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CLASS-PRESERVING AUTOMORPHISMS OF CERTAIN HNN EXTENSIONS OF BAUMSLAG-SOLITAR GROUPS

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Abstract. We show that, for any non-zero integers λ, μ, ν, ξ, class-preserving automorphisms of the group

$$G(\lambda,\mu,\nu,\xi)=\langle a,b,t : b^{-1}a^{\lambda}b=a^{\mu}, t^{-1}a^{\nu}t=b^{\xi}\rangle$$

are all inner. Hence, by using Grossman's result, the outer automorphism group of $G(\lambda,\pm\lambda,\nu,\xi)$ is residually finite.

1. Introduction

An automorphism α of a group G is called a *conjugating* (or *class-preserving*, or *point-wise inner*) *automorphism* if, for each $g\in G$, $\alpha(g)$ and g are conjugate in G. Clearly inner automorphisms are conjugating automorphisms. But there exist some groups admitting conjugating automorphisms which are not inner. For example, Burnside [7] constructed a group of order 3^6 admitting conjugating automorphisms which are not inner. Also Wall [26] constructed a group of order 32 having the same property. There are many nilpotent groups having conjugating automorphisms which are not inner [7, 24]. However conjugating automorphisms of free nilpotent groups are all inner [9].

In [13], Grossman defined that a group G has *Property A* if all conjugating automorphisms of G are inner. She showed that outer automorphism groups of finitely generated conjugacy separable groups with Property A are residually finite. This result has been used to study the residual finiteness of outer automorphism groups of certain groups. For example, outer automorphism groups of Fuchsian groups [1, 20], most of Seifert 3-manifold groups [2] and certain 1-relator groups [16, 17] are residually finite.

It is interesting to find some groups having Property A. Nontrivial free products of groups have Property A [1, 22]. Generalized free products of finitely

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generated nilpotent groups, amalgamating an infinite cyclic subgroup, have Property A [29]. Tree products of polycyclic-by-finite groups amalgamating central edge groups also have Property A [27]. However, there are generalized free products of finite nilpotent groups which do not have Property A [30]. Recently, it has been shown that HNN extensions with central associative subgroups have Property A [28]. In this note, we are interested in certain HNN extension of Baumslag-Solitar group having Property A.

The Baumslag-Solitar groups were introduced in [4]. For each pair λ and μ of non-zero integers, the Baumslag-Solitar group $BS(\lambda, \mu)$ is defined by the presentation

$$BS(\lambda, \mu) = \langle a, b : b^{-1}a^{\lambda}b = a^{\mu}\rangle.$$

It turns out that such groups have played a surprisingly useful role in combinatorial and more recently, geometric group theory. These groups have provided examples which mark boundaries between different classes of groups. For example, $BS(\lambda, \mu)$ is residually finite [19] if and only if $|\lambda| = |\mu|$ or $|\lambda| = 1$ or $|\mu| = 1$. It is known that $BS(\lambda, \mu)$ is solvable if and only if $|\lambda| = 1$ or $|\mu| = 1$. Recently, the solvable Baumslag-Solitar groups have served as a proving ground for many new ideas in combinatorial and geometric group theory ([10, 11, 14, 25]).

In this paper, we focus on the groups

$$G(\lambda, \mu, \nu, \xi) = \langle a, b, t : t^{-1}a^{\nu}t = b^{\xi},\ b^{-1}a^{\lambda}b = a^{\mu}\rangle,$$

where λ, μ, ν, ξ are all non-zero integers. This group $G(\lambda, \mu, \nu, \xi)$ is the HNN extension of the Baumslag-Solitar group $BS(\lambda, \mu)$ with associated subgroups $\langle a^{\nu}\rangle$ and $\langle b^{\xi}\rangle$. Baumslag [3] studied the group $G(1, 2, 1, 1)$ and showed that all its finite quotients are cyclic. Now, the group is called Baumslag-Gersten group (see [12, 21]). Brunner [6] and Borshchev and Moldvanskii [5] studied the group

$$G(l, m; k) = \langle a, t : t^{-1}a^{-k}ta^{l}t^{-1}a^{k}t = a^{m}\rangle$$

which is the same as the group $G(l, m, 1, k)$.

The main result in this paper is that the conjugating automorphisms of $G(\lambda, \mu, \nu, \xi)$ are all inner (Theorem 3.3). It is known that $G(\lambda, \mu, \nu, \xi)$ is residually finite [23] if and only if $|\lambda| = |\mu|$. Moreover, $G(\lambda, \mu, \nu, \xi)$ is conjugacy separable [15] if and only if $|\lambda| = |\mu|$. Hence, by using Grossman's result, the outer automorphism group of $G(\lambda, \pm\lambda, \nu, \xi)$ are residually finite (Theorem 3.4).

2. Preliminaries

Throughout this paper we use standard notation and terminology.

If $g \in G$, Inn_g denotes the inner automorphism of G induced by g.

$x \sim_G y$ means that x and y are conjugate in G, otherwise $x \nsim_G y$.

In order to make our statements more clear, we give the following definitions.

Definition 2.1. By a *conjugating* (*or class-preserving, or point-wise inner*) *automorphism* of a group G we mean an automorphism α of G which is such that, for each $g \in G$, there exists $k_g \in G$, depending on g, so that $\alpha(g) = k_g^{-1} g k_g$.

Definition 2.2 (Grossman [13]). A group G has *Property A* if, for each conjugating automorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1}gk$ for all $g \in G$.

Let $G = \langle B, t : t^{-1}Ht = K\rangle$ be an HNN extension of a base group B. Then every element $g \in G$ can be expressed in normal form ([18], p. 181),

$$g = a_0 t^{\epsilon_1} a_1 \cdots t^{\epsilon_n} a_n, \text{ where } \epsilon_i = \pm 1 \text{ and } a_i \in B,$$

satisfying the following:

(i) a_n is an arbitrary element of the base group B,
(ii) if $\epsilon_i = -1$, then a_{i-1} is a representative of a coset of K in B,
(iii) if $\epsilon_i = 1$, then a_{i-1} is a representative of a coset of H in B, and
(iv) there is no consecutive subsequence $t^{\epsilon}\ 1\ t^{-\epsilon}$.

In this note, we mean that a cyclically reduced element $g \in G$ is either $g \in A$ or $g = t^{\epsilon_1} a_1 \cdots t^{\epsilon_n} a_n$, where $t^{\epsilon_i} a_i \cdots t^{\epsilon_n} a_n t^{\epsilon_1} a_1 \cdots t^{\epsilon_{i-1}} a_{i-1}$ is reduced for each i.

We shall make extensive use of the following result due to D. J. Collins, which give us important information for conjugate elements in HNN extensions.

Theorem 2.3 (Collins [8]). *Let x and y be cyclically reduced elements of the HNN extension $G = \langle B, t : t^{-1}Ht = K\rangle$. Suppose that $x \sim_G y$. Then $\|x\| = \|y\|$, and one of the following holds.*

(1) $\|x\| = \|y\| = 0$ *and there is a finite sequence $z_1, z_2, \ldots, z_m$ of elements in $H \cup K$ such that $x \sim_B z_1 \sim_{B,t^*} z_2 \sim_{B,t^*} \cdots \sim_{B,t^*} z_m \sim_B y$, where $u \sim_{B,t^*} v$ means one of*: (i) *$u \sim_B v$, or* (ii) *$u \in H$ and $v = t^{-1}ut (\in K)$, or* (iii) *$u \in K$ and $v = tut^{-1} (\in H)$.*

(2) $\|x\| = \|y\| \geq 1$ *and $y \sim_{H \cup K} x^*$ where x^* is a cyclic permutation of x.*

Remark 2.4. In (1) of Theorem 2.3, if x or y is not conjugate to any element in $H \cup K$, then we have $x \sim_B y$.

Although the above observation (Remark 2.4) is very simple, it is important for our study of Baumslag-Solitar groups and HNN extensions of Baumslag-Solitar groups. We can see this from the following lemma.

Lemma 2.5. *Let ν, ξ be non-zero integers and $B = BS(\lambda, \mu)$ such that $\lambda > 0$ and $\mu \neq 0$.*

(1) *If $\lambda - \mu \neq \pm 1$, then, for $z = ba$, we have $z \not\sim_B a^{\nu i}$ and $z \not\sim_B b^{\xi i}$ for any integer i.*

(2) *If $\lambda - \mu = \pm 1$, then, for $z = b^2 a$, we have $z \not\sim_B a^{\nu i}$ and $z \not\sim_B b^{\xi i}$ for any integer i.*

Proof. (1) Let $\lambda - \mu \neq \pm 1$ and $z = ba$. Note ba is cyclically reduced of length 1 in the HNN extension B. Applying Theorem 2.3 for the HNN extension B, we have $z = ba \not\sim_B a^{\nu i}$ for any integer i.

Also, if $z = ba \sim_B b^{\xi i}$ for some integer i, then $\xi i = 1$ and $ba = a^{-c}b^{\xi i}a^c$ for some integer c by Theorem 2.3 for the HNN extension B. Hence $a^c \in \langle a^\lambda \rangle$. Let $a^c = a^{\lambda c_1}$ for some integer c_1. Then $ba = a^{-c}ba^c = a^{-\lambda c_1}ba^{\lambda c_1} = ba^{(\lambda-\mu)c_1}$. Hence $a = a^{(\lambda-\mu)c_1}$, which is impossible because of $\lambda - \mu \neq \pm 1$. Thus $z \not\sim_B b^{\xi i}$ for any integer i.

(2) Let $\lambda - \mu = \pm 1$ and $z = b^2 a$. As above, $z = b^2 a \not\sim_B a^{\nu i}$ for any integer i by applying Theorem 2.3 for the HNN extension B.

Also, if $z = b^2 a \sim_B b^{\xi i}$ for some integer i, then, as before, $\xi i = 2$ and $b^2 a = a^{-c}b^{\xi i}a^c$ for some integer c. Hence $a^c \in \langle a^\lambda \rangle$. Let $a^c = a^{\lambda c_1}$ for some integer c_1. Then we have $b^{-1}a^{\mu c_1}b = a^c a^{-1}$. Hence $a^{\mu c_1} \in \langle a^\lambda \rangle$. Since $\lambda - \mu = \pm 1$, λ and μ are relatively prime and again $c_1 = \lambda c_2$ for some integer c_2. Thus we have $b^{-1}a^{\mu c_1}b = b^{-1}a^{\mu\lambda c_2}b = a^{\mu\mu c_2}$. Hence $a^{\mu\mu c_2} = a^c a^{-1}$, that is $a = a^{c-\mu^2 c_2} = a^{(\lambda^2-\mu^2)c_2}$. Since $\lambda > 0$ and $\lambda - \mu = \pm 1$, we have $\lambda \geq 1$ and $\mu \geq 1$, whence $|\lambda^2 - \mu^2| = |(\lambda-\mu)(\lambda+\mu)| \geq 2$. Thus $a = a^{(\lambda^2-\mu^2)c_2}$ is impossible. So we get $z \not\sim_B b^{\xi i}$ for any integer i. □

Hence in the group $G = G(\lambda, \mu, \nu, \xi)$, Lemma 2.5 provides us with some element $z \in G$ which is not conjugate in B to any element in the associated subgroups in the HNN extension $G = \langle B, t : t^{-1}a^\nu t = b^\xi \rangle$. This element will play an important role later. The next lemma shows that such element is not unique in G.

Lemma 2.6. *Let ν, ξ and B be as in Lemma 2.5.*

(1) *If $\lambda - \mu \neq \pm 1$, then $b^2 a \not\sim_B a^{\nu i}$ and $b^2 a \not\sim_B b^{\xi i}$ for any integer i.*

(2) *If $\lambda - \mu = \pm 1$, then $b^3 a \not\sim_B a^{\nu i}$ and $b^3 a \not\sim_B b^{\xi i}$ for any integer i.*

Proof. (1) Let $\lambda - \mu \neq \pm 1$. By applying Theorem 2.3 for the HNN extension B, we can easily see that $b^2 a \not\sim_B a^{\nu i}$ for any integer i. Suppose $b^2 a \sim_B b^{\xi i}$ for some integer i. Also by Theorem 2.3, we have $\xi i = 2$ and $b^2 a = a^{-c}b^2 a^c$ for some $a^c \in \langle a^\lambda \rangle \cup \langle a^\mu \rangle$. Hence we have

$$b^{-2}a^c b^2 = a^{c-1}. \tag{2.1}$$

Case 1. Suppose $(\lambda, \mu) = 1$. From (2.1), $a^c \in \langle a^\lambda \rangle$. Hence $c = \lambda c_1$ for some integer c_1. Then (2.1) implies that $b^{-1}a^{\mu c_1}b = a^{c-1}$. Hence $a^{\mu c_1} \in \langle a^\lambda \rangle$. It follows from $(\lambda, \mu) = 1$ that $c_1 = \lambda c_2$ for some integer c_2. Hence we have $a^{\mu^2 c_2} = a^{c-1} = a^{\lambda^2 c_2 - 1}$. Thus $a = a^{(\lambda^2-\mu^2)c_2}$. But this is impossible, since $|\lambda^2 - \mu^2| = |(\lambda-\mu)(\lambda+\mu)| \neq 1$.

Case 2. Suppose $(\lambda, \mu) = d > 1$. As before, we have $c = \lambda c_1$ for some integer c_1 and from (2.1) $b^{-1}a^{\mu c_1}b = a^{c-1}$. Then $a^{\mu c_1} = a^{\lambda c_2}$ for some integer c_2 and we have $a^{\mu c_2} = a^{c-1} = a^{\lambda c_1 - 1}$. Hence $a = a^{\lambda c_1 - \mu c_2}$. But this is impossible, since $d > 1$ divides $\lambda c_1 - \mu c_2$.

Therefore, we get a contradiction in both cases, which means $b^2 a \not\sim_B b^{\xi i}$ for any integer i.

(2) Let $\lambda - \mu = \pm 1$. Then $(\lambda, \mu) = 1$. Clearly $b^3 a \not\sim_B a^{\nu i}$ for any integer i. Suppose $b^3 a \sim_B b^{\xi i}$ for some integer i. Applying Theorem 2.3 for the HNN

extension B, we have $\xi i = 3$ and $b^3 a = a^{-c} b^3 a^c$ for some $a^c \in \langle a^\lambda \rangle \cup \langle a^\mu \rangle$. Hence we have

$$b^{-3} a^c b^3 = a^{c-1}. \tag{2.2}$$

From (2.2), $a^c \in \langle a^\lambda \rangle$. Hence $c = \lambda c_1$ for some integer c_1. Then (2.2) implies that $b^{-2} a^{\mu c_1} b^2 = a^{c-1}$. Again $a^{\mu c_1} \in \langle a^\lambda \rangle$. It follows from $(\lambda, \mu) = 1$ that $c_1 = \lambda c_2$ for some integer c_2. Hence we have $b^{-1} a^{\mu^2 c_2} b = a^{c-1}$. Again $a^{\mu^2 c_2} \in \langle a^\lambda \rangle$. Since $(\lambda, \mu) = 1$, $c_2 = \lambda c_3$ for some integer c_3. Hence, we have $a^{\mu^3 c_3} = a^{c-1}$ and whence $a = a^{c - \mu^3 c_3} = a^{(\lambda^3 - \mu^3) c_3}$. Since $\lambda - \mu = \pm 1$ and $\lambda > 0$, we have $\lambda \geq 1$ and $\mu \geq 1$. Hence $|\lambda^3 - \mu^3| = |(\lambda - \mu)(\lambda^2 + \lambda\mu + \mu)| \geq 3$. Thus $a = a^{(\lambda^3 - \mu^3) c_3}$ is impossible. Therefore $b^3 a \not\sim_B b^{\xi i}$ for any integer i. □

3. The conjugating automorphisms of $G(\lambda, \mu, \nu, \xi)$

The following lemma is useful to study conjugating automorphisms of HNN extensions.

Lemma 3.1. *Let $G = \langle B, t : t^{-1} H t = K \rangle$ be an HNN extension of a base group B. Suppose there exists $z \in B$ such that $z \not\sim_B h$ and $z \not\sim_B k$ for any elements $h \in H$ and $k \in K$. If α is a conjugating automorphism of G such that $\alpha(z) = z$, then there exists an element $u \in B$ such that $\alpha(t) = u^{-1} t u$.*

Proof. Let $\alpha(t) = k_t^{-1} t k_t$ and let $k_t = a_0 t^{\epsilon_1} a_1 \cdots t^{\epsilon_n} a_n$, where $\epsilon_i = \pm 1$ and $a_i \in B$, be in normal form. We may assume that k_t is of the shortest length such that $\alpha(t) = k_t^{-1} t k_t$. Hence $a_0 \neq 1$ if $n \geq 1$.

Suppose $n \geq 1$. Since

$$t\, z \sim_G \alpha(t\, z) = \alpha(t) \alpha(z) = a_n^{-1} t^{-\epsilon_n} \cdots a_1^{-1} t^{-\epsilon_1} a_0^{-1}\, t\, a_0 t^{\epsilon_1} a_1 \cdots t^{\epsilon_n}\, a_n z,$$

we have

$$t\, z \sim_G t^{-\epsilon_n} \cdots t^{-\epsilon_1} a_0^{-1} t\, a_0\, t^{\epsilon_1} a_1 \cdots t^{\epsilon_n}\, a_n z\, a_n^{-1}. \tag{3.1}$$

If $\epsilon_1 = 1$, then $a_0 \notin H$ by the normal form of k_t. Since $z \not\sim_B h$ and $z \not\sim_B k$ for any elements $h \in H$ and $k \in K$, we have $a_n z\, a_n^{-1} \notin H \cup K$. Hence the R.H.S. of (3.1)

$$t^{-\epsilon_n} \cdots a_1^{-1} t^{-1} a_0^{-1}\, t\, a_0 t^1 a_1 \cdots t^{\epsilon_n}\, a_n z\, a_n^{-1}$$

is cyclically reduced of length $2n + 1$. Since the L.H.S. of (3.1) is of length 1, this is impossible by Theorem 2.3. Also, if $\epsilon_1 = -1$, then $a_0 \notin K$ by the normal form of k_t. Thus the R.H.S. of (3.1)

$$t^{-\epsilon_n} \cdots a_1^{-1} t^1 a_0^{-1}\, t\, a_0 t^{-1} a_1 \cdots t^{\epsilon_n}\, a_n z\, a_n^{-1}$$

is cyclically reduced of length $2n + 1$. Since the L.H.S. of (3.1) is of length 1, it is impossible by Theorem 2.3.

Therefore we must have $n = 0$ and $k_t = u \in B$, as required. □

From now on, the only groups considered are

$$B = BS(\lambda,\mu) = \langle a, b : b^{-1}a^{\lambda}b = a^{\mu}\rangle \text{ and}$$
$$G = G(\lambda,\mu,\nu,\xi) = \langle B, t : t^{-1}a^{\nu}t = b^{\xi}\rangle,$$

where λ,μ,ν,ξ are non-zero integers. Clearly, we only need to consider the case that $\lambda > 0$ and $\nu > 0$.

By Lemma 2.5, there exists an element z which is not conjugate to any elements in the associated subgroups of the HNN extension $G=\langle B, t : t^{-1}a^{\nu}t=b^{\xi}\rangle$. Then we can make use of Lemma 3.1.

Lemma 3.2. *Let $G = G(\lambda,\mu,\nu,\xi)$. Suppose α is a conjugating automorphism of G such that $\alpha(z) = z$, where (1) $z = ba$ if $\lambda-\mu \neq \pm 1$ and (2) $z = b^2a$ if $\lambda-\mu = \pm 1$. Then there exists an element $w \in B$ such that if $\overline{\alpha} = \mathrm{Inn}_w \circ \alpha$, then $\overline{\alpha}(z) = z$, $\overline{\alpha}(t) = t$ and $\overline{\alpha}(a), \overline{\alpha}(b) \in B$.*

Proof. Let α be a conjugating automorphism of G such that $\alpha(z) = z$, where $z = ba$ or $z = b^2a$ depending on $\lambda-\mu \neq \pm 1$ or $\lambda-\mu = \pm 1$. By Lemma 3.1 and Lemma 2.5, there exists an element $w \in B$ such that $\alpha(t) = w^{-1}tw$. Since $tz \sim_G \alpha(tz) = w^{-1}tw\,z \sim_G t\,wzw^{-1}$, by Theorem 2.3 $t\,wzw^{-1} = x^{-1}(tz)^*x$ for some $x \in \langle a^{\nu}\rangle \cup \langle b^{\xi}\rangle$, where $(tz)^*$ is a cyclic permutation of tz. Hence we have either $t\,wzw^{-1} = x^{-1}(tz)x$ or $t\,wzw^{-1} = x^{-1}(zt)x$.

First, suppose

$$t\,wzw^{-1} = x^{-1}(tz)x. \tag{3.2}$$

Then $t^{-1}xt = zxwz^{-1}w^{-1}$. It follows that $x = a^{\nu i}$ for some integer i and $b^{\xi i} = zxwz^{-1}w^{-1}$. Hence $wzw^{-1} = b^{-\xi i}zx$ and $z \sim_B b^{-\xi i}zx$.

(1) $\lambda-\mu \neq \pm 1$ and $z = ba$. Then $z = ba \sim_B b^{1-\xi i}a^{1+\nu i}$. Since ba and $b^{1-\xi i}a^{1+\nu i}$ are cyclically reduced in the HNN extension B, by Theorem 2.3 we have $1-\xi i = 1$. Thus $i = 0$ which is $x = 1$. By (3.2), we get $wzw^{-1} = z$.

(2) $\lambda-\mu = \pm 1$ and $z = b^2a$. Then $z = b^2a \sim_B b^{2-\xi i}a^{1+\nu i}$. As before we have $2-\xi i = 2$. Thus $i = 0$ which is $x = 1$. By (3.2), we also get $wzw^{-1} = z$.

Second, suppose $t\,wzw^{-1} = x^{-1}(zt)x$. Let $z^{-1}x = x_1$. Then we have $t\,wzw^{-1} = x_1^{-1}(tz)x_1$ and $x_1 \in \langle a^{\nu}\rangle$. Thus this case is similar to (3.2).

Let $\overline{\alpha} = \mathrm{Inn}_{w^{-1}} \circ \alpha$. Then $\overline{\alpha}$ is a conjugating automorphism of G such that $\overline{\alpha}(z) = z$ and $\overline{\alpha}(t) = t$.

Now we shall show that $\overline{\alpha}(a) \in B$. Note that $az \sim_G \overline{\alpha}(az) = \overline{\alpha}(a)z = k_a^{-1}ak_a\,z$, where $k_a = b_0t^{\epsilon_1}b_1\cdots t^{\epsilon_m}b_m$, $\epsilon_i = \pm 1$ and $b_i \in B$, is in normal form. Hence

$$a\,z \sim_G t^{-\epsilon_m}\cdots b_1^{-1}t^{-\epsilon_1}b_0^{-1}a\,b_0\,t^{\epsilon_1}b_1\cdots t^{\epsilon_m}b_m z\,b_m^{-1}. \tag{3.3}$$

By Lemma 2.5, $z \not\sim_B a^{\nu i}$ and $z \not\sim_B b^{\xi i}$ for any integer i. Hence $b_m z\,b_m^{-1} \notin \langle a^{\nu}\rangle \cup \langle b^{\xi}\rangle$. If $t^{-\epsilon_m}\cdots b_1^{-1}t^{-\epsilon_1}b_0^{-1}ab_0t^{\epsilon_1}b_1\cdots t^{\epsilon_m} \notin B$, then the R.H.S. of (3.3) is cyclically reduced of even length > 0. Since the L.H.S. of (3.3) is of length 0, we have a contradiction. Hence we have $t^{-\epsilon_m}\cdots t^{-\epsilon_1}b_0^{-1}a\,b_0\,t^{\epsilon_1}\cdots t^{\epsilon_m} \in B$

and

$$\overline{\alpha}(a) = b_m^{-1}(t^{-\epsilon_m} \cdots b_1^{-1} t^{-\epsilon_1} b_0^{-1} a b_0 t^{\epsilon_1} \cdots t^{\epsilon_m}) b_m \in B,$$

as required.

Similarly, $\overline{\alpha}(b) \in B$. □

Now we can prove the main result:

Theorem 3.3. *The group $G = G(\lambda, \mu, \nu, \xi)$ has Property A.*

Proof. Let α be a conjugating automorphism of G. Without loss of generality we may assume $\alpha(z) = z$, where $z = ba$ (if $\lambda - \mu \neq \pm 1$) or $z = b^2 a$ (if $\lambda - \mu = \pm 1$). By Lemma 3.2 we may assume $\alpha(z) = z$, $\alpha(t) = t$, $\alpha(a) = u \in B$ and $\alpha(b) = w \in B$. We shall prove that $\alpha(b) = b$, which implies $\alpha(a) = a$. Hence α is the identity, which means G has Property A.

Since $tb \sim_G \alpha(tb) = tw$, by Theorem 2.3, we get $t\,w = x^{-1}(tb)^* x$ for some $x \in \langle a^\nu \rangle \cup \langle b^\xi \rangle$, where $(tb)^*$ is a cyclic permutation of tb. Hence we have either (a) $t\,w = x^{-1}(tb)x$ or (b) $t\,w = x^{-1}(bt)x$.

(a) If $t\,w = x^{-1}(tb)x$, then $t^{-1}xt = bxw^{-1}$. It follows that $x = a^{\nu i}$ and $b^{\xi i} = bxw^{-1}$ for some integer i. Thus $w = b^{1-\xi i} a^{\nu i}$.

(b) If $t\,w = x^{-1}(bt)x$, then $t^{-1}b^{-1}xt = xw^{-1}$. Hence $b^{-1}x \in \langle a^\nu \rangle$, say $b^{-1}x = a^{\nu i}$ for some integer i. Then we have $b^{\xi i} = xw^{-1} = ba^{\nu i}w^{-1}$ and $w = b^{1-\xi i} a^{\nu i}$ as before.

Case 1. $\lambda - \mu \neq \pm 1$ and $z = ba$. Since $b^2 a \sim_G \alpha(b^2 a) = \alpha(b)\alpha(ba) = wba$, there exist $z_i \in \langle a^\nu \rangle \cup \langle b^\xi \rangle$ such that

$$b^2 a \sim_B z_1 \sim_{B,t^*} z_2 \sim_{B,t^*} \cdots \sim_{B,t^*} z_m \sim_B wba. \tag{3.4}$$

By Lemma 2.6, $b^2 a \not\sim_B z_1 \in \langle a^\nu \rangle \cup \langle b^\xi \rangle$. By Remark 2.4, we have $b^2 a \sim_B wba = b^{1-\xi i} a^{\nu i} ba$. It follows that $1 - \xi i = 1$, that is, $i = 0$. Hence $x = 1$ and $w = b$, thus $\alpha(b) = b$, as required.

Case 2. $\lambda - \mu = \pm 1$ and $z = b^2 a$. Since $b^3 a \sim_G \alpha(b^3 a) = \alpha(b)\alpha(b^2 a) = wb^2 a$, there exist $z_i \in \langle a^\nu \rangle \cup \langle b^\xi \rangle$ such that

$$b^3 a \sim_B z_1 \sim_{B,t^*} z_2 \sim_{B,t^*} \cdots \sim_{B,t^*} z_m \sim_B wb^2 a. \tag{3.5}$$

By Lemma 2.6, $b^3 a \not\sim_B z_1 \in \langle a^\nu \rangle \cup \langle b^\xi \rangle$. By Remark 2.4, we have $b^3 a \sim_B wb^2 a = b^{1-\xi i} a^{\nu i} b^2 a$. It follows that $1 - \xi i = 1$, that is, $i = 0$. Hence $x = 1$ and $w = b$, thus $\alpha(b) = b$ as required. □

It is known that $G(\lambda, \mu, \nu, \xi)$ is conjugacy separable [15] if and only if $|\lambda| = |\mu|$. In [13], Grossman proved that outer automorphism groups of finitely generated conjugacy separable groups with Property A are residually finite. Using this, we can derive the following:

Theorem 3.4. *The outer automorphism group of $G(\lambda, \pm\lambda, \nu, \xi)$ is residually finite.*

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