- For a ball $B \subset \mathbb{R}^n$ and $c > 0$, we use cB to denote the ball concentric with B whose radius is c times of $B's$.

2. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. The techniques in our arguments are taken from [5, 16], which originate from [7].

Proof of Theorem 1.1. By symmetry of the kernel of $[b, \mathcal{I}_\alpha]_i$, we will give our arguments to $[b, \mathcal{I}_\alpha]_1$. For $\delta > 0$, let $B((y_0, z_0), \delta\sqrt{n}) \subset \mathbb{R}^{2n}$ be the ball for which we can express $(|y| + |z|)^{2n-\alpha}$ as an absolutely convergent Fourier series of the form

$$(|y| + |z|)^{2n-\alpha} = \sum_{j=0}^{\infty} a_j e^{iv_j \cdot (y,z)},$$

which $\sum_{j=0}^{\infty} |a_j| < \infty$ in the neighborhood $|y - y_0| + |z - z_0| \leq 2\delta\sqrt{n}$.

The specific vectors v_j will not play a role in the proof. We will express them as $v_j = (v_j^1, v_j^2) \in \mathbb{R}^n \times \mathbb{R}^n$. Note that due to the homogeneity of $(|y|+|z|)^{2n-\alpha}$, we can take (y_0, z_0) such that $|(y_0, z_0)| > 2\sqrt{n}$ and $\delta < 1$ such that $\bar{B} \cap \{0\} = \emptyset$.

Set $y_1 = y_0\delta^{-1}$ and $z_1 = z_0\delta^{-1}$, and note that

$$|y - y_1| + |z - z_1| < 2\sqrt{n} \Rightarrow |\delta y - y_0| + |\delta z - z_0| \leq 2\delta\sqrt{n},$$

and so for all (y, z) satisfying the inequality on the left. We have

$$(|y| + |z|)^{2n-\alpha} = (|\delta y| + |\delta z|)^{2n-\alpha}\delta^{-2n+\alpha}$$

$$= \delta^{-2n+\alpha} \sum_{j=0}^{\infty} a_j e^{i\delta v_j \cdot (y,z)}.$$

Let $Q = Q(x_0, r_0)$ be an arbitrary cube in $\mathbb{R}^n$. Set $\tilde{y} = x_0 - r_0 y_1$, $\tilde{z} = x_0 - r_0 z_1$ and take $Q' = Q(\tilde{y}, r_0) \subset \mathbb{R}^n$ and $Q'' = Q(\tilde{z}, r_0) \subset \mathbb{R}^n$. Then for any $x \in Q$ and $y \in Q'$, we have

$$|\frac{x-y}{r_0} - y_1| = |\frac{x-y}{r_0} - \frac{x_0 - \tilde{y}}{r_0}| \leq |\frac{x - x_0}{r_0}| + |\frac{\tilde{y} - y}{r_0}| \leq \sqrt{n}.$$

The same estimate holds for $x \in Q$ and $z \in Q''$, and so we have

$$|\frac{x-y}{r_0} - y_1| + |\frac{x-z}{r_0} - z_1| \leq 2\sqrt{n}.$$

Let $\sigma(x) = \operatorname{sgn}\,(b(x) - b_{Q'})$. Then

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}| dx \\
&= \int_Q (b(x) - b_{Q'}) \sigma(x) dx \\
&= \frac{1}{|Q'|} \int_Q \int_{Q'} (b(x) - b(y)) dy \sigma(x) dx \\
&= \frac{1}{|Q''|} \frac{1}{|Q'|} \int_Q \int_{Q'} \int_{Q''} (b(x) - b(y)) \sigma(x) dz dy dx \\
&= r_0^{-2n} \int_{\mathbb{R}^{3n}} \frac{b(x) - b(y)}{(|x-y| + |x-z|)^{2n-\alpha}} r_0^{2n-\alpha} \Big(\Big|\frac{x-y}{r_0}\Big| + \Big|\frac{x-z}{r_0}\Big| \Big)^{2n-\alpha} \\
&\qquad \times \sigma(x) \chi_Q(x) \chi_{Q'}(y) \chi_{Q''}(z) dz dy dx \\
&= r_0^{-\alpha} \int_{\mathbb{R}^{3n}} \frac{b(x) - b(y)}{(|x-y| + |x-z|)^{2n-\alpha}} \delta^{-2n+\alpha} \sum_{j=0}^{\infty} a_j e^{i\delta v_j \cdot (\frac{x-y}{r_0}, \frac{x-z}{r_0})} \\
&\qquad \times \sigma(x) \chi_Q(x) \chi_{Q'}(y) \chi_{Q''}(z) dz dy dx.
\end{aligned}$$

Let $f_j(y) = e^{-i\frac{\delta}{r_0} v_j^1 \cdot y} \chi_{Q'}(y)$, $g_j(z) = e^{-i\frac{\delta}{r_0} v_j^2 \cdot z} \chi_{Q''}(z)$ and

$$h_j(x) = e^{i\frac{\delta}{r_0} v_j \cdot (x,x)} \sigma(x) \chi_Q(x).$$

Then $f_j \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g_j \in L^{p_2,\lambda_2}(\mathbb{R}^n)$ and by Hölder's inequality and the assumption the boundedness of $[b, \mathcal{I}_\alpha]_1$ from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}| dx \\
&= r_0^{-\alpha} \delta^{-2n+\alpha} \int_{\mathbb{R}^{3n}} \frac{b(x) - b(y)}{(|x-y| + |x-z|)^{2n-\alpha}} \sum_{j=0}^{\infty} a_j f_j(y) g_j(z) h_j(x) dy dz dx
\end{aligned}$$

$$= r_0^{-\alpha}\delta^{-2n+\alpha}\sum_{j=0}^{\infty}a_j\int_{\mathbb{R}^n}h_j(x)\int_{\mathbb{R}^{2n}}\frac{b(x)-b(y)}{(|x-y|+|x-z|)^{2n-\alpha}}f_j(y)g_j(z)dydzdx$$
$$= r_0^{-\alpha}\delta^{-2n+\alpha}\sum_{j=0}^{\infty}a_j\int_Q h_j(x)[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)dx$$
$$\le r_0^{-\alpha}\delta^{-2n+\alpha}\sum_{j=0}^{\infty}|a_j|\left(\int_Q|h_j(x)|^{q'}dx\right)^{1/q'}\left(\int_Q|[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)|^q dx\right)^{1/q}$$
$$= r_0^{-\alpha}\delta^{-2n+\alpha}\sum_{j=0}^{\infty}|a_j||Q|^{1/q'}r_0^{\lambda/q}\left(\frac{1}{r_0^{\lambda}}\int_Q|[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)|^q dx\right)^{1/q}$$
$$\le Cr_0^{-\alpha}\delta^{-2n+\alpha}\sum_{j=0}^{\infty}|a_j||Q|^{1/q'}r_0^{\lambda/q}\|f_j\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g_j\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}.$$

Note that for any $t\in\mathbb{R}^n$ and $0<d<\infty$,

$$\frac{1}{d^{\lambda_1}}\int_{Q_d}|\chi_{Q'(y)}|^{p_1}dy \le C\frac{1}{r_0^{\lambda_1}}\int_{Q'}|\chi_{Q'(y)}|^{p_1}dy, \tag{4}$$

where $Q_d=Q(t,d)$ and $C=2^n$. In fact, we need only consider the case of $t=\tilde{y}$ and $d<r_0$. The other cases are simply. In this case, there exists a $k\in\mathbb{N}$, such that $2^{k-1}d<r_0\le 2^k r_0$, we get

$$\frac{|Q'\bigcap Q_d|}{d^{\lambda_1}}=\frac{|Q_d|}{d^{\lambda_1}}<\frac{|Q'|}{2^{(k-1)n}d^{\lambda_1}}\le\frac{2^{k\lambda_1}|Q'|}{2^{(k-1)n}r_0^{\lambda_1}}\le 2^n\frac{1}{r_0^{\lambda_1}}\int_{Q'}|\chi_{Q'(y)}|^{p_1}dy.$$

The same estimate for $\|g_j\|_{L^{p_2,\,\lambda_2}}(\mathbb{R}^n)$, hence

$$\int_Q|b(x)-b_{Q'}|dx$$
$$\le C\delta^{-2n+\alpha}\sum_{j=0}^{\infty}|a_j|r_0^{-\alpha}r_0^{n(1-1/q)}r_0^{-(\lambda_1/p_1+\lambda_2/p_2)}|Q'|^{1/p_1}|Q''|^{1/p_2}r_0^{\lambda/q}$$
$$= C\delta^{-2n+\alpha}\sum_{j=0}^{\infty}|a_j|r_0^n$$
$$\le C|Q|.$$

Recall that $|Q|^{-1}\int_Q|b(x)-b_Q|dx\le 2|Q|^{-1}\int_Q|b(x)-c|dx$ for any c, and so this gives us that for any arbitrary $Q\subset\mathbb{R}^n$

$$\frac{1}{|Q|}\int_Q|b(x)-b_Q|dx\le\frac{2}{|Q|}\int_Q|b(x)-b_{Q'}|dx\le C,$$

which implies that $b\in BMO(\mathbb{R}^n)$ and completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. The following characterization of $CMO(\mathbb{R}^n)$ will play key role in our arguments.

Lemma 3.1 (cf. [25])**.** *A function $b \in BMO(\mathbb{R}^n)$ is in $CMO(\mathbb{R}^n)$, if and only if b satisfies the following three conditions*:

(1) $\lim_{a\to 0} \sup_{|Q|=a} \frac{1}{|Q|}\int_Q |b(x)-b_Q|dx = 0,$

(2) $\lim_{a\to \infty} \sup_{|Q|=a} \frac{1}{|Q|}\int_Q |b(x)-b_Q|dx = 0,$

(3) $\lim_{|y|\to \infty} \frac{1}{|Q|}\int_Q |b(x+y)-b_Q|dx = 0$ *for each* Q.

Proof of Theorem 1.2. Note that a compact operator is bounded, by Theorem 1.1, we know that the symbol b of a compact operator must be at least in $BMO(\mathbb{R}^n)$. In what follows, we will prove that $b \in CMO(\mathbb{R}^n)$.

Employing the ideas of [6, 9], our approach is as follows: Under the assumption of that $[b, \mathcal{I}_\alpha]_i$ is a compact operator from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$ for $i = 1$ or 2, we will show that if b fails to satisfy one of the conditions (1)-(3) in Lemma 3.1, then one can construct sequences of functions $\{f_j\}_{j=1}^\infty$ uniformly bounded on $L^{p_1,\lambda_1}(\mathbb{R}^n)$ and $\{g_j\}_{j=1}^\infty$ uniformly bounded on $L^{p_2,\lambda_2}(\mathbb{R}^n)$, such that $\{[b, \mathcal{I}_\alpha]_i(f_j, g_j)\}_{j=1}^\infty$ has no convergent subsequence in $L^{q,\lambda}(\mathbb{R}^n)$, which contradicts the compactness assumption. It then follows that if $[b, \mathcal{I}_\alpha]_i$ is compact, the symbol b must satisfy all three conditions and hence be an element of $CMO(\mathbb{R}^n)$.

By the symmetry of the kernel of $[b, \mathcal{I}_\alpha]_i$ again, we will give the arguments only to $[b, \mathcal{I}_\alpha]_1$. Before constructing the sequence, we make some preliminaries.

Assume that $b \in BMO(\mathbb{R}^n)$with $\|b\|_* = 1$. Then there exist $\epsilon > 0$ and a sequence of cubes $\{Q_j(y_j, d_j)\}_j^\infty$ such that for every j,

$$\frac{1}{|Q_j|}\int_{Q_j} |b(y)-b_{Q_j}|dy > \epsilon. \tag{5}$$

We define

$$f_j(y) = |Q_j|^{-(n-\lambda_1)/(np_1)}\big(\operatorname{sgn}\,(b(y)-b_{Q_j}) - c_0\big)\chi_{Q_j}(y), \tag{6}$$

where $c_0 = |Q_j|^{-1}\int_{Q_j} \operatorname{sgn}\,(b(y)-b_{Q_j})dy$. It is easy to check that $|c_0| < 1$ and $\{f_j\}$ has the following properties

$$\operatorname{supp} f_j \subset Q_j, \tag{7}$$

$$f_j(y)(b(y)-b_{Q_j}) \geq 0, \tag{8}$$

$$\int_{\mathbb{R}^n} f_j(y)dy = 0, \tag{9}$$

$$\int_{\mathbb{R}^n} f_j(y)(b(y)-b_{Q_j})dy = |Q_j|^{-(n-\lambda_1)/(np_1)}\int_{Q_j} |b(y)-b_{Q_j}|dy, \tag{10}$$

$$|f_j(y)| \le 2|Q_j|^{-(n-\lambda_1)/(np_1)}. \tag{11}$$

(11) gives us that $\{\|f_j\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\}_{j=1}^{\infty}$ is bounded uniformly. In fact, for any $t \in \mathbb{R}^n$,

$$\left(\frac{1}{R^{\lambda_1}}\int_{B(t,R)}|f_j(y)|^{p_1}dy\right)^{1/p_1} \le \begin{cases} C_1\left(\frac{R}{d_j}\right)^{(n-\lambda_1)/p_1} \le C_1, & 0<R\le d_j; \\ \left(\frac{1}{R^{\lambda_1}}\int_{Q_j}|f_j(y)|^{p_1}dy\right)^{1/p_1} & \\ \le C_1\left(\frac{d_j}{R}\right)^{\lambda_1/p_1} \le C_1, & R>d_j>0, \end{cases}$$

where C_1 is independent of j, R, t.

For the other functions, we will simply define

$$g_j = \frac{\chi_{Q_j}}{|Q_j|^{(n-\lambda_2)/(p_2 n)}}, \tag{12}$$

which satisfies

$$\left(\frac{1}{R^{\lambda_2}}\int_{B(t,R)}|g_j(y)|^{p_2}dy\right)^{1/p_2} \le \begin{cases} C_2\left(\frac{R}{d_j}\right)^{(n-\lambda_2)/p_2} \le C_2, & 0<R\le d_j; \\ \left(\frac{1}{R^{\lambda_2}}\int_{Q_j}|g_j(z)|^{p_2}dz\right)^{1/p_2} & \\ \le C_2\left(\frac{d_j}{R}\right)^{\lambda_2/p_2} \le C_2, & R>d_j>0, \end{cases}$$

where C_2 is independent of j, R, t. Thus the sequence $\{[b,\mathcal{I}_\alpha]_1(f_j,g_j)\}_{j=1}^{\infty}$ is also bounded in $L^{q,\lambda}(\mathbb{R}^n)$.

Next we establish several technical estimates. For a cube Q_j with centered y_j and satisfying (5) for some $\epsilon>0$, f_j, g_j as above, and all $x \in (2\sqrt{n}Q_j)^c$ the following point-wise estimates hold:

$$|\mathcal{I}_\alpha((b-b_{Q_j})f_j,g_j)(x)| \le C|Q_j|^{2-(n-\lambda)/(np)}|x-y_j|^{-2n+\alpha}, \tag{13}$$

$$|\mathcal{I}_\alpha((b-b_{Q_j})f_j,g_j)(x)| \ge C|Q_j|^{2-(n-\lambda)/(np)}|x-y_j|^{-2n+\alpha}\epsilon, \tag{14}$$

$$|\mathcal{I}_\alpha(f_j,g_j)(x)| \le C|Q_j|^{2-(n-\lambda)/(np)+1/n}|x-y_j|^{-2n+\alpha-1}, \tag{15}$$

where the constants involved are independent of b, f_j, g_j and ϵ.

To prove (13), we use that $|x-y_j| \approx |x-y|$ for all $y \in Q_j$, and that $\|b\|_* = 1$ to obtain

$$\begin{aligned} &|\mathcal{I}_\alpha((b-b_{Q_j})f_j,g_j)(x)| \\ &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(y)-b_{Q_j})f_j(y)g_j(z)}{(|x-y|+|x-z|)^{2n-\alpha}}dydz\right| \\ &= \left|\int_{Q_j}\int_{Q_j}\frac{(b(y)-b_{Q_j})f_j(y)g_j(z)}{(|x-y|+|x-z|)^{2n-\alpha}}dydz\right| \\ &\le |Q_j|^{-(n-\lambda_1)/(np_1)-(n-\lambda_2)/(np_2)}|x-y_j|^{-2n+\alpha}\int_{Q_j}\int_{Q_j}|b(y)-b_{Q_j}|dydz \\ &\le C|Q_j|^{2-(n-\lambda)/(np)}|x-y_j|^{-2n+\alpha}\|b\|_* \end{aligned}$$

$$\le C|Q_j|^{2-(n-\lambda)/(np)}|x-y_j|^{-2n+\alpha}.$$

By (5), (8) and (10), we have

$$\begin{aligned}
&|\mathcal{I}_\alpha((b-b_{Q_j})f_j,g_j)(x)|\\
&=\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(y)-b_{Q_j})f_j(y)g_j(z)}{(|x-y|+|x-z|)^{2n-\alpha}}dydz\right|\\
&\ge C|Q_j|^{1-(n-\lambda_2)/(np_2)}|x-y_j|^{-2n+\alpha}\left|\int_{Q_j}(b(y)-b_{Q_j})f_j(y)dy\right|\\
&=C|Q_j|^{1-(n-\lambda_1)/(np_1)-(n-\lambda_2)/(np_2)}|x-y_j|^{-2n+\alpha}\int_{Q_j}|b(y)-b_{Q_j}|dy\\
&\ge C|Q_j|^{2-(n-\lambda)/(np)}|x-y_j|^{-2n+\alpha}\epsilon,
\end{aligned}$$

which gives (14). Finally, using that f_j has mean zero we obtain (15) in the following way,

$$\begin{aligned}
&|\mathcal{I}_\alpha(f_j,g_j)(x)|\\
&=\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\frac{f_j(y)g_j(z)}{(|x-y|+|x-z|)^{2n-\alpha}}-\frac{f_j(y)g_j(z)}{(|x-y_j|+|x-z|)^{2n-\alpha}}\right)dydz\right|\\
&\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|y-y_j|}{(|x-y_j|+|x-z|)^{2n-\alpha+1}}f_j(y)g_j(z)|dydz\\
&\le C\frac{|Q_j|^{1/n}}{|x-y_j|^{2n-\alpha+1}}\int_{Q_j}\int_{Q_j}|f_j(y)g_j(z)|dydz\\
&\le C|Q_j|^{2-(n-\lambda)/(np)+1/n}|x-y_j|^{-2n+\alpha-1}.
\end{aligned}$$

Following [25] and [9], we now use the above point-wise estimates (13)–(15) to prove some $L^{q,\lambda}(\mathbb{R}^n)$ inequalities for $[b,\mathcal{I}_\alpha]_1(f_j,g_j)$. We may get the following claims.

Claim 1. There exist constants $\gamma_2>\gamma_1>2$ and $\gamma_3>0$, which are depending only on $p_1,p_2,n,\epsilon,\lambda$ and b, such that

$$\left(\int_{\gamma_1 d_j<|x-y_j|<\gamma_2 d_j}|[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)|^q dx\right)^{1/q}\ge\gamma_3|Q_j|^{\lambda/(nq)},\tag{16}$$

and

$$\left(\int_{|x-y_j|>\gamma_2 d_j}|[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)|^q dx\right)^{1/q}\le\frac{\gamma_3}{4}|Q_j|^{\lambda/(nq)}.\tag{17}$$

Claim 2. There exists a constant $0<\beta<\gamma_2$ depending only on $p_1,p_2,n,\epsilon,\lambda$ and b, such that

$$\left(\int_E|[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)|^q dx\right)^{1/q}\le\frac{\gamma_3}{4}|Q_j|^{\lambda/(nq)}\tag{18}$$

holds for every measurable set E satisfying

$$E \subset \{x : \gamma_1 d_j < |x - y_j| < \gamma_2 d_j\} \text{ and } \frac{|E|}{|Q_j|} < \beta^n.$$

Now we prove (16)-(18). Starting $\tilde{\gamma_1} > \max\{16, n\}$, using (15) and the fact that $2n - \alpha - n/q > 0$, $1/p_1 + 1/p_2 < 2$. Since $|b_{2Q} - b_Q| \leq C\|b\|_* = C$, by $\|b\|_* = 1$ we have

$$\left(\int_{2^s d_j < |x-y_j| < 2^{s+1} d_j} |b(x) - b_{Q_j}|^q dx \right)^{1/q} \leq C s 2^{sn/q} |Q_j|^{1/q}.$$

Then

$$\begin{aligned}
&\left(\int_{|x-y_j| > \tilde{\gamma_1} d_j} |(b(x) - b_{Q_j}) \mathcal{I}_\alpha(f_j, g_j)(x)|^q dx \right)^{1/q} \\
&\leq C|Q_j|^{2-(n-\lambda)/(np)+1/n} \sum_{s=\llcorner \log_2 \tilde{\gamma_1} \lrcorner}^{\infty} \left(\int_{2^s d_j < |x-y_j| < 2^{s+1} d_j} \tfrac{|b(x)-b_{Q_j}|^q}{|x-y_j|^{q(2n-\alpha+1)}} dx \right)^{1/q} \\
&\leq C|Q_j|^{2-(n-\lambda)/(np)+1/n} \sum_{s=\llcorner \log_2 \tilde{\gamma_1} \lrcorner}^{\infty} 2^{-s(2n-\alpha+1)} |Q_j|^{-2+\alpha/n-1/n} \\
&\qquad \times \left(\int_{2^s d_j < |x-y_j| < 2^{s+1} d_j} |b(x) - b_{Q_j}|^q dx \right)^{1/q} \\
&\leq C|Q_j|^{\alpha/n-(n-\lambda)/(np)} \sum_{s=\llcorner \log_2 \tilde{\gamma_1} \lrcorner}^{\infty} 2^{-s(2n-\alpha+1)} s 2^{sn/q} |Q_j|^{1/q} \\
&\leq C|Q_j|^{\lambda/(nq)} \sum_{s=\llcorner \log_2 \tilde{\gamma_1} \lrcorner}^{\infty} 2^{-s(2n-\alpha-n/q+1/2)},
\end{aligned}$$

where we have used that $s \leq 2^{s/2}$ for $4 \leq \llcorner \log_2 \tilde{\gamma_1} \lrcorner \leq s$. Thus we obtain that

$$\begin{aligned}
&\left(\int_{|x-y_j| > \tilde{\gamma_1} d_j} |(b(x) - b_{Q_j}) \mathcal{I}_\alpha(f_j, g_j)(x)|^q dx \right)^{1/q} \\
&\leq C|Q_j|^{\lambda/(nq)} \tilde{\gamma_1}^{-2n+\alpha+n/q-1/2}.
\end{aligned} \tag{19}$$

Next, for $\tilde{\gamma_2} > \tilde{\gamma_1}$, using (14) and (19), we have the following

$$\begin{aligned}
&\left(\int_{\tilde{\gamma_1} d_j < |x-y_j| < \tilde{\gamma_2} d_j} |[b, \mathcal{I}_\alpha]_1(f_j, g_j)(x)|^q dx \right)^{1/q} \\
&\geq \left(\int_{\tilde{\gamma_1} d_j < |x-y_j| < \tilde{\gamma_2} d_j} |\mathcal{I}_\alpha((b - b_{Q_j}) f_j, g_j)(x)|^q dx \right)^{1/q} \\
&\quad - \left(\int_{\tilde{\gamma_1} d_j < |x-y_j|} |(b(x) - b_{Q_j}) \mathcal{I}_\alpha(f_j, g_j)(x)|^q dx \right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\geq C\epsilon|Q_j|^{2-(n-\lambda)/(np)}\bigg(\int_{\tilde{\gamma_1}d_j<|x-y_j|<\tilde{\gamma_2}d_j}\frac{1}{|x-y_j|^{q(2n-\alpha)}}dx\bigg)^{1/q}\\
&\qquad - C|Q_j|^{\lambda/(nq)}\tilde{\gamma_1}^{-2n+\alpha+n/q-1/2}\\
&\geq C\epsilon|Q_j|^{\lambda/(nq)}(\tilde{\gamma_1}^{-2nq+\alpha q+n}-\tilde{\gamma_2}^{-2nq+\alpha q+n})^{1/q}\\
&\qquad - C|Q_j|^{\lambda/(nq)}\tilde{\gamma_1}^{-2n+\alpha+n/q-1/2}. \qquad (20)
\end{aligned}
$$

Similarly, from (13) and (19), we have

$$
\begin{aligned}
&\bigg(\int_{|x-y_j|>\tilde{\gamma_2}d_j}|[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)|^q dx\bigg)^{1/q}\\
&\leq \bigg(\int_{|x-y_j|>\tilde{\gamma_2}d_j}|\mathcal{I}_\alpha((b-b_{Q_j})f_j,g_j)(x)|^q dx\bigg)^{1/q}\\
&\quad + \bigg(\int_{|x-y_j|>\tilde{\gamma_2}d_j}|(b(x)-b_{Q_j})\mathcal{I}_\alpha(f_j,g_j)(x)|^q dx\bigg)^{1/q}\\
&\leq C|Q_j|^{2-(n-\lambda)/pn}\bigg(\int_{|x-y_j|>\tilde{\gamma_2}d_j}\frac{1}{|x-y_j|^{(2n-\alpha)q}}dx\bigg)^{1/q}\\
&\quad + C|Q_j|^{\lambda/(nq)}\tilde{\gamma_2}^{-2n+\alpha+n/q-1/2}\\
&\leq C|Q_j|^{\lambda/(qn)}\tilde{\gamma_2}^{-2n+\alpha+n/q}+C|Q_j|^{\lambda/(nq)}\tilde{\gamma_2}^{-2n+\alpha+n/q-1/2}. \qquad (21)
\end{aligned}
$$

Using (20) and (21) $\big($since $1<p<n/\alpha$, $1/q=1/p-\alpha/(n-\lambda), n/q+\alpha<n\big)$, we can select γ_1,γ_2 in place $\tilde{\gamma_1},\tilde{\gamma_2}$, with $\gamma_2>\gamma_1$, such that (16) and (17) are verified for some $\gamma_3>0$.

We now verify (18). Let $E\subset\{x:\gamma_1 d_j<|x-y_j|<\gamma_2 d_j\}$ be an arbitrary measurable set. Then by (13) and (15), we get

$$
\begin{aligned}
&\bigg(\int_E|[b,\mathcal{I}_\alpha]_1(f_j,g_j)(x)|^q dx\bigg)^{1/q}\\
&\leq |Q_j|^{2-(n-\lambda)/(np)}\bigg(\int_E|x-y_j|^{-q(2n-\alpha)}dx\bigg)^{1/q}\\
&\quad + |Q_j|^{2-(n-\lambda)/(np)+1/n}\bigg(\int_E\frac{|b(x)-b_{Q_j}|^q}{|x-y_j|^{q(2n-\alpha+1)}}dx\bigg)^{1/q}\\
&\leq C|Q_j|^{\lambda/(nq)}\bigg\{\frac{|E|^{1/q}}{|Q_j|^{1/q}}+\bigg(\frac{1}{|Q_j|}\int_E|b(x)-b_{Q_j}|^q dx\bigg)^{1/q}\bigg\}. \qquad (22)
\end{aligned}
$$

On the other hand, by the same arguments as in [9, p. 309], we can show that there exists some positive constant c_1 depending on γ_1 and γ_2 and b such that

$$
\frac{1}{|Q_j|}\int_E|b(x)-b_{Q_j}|^q dx\leq C\frac{|E|}{|Q_j|}\bigg(1+\log\big(\frac{c_1|Q_j|}{|E|}\big)\bigg)^{\llcorner q\lrcorner+1}.
$$

This together with (22), if we take $0 < \beta < \min(c_1^{1/n}, \gamma_2)$, implies that (18) holds.

We are left with constructing the sequences to lead to a contradiction. The arguments are again borrowed from [6, 9].

Case 1. If b does not satisfy (1), then there exist some $\epsilon > 0$ and sequence $\{Q_j\}$ with $\lim_{j\to\infty} |Q_j| = 0$ such that for every j, (5) holds.

By $\lim_{j\to\infty} d_j = 0$, we may choose a subsequence $\{Q_{j_k}^{(1)} = Q(y_{j_k}, q_{j_k}^{(1)})\}$, such that their radius $\{d_{j_k}^{(1)}\}$ satisfying

$$\frac{d_{j_{k+1}}^{(1)}}{d_{j_k}^{(1)}} < \frac{\beta}{\gamma_2}. \tag{23}$$

We also let $\{f_{j_k}\}$ and $\{g_{j_k}\}$ be the subsequence associated to the selected cubes $\{Q_{j_k}^{(1)}\}$ as defined earlier on. For fixed k and m, we define the following sets

$$G = \{x : \gamma_1 d_{j_k}^{(1)} < |x - y_{j_k}| < \gamma_2 d_{j_k}^{(1)}\},$$

$$G_1 = G \setminus \{x : |x - y_{j_{k+m}}| \le \gamma_2 d_{j_{k+m}}^{(1)}\},$$

and

$$G_2 = \{x : |x - y_{j_{k+m}}| > \gamma_2 d_{j_{k+m}}^{(1)}\},$$

where γ_1 and γ_2 are defined as before. Note that

$$G_1 \subset B(y_{j_k}, \gamma_2 d_{j_k}^{(1)}) \cap G_2, \tag{24}$$

and

$$G_1 = G \setminus (G_2^c \cap G). \tag{25}$$

Also, by construction and our choice of $Q_{j_k}^{(1)}$, one can easily see that

$$\frac{|G_2^c \cap G|}{|Q_{j_k}|} \le \frac{(\gamma_2 d_{j_{k+m}}^{(1)})^n}{(d_{j_k}^{(1)})^n} \le \gamma_2^n \left(\frac{\beta^n}{\gamma_2^n}\right)^m < \beta^n. \tag{26}$$

It follows that

$$\begin{aligned}
&\left(\int_{B(y_{j_k}, \gamma_2 d_{j_k}^{(1)})} |[b, \mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x) - [b, \mathcal{I}_\alpha]_1(f_{j_{k+m}}, g_{j_{k+m}})(x)|^q dx\right)^{1/q} \\
\ge& \left(\int_{G_1} |[b, \mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x)|^q dx\right)^{1/q} - \left(\int_{G_2} |[b, \mathcal{I}_\alpha]_1(f_{j_{k+m}}, g_{j_{k+m}})(x)|^q dx\right)^{1/q} \\
\ge& \left(\int_G |[b, \mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x)|^q dx - \int_{G_2^c \cap G} |[b, \mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x)|^q dx\right)^{1/q} \\
&\quad - \frac{\gamma_3}{4} |Q_{j_{k+m}}^{(1)}|^{\lambda/(nq)} \\
\ge& \left(\gamma_3^q |Q_{j_k}^{(1)}|^{\lambda/n} - \int_{G_2^c \cap G} |[b, \mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x)|^q dx\right)^{1/q} - \frac{\gamma_3}{4} |Q_{j_{k+m}}^{(1)}|^{\lambda/(nq)}.
\end{aligned}$$

By (26) and applying (18) with $E := G_2^c \cap G$, we have

$$\int_{G_2^c\cap G} |[b,\mathcal{I}_\alpha]_1(f_{j_k},g_{j_k})(x)|^q dx \le (\frac{\gamma_3}{4})^q |Q_{j_k}^{(1)}|^{\lambda/n}. \tag{27}$$

This together with (26) and the fact that $|Q_{j_{k+m}}^{(1)}| < |Q_{j_k}^{(1)}|$ for any $m \in \mathbb{N}$, yields that there exists $\delta_0 = \delta_0(\gamma_3, q, n) > 0$ such that

$$\begin{aligned}&\left(\int_{B(y_{j_k},\gamma_2 d_{j_k}^{(1)})} |[b,\mathcal{I}_\alpha]_1(f_{j_k},g_{j_k})(x) - [b,\mathcal{I}_\alpha]_1(f_{j_{k+m}},g_{j_{k+m}})(x)|^q dx\right)^{1/q}\\ &\ge \left(\gamma_3^q |Q_{j_k}^{(1)}|^{\lambda/n} - (\frac{\gamma_3}{4})^q |Q_{j_k}^{(1)}|^{\lambda/n}\right)^{1/q} - \frac{\gamma_3}{4}|Q_{j_{k+m}}^{(1)}|^{\lambda/(nq)}\\ &\ge \delta_0 |Q_{j_k}^{(1)}|^{\lambda/(qn)}.\end{aligned}$$

Thus

$$\left(\frac{1}{d_{j_k}^{(1)\lambda}}\int_{B(y_{j_k},\gamma_2 d_{j_k}^{(1)})} |[b,\mathcal{I}_\alpha]_1(f_{j_k},g_{j_k})(x) - [b,\mathcal{I}_\alpha]_1(f_{j_{k+m}},g_{j_{k+m}})(x)|^q dx\right)^{1/q} \ge \delta,$$

where $\delta = \delta(\delta_0, n, q, \lambda)$ and δ is independent on m.

Hence, $[b,\mathcal{I}_\alpha]_1$ is not a compact operator from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. This contradiction shows that b must satisfy the condition (1) of Lemma 3.1.

Case 2. If b does not satisfy (2), then there also exist ϵ and sequence of cubes $\{Q_j\}$, this time with $|Q_j| \to \infty$ as $j \to \infty$ such that (3.1) holds, too. This time we choose the subsequence $\{Q_{j_i}^{(2)} = Q(y_{j_i}, d_{j_i}^{(2)})\}$ so that

$$\frac{d_{j_i}^{(2)}}{d_{j_{i+1}}^{(2)}} < \frac{\beta}{\gamma_2}.$$

We can use a similar method as in the previous case, but the diameters are increasing, so the sets of definition in a reversed order. Thus, for fixed m, i, we have

$$\begin{aligned}\tilde{G} &= \{x : \gamma_1 d_{j_{i+m}}^{(2)} < |x - y_{j_{i+m}}| < \gamma_2 d_{j_{i+m}}^{(2)}\},\\ \tilde{G}_1 &= \tilde{G} \setminus \{x : |x - y_{j_i}| \le \gamma_2 d_{j_i}^{(2)}\},\\ \tilde{G}_2 &= \{x : |x - y_{j_i}| > \gamma_2 d_{j_i}^{(2)}\}.\end{aligned}$$

As before, (24) and (25) hold, and from this, the calculations are identical to those in Case 1.

Case 3. It remains to show that b must satisfying (3). In fact, in this case, if b does not satisfy (3), then there exists a cube Q with its diameter d and a sequence $\{y_j\}$ with $\lim_{j\to\infty} y_j = \infty$, such that (5) holds for the sequence $\{Q_j := Q + y_j\}$. Thus, if we take the sequences $\{f_j\}$ and $\{g_j\}$ defined by (6) and (12), respectively. Then (16) and (17) hold still. Now we denote

$$B_j = \{x \in \mathbb{R}^n : |x - y_j| < \gamma_2 d\}$$

and choose a subsequence $\{B_{j_k} = B(y_{j_k}, \gamma_2 d)\}$ such that $B_{j_k} \cap B_{j_l} = \emptyset$ for $l \neq k$. For selected j_k, let f_{j_k} and g_{j_k} be the functions associated with Q_{j_k}. We also define

$$\begin{aligned}\tilde{\tilde{G}} &= \{x : \gamma_1 d < |x - y_{j_k}| < \gamma_2 d\},\\ \tilde{\tilde{G}}_1 &= \tilde{\tilde{G}} \setminus \{x : |x - y_{j_{k+m}}| \leq \gamma_2 d\},\\ \tilde{\tilde{G}}_2 &= \{x : |x - y_{j_{k+m}}| > \gamma_2 d\};\end{aligned}$$

We see that $\tilde{\tilde{G}}_1 = \tilde{\tilde{G}} - \tilde{\tilde{G}}_2^c = \tilde{\tilde{G}}$, by $B_{j_k} \cap B_{j_{k+m}} = \emptyset$. Thus, for any $k, m \in \mathbb{N}$, by (16) and (17) we get

$$\begin{aligned}&\left(\int_{B_{j_k}} |[b,\mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x) - [b,\mathcal{I}_\alpha]_1(f_{j_{k+m}}, g_{j_{k+m}})(x)|^q dx\right)^{1/q}\\ &\geq \left(\int_{\tilde{\tilde{G}}} |[b,\mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x)|^q\right)^{1/q} - \left(\int_{\tilde{\tilde{G}}_2} |[b,\mathcal{I}_\alpha]_1(f_{j_{k+m}}, g_{j_{k+m}})(x)|^q dx\right)^{1/q}\\ &\geq \gamma_3 |Q|^{\lambda/nq} - \frac{\gamma_3}{4}|Q|^{\lambda/nq}\\ &> \frac{\gamma_3}{2}|Q|^{\lambda/nq}.\end{aligned}$$

Hence,

$$\left(\frac{1}{d^\lambda}\int_{B_{j_k}} |[b,\mathcal{I}_\alpha]_1(f_{j_k}, g_{j_k})(x) - [b,\mathcal{I}_\alpha]_1(f_{j_{k+m}}, g_{j_{k+m}})(x)|^q dx\right)^{1/q} \geq C\gamma_3.$$

This contradicts the compactness of $[b,\mathcal{I}_\alpha]_1$ from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. So b must also satisfy the condition (3) in Lemma 3.1. Theorem 1.2 is proved. □

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