

## A GEOMETRIC APPROACH TO THE STUDY OF AUTOMORPHISM GROUPS

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ABSTRACT. In this paper we study questions about automorphism groups of domains in  $\mathbb{C}^n$ , formulating the ideas entirely in terms of metric geometry. We also provide some applications.

### 1. Introduction

In this paper a *domain* is a connected, open set. We do our work in  $\mathbb{C}^n$ . If  $\Omega \subseteq \mathbb{C}^n$  is a domain, then an *automorphism* of  $\Omega$  is a biholomorphic selfmap. The collection of automorphisms of  $\Omega$  forms a group—denoted  $\text{Aut}(\Omega)$ —with the binary operation of composition of mappings. The topology on this group is uniform convergence on compact sets (also known as the compact-open topology). In fact, for a bounded domain, this group is always a real Lie group and never a complex Lie group (see [8]).

The most classical version of Bun Wong’s theorem [16] says this:

**Theorem 1.1.** *Let  $\Omega \subseteq \mathbb{C}^n$  be a smoothly bounded, strongly pseudoconvex domain. Suppose that the automorphism group of  $\Omega$  is noncompact. Then  $\Omega$  is biholomorphic to the unit ball  $B$  in  $\mathbb{C}^n$ .*

Later, Jean-Pierre Rosay [13] generalized the result as follows:

**Theorem 1.2.** *Let  $\Omega \subseteq \mathbb{C}^n$  be any bounded domain. Suppose that  $P \in \partial\Omega$  has a boundary neighborhood  $U$  that is strongly pseudoconvex. Further assume that there is a point  $X \in \Omega$  and automorphisms  $\varphi_j$  of  $\Omega$  so that  $\varphi_j(X) \rightarrow P$  as  $j \rightarrow \infty$ . Then  $\Omega$  is biholomorphic to the unit ball  $B$  in  $\mathbb{C}^n$ .*

What is nice about Rosay’s formulation is that it emphasizes that local geometric properties of  $\partial\Omega$  at  $P$  yield results about the global geometry of  $\Omega$ . Later work—see, for instance, [6]—capitalized on this insight.

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More recent work (see [7], [14], [15]) has given a metric version of the Wong/Rosay theorem. Instead of being about automorphisms, these works are about isometries of the Kobayashi metric.

One of our goals in the present work is to present and prove a version of this classical result that makes no reference to automorphisms or to strong pseudoconvexity.

## 2. Bergman metric balls

In this section we shall restrict attention to smoothly bounded, *strongly pseudoconvex* domains. We equip our domain  $\Omega$  with the Bergman metric (see [10, Ch. 1] for definition and discussion). We denote this metric by  $F_B^\Omega(\cdot, \cdot)$ .

**Definition 2.1.** With notation as above, we say that the domain  $\Omega$  satisfies the *metric uniformization property* if:

- There is a point  $P \in \Omega$  and a sequence of positive real numbers  $r_j \rightarrow +\infty$  and a collection of isometries  $\varphi_j$  on the metric balls  $\beta(P, r_j)$  so that
- The image of each  $\varphi_j$  is a metric ball  $\beta(P'_j, r_j)$  which is of Euclidean distance less than  $\delta_j > 0$  from the boundary, and  $\delta_j \rightarrow 0$  as  $j \rightarrow \infty$ .

Now we have our first fundamental result.

**Theorem 2.2.** *Let  $\Omega$  be a smoothly bounded, strongly pseudoconvex domain in  $\mathbb{C}^n$ . Suppose that  $\Omega$  has the metric uniformization property. Then  $\Omega$  is biholomorphic to the unit ball  $B$  in  $\mathbb{C}^n$ .*

*Remark 2.3.* It should be understood here that, in case  $\Omega$  is bounded and has noncompact automorphism group, then an old result of Cartan (see [10, Ch. 11]) guarantees that there exist a point  $Q \in \partial\Omega$  and a sequence  $\psi_j$  of holomorphic automorphisms of  $\Omega$  so that  $\psi_j(P) \rightarrow Q \in \partial\Omega$  for any point  $P \in \Omega$ . And of course  $Q$  is strongly pseudoconvex. It is well known—see [10, Ch. 11] and also [2]—that a strongly pseudoconvex point is well approximated by biholomorphic images of the unit ball  $B$ . The proof of the theorem will make these points clear.

*Proof of Theorem 2.2.* Of course the boundary of each metric ball is smooth (because the metric is smooth). When  $R$  is very large then  $\beta(P, R)$  will in fact have strongly pseudoconvex boundary. This is because, by Fefferman's regularity theory for Bergman metric geodesics (see [2]),  $\partial\beta(P, R)$  will approach  $\partial\Omega$  smoothly as  $R \rightarrow \infty$ . Therefore  $\varphi_j(\partial\beta(P, r_j))$  will also be strongly pseudoconvex.

It follows from an old result of Kobayashi and Nomizu (see [9]) that any isometry of strongly pseudoconvex domains in the Bergman metric is in fact either biholomorphic or conjugate biholomorphic. Thus the mappings  $\varphi_j$  in

the definition of “metric uniformization property” may be taken to be biholomorphic.

Passing to a subsequence, we may as well suppose that the balls  $\beta(P'_j, r_j)$  converge in the Euclidean sense to a boundary point  $Q$ . Now it is well known [2], just by a sophisticated diagonalization of the Levi form, that a neighborhood of the strongly pseudoconvex boundary point  $Q$  can be mapped biholomorphically to the ball with a mapping  $\Psi_j$  so that the boundary of the image has 4th order contact with  $\partial B$ . It is further known exactly the shape of a metric ball near a strongly pseudoconvex boundary point (see [10, Ch. 8]). It is of size  $C\delta$  in the (complex) normal directions and of size  $C\sqrt{\delta}$  in the (complex) tangential directions. Here  $\delta$  is the distance of the ball to the boundary.

So our situation is that we have a biholomorphic map  $\varphi_j$  from the strongly pseudoconvex metric ball  $\beta(P, r_j)$  to the strongly pseudoconvex metric ball  $\beta(P', r_j)$  and a geometrically normalizing holomorphic mapping  $\Psi_j$  from a neighborhood of  $\beta(P', r_j)$  to the unit ball  $B$ . Now it is a simple calculation to see that a Möbius transformation  $\Lambda_j$  of the ball will take  $\Psi_j(\varphi_j(\beta(P, r_j)))$  to a superset of a large ball inside  $B$ . In fact we could fix any large compact set  $K \subseteq B$  and, by choosing  $j$  large enough, be sure that  $\Lambda_j(\Psi_j(\varphi_j(\beta(P, r_j))))$  covers  $K$ .

But, since the  $\beta(P, r_j)$  exhaust  $\Omega$  and the compact sets  $K$  exhaust  $B$ , we see in the limit that the maps  $\Lambda_j \circ \Psi_j \circ \varphi_j$  converge to a biholomorphism of  $\Omega$  to  $B$ . That is the desired result.  $\square$

### 3. Geometric characterization of boundary orbit accumulation points

A notable theorem of Greene and Krantz (see [3, §1.5]) says this:

**Theorem 3.1.** *Let  $\Omega \subseteq \mathbb{C}^n$  be a smoothly bounded domain. Let  $P \in \partial\Omega$  be a boundary orbit accumulation point in the sense that there are  $\varphi_j \in \text{Aut}(\Omega)$  and a point  $X \in \Omega$  so that  $\varphi_j(X) \rightarrow P$  as  $j \rightarrow \infty$ . Then  $P$  must be a point of Levi pseudoconvexity.*

The original proof of this theorem was just classical function theory. What we want to do here is to give a geometric view of the matter.

**Proposition 3.2.** *Let  $\Omega \subseteq \mathbb{C}^n$  be a smoothly bounded domain. Let  $P \in \partial\Omega$  be a boundary orbit accumulation point. Then  $P$  is a point of completeness for the Carathéodory metric on  $\Omega$ .*

*Remark 3.3.* In fact we could use most any metric in the statement of this result. It is valid, in particular, for both the Bergman and the Kobayashi metrics. An examination of the proof validates this assertion.

*Proof of Proposition 3.2.* Let  $X \in \Omega$  and  $\varphi_j \in \text{Aut}(\Omega)$  be such that  $\varphi_j(X) \rightarrow P$ . Fix an  $r > 0$ . Consider the metric balls  $\beta(\varphi_j(X), r)$ . Passing to a subsequence if necessary, we may suppose that these balls are pairwise disjoint.

Consider a curve that passes through the points  $\{\varphi_j(X)\}$  and which terminates at  $P$ . Then it follows that the curve is of infinite length in the Carathéodory metric. It is in this sense that  $P$  is a point of completeness for the metric.  $\square$

**Proposition 3.4.** *Let  $\Omega \subseteq \mathbb{C}^n$  be a smoothly bounded domain. Let  $P \in \partial\Omega$  be a point of completeness for the Carathéodory metric. Then  $P$  is a point of Levi pseudoconvexity.*

*Proof.* Suppose not. Then the Levi form at  $P$  has a negative eigenvalue. By the Hartogs extension phenomenon, it follows that there is an open neighborhood of  $P$  so that every holomorphic function on  $U \cap \Omega$  analytically continues to  $U$ . But this implies that the Carathéodory distance from an interior point  $q \in \Omega$  to  $P$  is finite. So  $P$  is *not* a point of completeness for the metric.  $\square$

Putting together Propositions 3.2 and 3.4 yields Theorem 3.1.

#### 4. Semicontinuity of automorphism groups

A noted theorem of Greene and Krantz (see [4]) is described below. We first need to define an idea.

A smoothly bounded domain is conveniently thought of as given by a defining function  $\rho \equiv \rho_\Omega$ :

$$\Omega = \{z \in \mathbb{C}^n : \rho_\Omega(z) < 0\},$$

where  $\nabla\rho_\Omega \neq 0$  on  $\partial\Omega$ . We may define a subbasis for a topology on the collection of all smoothly bounded strongly pseudoconvex domains by (for  $\epsilon > 0$  small)

$$\mathcal{U}_{\Omega_0, \epsilon} = \{\Omega \subseteq \mathbb{C}^n \text{ strongly pseudoconvex; } \|\rho_\Omega - \rho_{\Omega_0}\|_{C^\infty} < \epsilon\}.$$

We call this topology *the  $C^\infty$  topology on the collection of smoothly bounded strongly pseudoconvex domains*. Now we have:

**Theorem 4.1.** *Let  $\Omega_0$  be a smoothly bounded, strongly pseudoconvex domain. Then there is a neighborhood  $\mathcal{O}$  of  $\Omega_0$  in the  $C^\infty$  topology on the collection of smoothly bounded, strongly pseudoconvex domains so that if  $\Omega \in \mathcal{O}$ , then*

- $\text{Aut}(\Omega)$  is a subgroup of  $\text{Aut}(\Omega_0)$ ;
- there is a  $C^\infty$  (not biholomorphic) mapping  $\alpha : \Omega \rightarrow \Omega_0$  so that the map

$$\text{Aut}(\Omega) \ni \varphi \mapsto \alpha \circ \varphi \circ \alpha^{-1} \in \text{Aut}(\Omega_0)$$

*is a univalent group homomorphism.*

Of course the proof of this result is too complicated to reproduce here. But we wish to make some remarks about the nature of the proof.

The inspiration for the above theorem comes from a fundamental work [1] of David Ebin. Ebin's result, reformulated for the purposes here, may be stated as follows:

**Theorem 4.2.** *Let  $M$  be a compact Riemannian manifold equipped with a metric  $g$ . Let  $G$  denote the group of isometries of  $M$  with respect to the metric  $g$ . There is an  $\epsilon > 0$  so that if  $\tilde{g}$  is another metric on  $M$  with  $\|g - \tilde{g}\| < \epsilon$ , then the isometry group  $\tilde{G}$  of  $M$  in the metric  $\tilde{g}$  is a subgroup of the isometry group  $G$  of  $M$  in the metric  $g$ . Also there is a  $C^\infty$  diffeomorphism  $\alpha$  of  $M$  to itself so that the mapping*

$$\tilde{G} \ni \varphi \mapsto \alpha \circ \varphi \circ \alpha^{-1} \in G$$

*is a univalent group homomorphism.*

The proof of Ebin's theorem is a sophisticated application of the implicit function theorem. In particular, he proves the semicontinuity by constructing a slice.

The strategy of Greene and Krantz was to reduce the question for automorphism groups of strongly pseudoconvex domains to that of isometry groups for Riemannian manifolds (see also [11], where the result is generalized). They proceeded as follows:

- (i) They constructed on  $\overline{\Omega}$  a biholomorphically invariant metric which has a product structure near the boundary.
- (ii) They considered the metric double  $\widehat{M}$  of the domain from (i) equipped with the special metric.
- (iii) They verified that the isometries of  $\widehat{M}$  are just the biholomorphic mappings or the conjugate biholomorphic mappings.
- (iv) They applied Ebin's theorem to the isometry group of  $\widehat{M}$ .
- (v) They analyzed the result of (iv) and derived therefrom the semicontinuity theorem for automorphism groups stated above.

So we see that the Greene-Krantz semi-continuity theorem for automorphism groups is, by its very nature, a theorem of Riemannian geometry. Thus it is very much in the spirit of the present paper.

### 5. Openness of the set of rigid domains

A *rigid domain* is one whose automorphism group consists of the identity alone. It is an immediate corollary of Theorem 4.2 that the set of smoothly bounded, rigid, strongly pseudoconvex domains is open. We present in the present section another way to view the matter.

Let  $\Omega_0$  be a rigid domain. If  $\Phi$  is any non-identity holomorphic map of  $\Omega$  to itself, then there are points  $P$  and  $Q$  in  $\Omega_0$  so that  $d_{\Omega_0}(P, Q) \neq d_{\Omega_0}(\Phi(P), \Phi(Q))$ . Here  $d_{\Omega_0}$  is distance in the Bergman metric on  $\Omega_0$ . If  $\Omega$  is a domain that is close to  $\Omega_0$  in the  $C^\infty$  topology, then the Bergman metric on  $\Omega$  will be close to the Bergman metric on  $\Omega_0$  (see [5]). In particular, we can conclude that, if  $\Phi$  is any holomorphic map of  $\Omega$  to itself, then there are points  $P$  and  $Q$  in  $\Omega_0$  so that  $d_\Omega(P, Q) \neq d_\Omega(\Phi(P), \Phi(Q))$ . Here  $d_\Omega$  is distance in the Bergman metric on  $\Omega$ . We conclude that  $\Omega$  too is rigid.

## 6. Closedness of the biholomorphic equivalence classes

Fix a smoothly bounded, strongly pseudoconvex domain  $\Omega_0$ . Let  $\mathcal{B}_{\Omega_0}$  be the collection of smoothly bounded, strongly pseudoconvex domains which are biholomorphically equivalent to  $\Omega_0$ . Equip  $\mathcal{B}_{\Omega_0}$  with the usual  $C^\infty$  topology on domains. We claim that  $\mathcal{B}_{\Omega_0}$  is closed.

To see this, let  $\{\Omega_j\} \subseteq \mathcal{B}_{\Omega_0}$  be a sequence that converges to a limit domain  $\Omega^*$ . Our claim is that  $\Omega^* \in \mathcal{B}_{\Omega_0}$ . Let  $\varphi_j : \Omega_j \rightarrow \Omega_0$  be a biholomorphism. Equip each domain with the Kobayashi metric. Then each  $\varphi_j$  is a Lipschitz mapping with Lipschitz norm at most 1. We can then apply the Ascoli-Arzelà theorem, along with the usual diagonalization on an exhausting sequence of compact subsets, to determine that there is a subsequence  $\varphi_{j_k}$  that converges uniformly on compact sets to a limit mapping  $\varphi^*$ . But it is also plain that  $\varphi^* : \Omega^* \rightarrow \Omega_0$ . And Hurwitz's theorem guarantees that this mapping is in fact a biholomorphism. We conclude, then, that  $\Omega^* \in \mathcal{B}_{\Omega_0}$ .

## 7. Concluding remarks

It has been a truism in the function theory of several complex variables for quite some time now that complex analytic situations can often be clarified by casting them in the language of metric geometry. The purpose of this paper has been to illustrate this idea by way of several results in the theory of automorphism groups of domains.

We look forward to future work developing these ideas.

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