

GENERALIZED CAYLEY GRAPH OF UPPER TRIANGULAR MATRIX RINGS

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ABSTRACT. Let R be a commutative ring with the non-zero identity and n be a natural number. Γ_R^n is a simple graph with $R^n \setminus \{0\}$ as the vertex set and two distinct vertices X and Y in R^n are adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose entries on the main diagonal are non-zero such that $AX^t = Y^t$ or $AY^t = X^t$, where, for a matrix B , B^t is the matrix transpose of B . Γ_R^n is a generalization of Cayley graph. Let $T_n(R)$ denote the $n \times n$ upper triangular matrix ring over R . In this paper, for an arbitrary ring R , we investigate the properties of the graph $\Gamma_{T_n(R)}^n$.

1. Introduction

The investigation of graphs related to algebraic structures is a very large and growing area of research. One of the most important classes of graphs considered in this framework is that of Cayley graphs. These graphs have been considered, for example in [2], [7], [10] and [11]. Let us refer the readers to the survey article [14] for extensive bibliography devoted to various applications of Cayley graphs. Several other classes of graphs associated with algebraic structures have been also actively investigated. For example, see [1], [3], [5], [6], [12] and [13].

Let G be a semigroup, and let S be a nonempty subset of G . The Cayley graph of a semigroup G relative to S , which is denoted by $\text{Cay}(G, S)$, is defined as the graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (x, y) such that $sx = y$ for some $s \in S$ (see [7]). In particular, the Cayley graphs of semigroups are related to automata theory, as explained in [9] and the monograph [8].

In [2], for a commutative ring R with the identity element $1 \neq 0$, the authors introduced and studied a simple graph, denoted by Γ_R^n , with $R^n \setminus \{0\}$ as the vertex set and two distinct vertices X and Y in R^n are adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose all entries on the

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main diagonal are non-zero such that $AX^t = Y^t$ or $AY^t = X^t$, where, for a matrix B , B^t is the matrix transpose of B . Here, R^n is the Cartesian product of n copies of R . Clearly, in the case that $n = 1$, Γ_R^1 is the undirected Cayley graph $\text{Cay}(H, S)$, where $H = S = R \setminus \{0\}$. So Γ_R^n is a generalization of Cayley graphs.

Let $T_n(R)$ be the $n \times n$ upper triangular matrix ring over an arbitrary ring R . When there is no confusion, let us denote $T_n(R)$ by T . In this paper for a natural number n , we study the graph Γ_T^n . Γ_T^n is an undirected graph with vertex set

$$\{(X_1, X_2, \dots, X_n); X_i \in T_n(R), 1 \leq i \leq n\},$$

and two distinct vertices $\mathcal{X} = (X_1, X_2, \dots, X_n)$ and $\mathcal{Y} = (Y_1, Y_2, \dots, Y_n)$ in Γ_T^n are adjacent if and only if there exists an $n \times n$ lower triangular matrix A over an arbitrary ring R whose all entries on the main diagonal are non-zero such that $A\mathcal{X}^t = \mathcal{Y}^t$ or $A\mathcal{Y}^t = \mathcal{X}^t$. We denote the set of zero-divisors and unit elements of R by $ZD(R)$ and $U(R)$, respectively (cf. [15]).

We use the standard terminology of graphs contained in [4]. Throughout the paper, all graphs are simple and connected graphs. Suppose that G is a graph with vertex set $V(G)$. The *distance* between the vertices u and v of $V(G)$ is defined as the length of a minimal path connected them, denoted by $d(u, v)$. The *diameter* of a graph G is $\text{diam}(G) = \sup\{d(u, v); u, v \in V(G) \text{ and } u \neq v\}$. The *girth* of G is the length of the shortest cycle in G , and denoted by $\text{gr}(G)$. The *degree* of a vertex u in a graph G , denoted by $\text{deg}_G(u)$, is the number of edges of G incident with u . The graph G is *complete* if each pair of distinct vertices is joined by an edge. We use K_n to denote the complete graph with n vertices. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. A dominating set for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D . The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G . For a vertex a of G , we denote the neighbourhood of a by $N(a) = \{x \in V(G) : d(a, x) = 1\}$. Let G be a graph and $a, b, c \in V(G)$, such that $N(a) \subsetneq N(b) \subseteq N(c)$. Then G is not End-regular. Let G_1 and G_2 be subgraphs of G . We say that G_1 and G_2 are *disjoint* if they have no vertex and no edge in common. The *union* of two disjoint graphs G_1 and G_2 , which is denoted by $G_1 \cup G_2$, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *independence number* of a graph G is the cardinality of the largest independent set. The independence number is denoted by $\alpha(G)$. The *chromatic number* of a graph G , which is denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices have the same colors. Also, the *edge chromatic number* of a graph G , which is denoted by $\chi'(G)$, is the smallest number of colors such that one can associate colors to edges of G so that every pair of distinct edges meeting at a common vertex are assigned two different colors.

In Section 2, we study some basic properties of Γ_T^n . In Section 3, we describe the graph $\Gamma_{T_2(R)}^1$ in detail. Moreover, in the case that R is a field, we completely investigate the properties of the graph $\Gamma_{T_2(R)}^1$.

2. Basic properties

In this section, for a finite ring R , we study the cardinality of the sets of vertices and edges of the graph Γ_T^n . Also, we study the connectivity and diameter of the graph Γ_T^n , where n is a positive integer with $n > 1$.

For i with $1 \leq i \leq n$, we use the notation \mathcal{T}^i to denote the set of all vertices whose first non-zero components are in the i th place. In the following remark, we bring some results about the adjacencies in the graph Γ_T^n from [2].

Remark 2.1. Assume that $\mathcal{X} = (X_1, X_2, \dots, X_n)$ and $\mathcal{Y} = (Y_1, Y_2, \dots, Y_n)$ are two distinct vertices in Γ_T^n . Then,

- (i) If X_1 is unit (or invertible) and Y_1 is non-zero, then \mathcal{X} and \mathcal{Y} are adjacent in Γ_T^n .
- (ii) If $\mathcal{X} \in \mathcal{T}^i$ such that its i th component is a unit matrix, then \mathcal{X} is adjacent to \mathcal{Y} for all $\mathcal{Y} \in \mathcal{T}^i$.
- (iii) If $X_1 = 0 = Y_1$, then there is a vertex $\mathcal{Z} = (Z_1, Z_2, \dots, Z_n)$ such that Z_1 is non-zero and \mathcal{Z} is adjacent to both vertices \mathcal{X} and \mathcal{Y} .
- (iv) If $X_1 \neq 0$, then \mathcal{X} is adjacent to (X_1, I_n, \dots, I_n) .

Proposition 2.2. *Let R be a finite ring with $|R| = k$. Then*

$$|V(\Gamma_T^n)| = \left(k^{\frac{n(n+1)}{2}}\right)^n - 1.$$

Proof. Since we have $|T| = k^n \cdot k^{n-1} \cdot \dots \cdot k = k^{(1+2+\dots+n)} = k^{\frac{n(n+1)}{2}}$, the result holds. □

Proposition 2.3. *Let R be a finite ring with $|R| = k$ and $|ZD(R)| = d$. Then the number of edges of Γ_T^n is at least*

$$\left[\sum_{i=1}^n \left(\frac{|V_i| \cdot |V_i - 1|}{2} + |V_i| \cdot |V'_i| \right) + [|ZD(T)|(|T|^{n-1} - 1)] \right],$$

where $|V_i|$ is the number of vertices of \mathcal{T}^i in Γ_T^n whose first unit components are in the i th place, and $|V'_i|$ is the number of vertices of \mathcal{T}^i in Γ_T^n whose first non-zero and non-unit components are in the i th place.

Proof. Assume that $|R| = k$, $|ZD(R)| = d$ and $|U(R)| = k - d$. Clearly

$$|U(T)| = (k - d)^n (k^{n-1} \cdot k^{n-2} \cdot \dots \cdot k) = (k - d)^n \cdot k^{\frac{n(n-1)}{2}}.$$

Also, the number of non-zero and non-unit elements of T is

$$|U'(T)| = k^{\frac{n(n+1)}{2}} - (k - d)^n \cdot k^{\frac{n(n-1)}{2}} - 1.$$

For each i with $1 \leq i \leq n$, we have

$$|V_i| = |U(T)| \cdot |T|^{n-i} = (k-d)^n \cdot k^{\frac{n(n-1)}{2}} \cdot (k^{\frac{n(n+1)}{2}})^{n-i},$$

and

$$|V'_i| = |U'(T)| \cdot |T|^{n-i} = \left(k^{\frac{n(n+1)}{2}} - (k-d)^n \cdot k^{\frac{n(n-1)}{2}} - 1\right) (k^{\frac{n(n+1)}{2}})^{n-i}.$$

Thus, in this case, by Remark 2.1(i), we conclude that the number of edges between V_i 's and V'_i 's are at least

$$\sum_{i=1}^n \left(\frac{|V_i|(|V_i| - 1)}{2} + |V_i| \cdot |V'_i|\right).$$

Also, the number of vertices whose first components are zero, is equal to

$$k^{\frac{n^2(n+1)}{2}} - 1 - \left(k^{\frac{n^2(n+1)}{2}} - k^{\frac{n(n+1)(n-1)}{2}}\right) = k^{\frac{n(n-1)(n+1)}{2}} - 1,$$

and by Remark 2.1(iii), these vertices are adjacent to vertex (B, I_n, \dots, I_n) , where $B \in ZD(T)$. Hence, the number of edges of Γ_T^n is at least

$$\left[\sum_{i=1}^n \left(\frac{|V_i| \cdot (|V_i| - 1)}{2} + |V_i| \cdot |V'_i|\right)\right] + (|ZD(T)| \cdot (|T|^{n-1} - 1)). \quad \square$$

Theorem 2.4. *The graph Γ_T^n is connected and $\text{diam}(\Gamma_T^n) \in \{2, 3\}$.*

Proof. Since T is not an integral domain, by [2, Theorem 2.8], we have Γ_T^n is connected and $\text{diam}(\Gamma_T^n) \in \{2, 3\}$. \square

Lemma 2.5. *Assume that R is an arbitrary ring. Then $R = ZD(R) \cup U(R)$ if and only if $T = ZD(T) \cup U(T)$.*

Proof. Let $R = ZD(R) \cup U(R)$. Then clearly $T = ZD(T) \cup U(T)$. Now, suppose that $T = ZD(T) \cup U(T)$ and $R \neq ZD(R) \cup U(R)$. Then there exists $0 \neq a \in R \setminus (ZD(R) \cup U(R))$. So there exists an $n \times n$ upper triangular matrix A over R with entries on the main diagonal are $a \neq 0$, and so $A \notin ZD(T)$. Hence $A \in U(T)$. Therefore there exists an $n \times n$ upper triangular matrix B over R such that $AB = I_n$. So $ab_{11} = 1$, which implies that $a \in U(R)$, and this is impossible. \square

The following proposition immediately follows from Theorem 2.4 and Lemma 2.5 in conjunction with [2, Theorem 2.10].

Proposition 2.6. *In the graphs Γ_R^n and Γ_T^n we have $\text{diam}(\Gamma_R^n) = \text{diam}(\Gamma_T^n)$.*

Lemma 2.7. *If R is a field, then $\text{diam}(\Gamma_T^n) = 2$.*

Proof. Let $\mathcal{X} = (X_1, X_2, \dots, X_n)$ and $\mathcal{Y} = (Y_1, Y_2, \dots, Y_n)$ be two vertices of Γ_T^n . By Remark 2.1(i), if $X_1 \neq 0 \neq Y_1$, then $d(\mathcal{X}, \mathcal{Y}) = 1$. If $X_1 = 0$ and $Y_1 \neq 0$, then, by Remark 2.1(iii), there exists a vertex $\mathcal{Z} = (Z_1, Z_2, \dots, Z_n)$ with $Z_1 \neq 0$ such that \mathcal{X} and \mathcal{Z} are adjacent. Also, by Remark 2.1(i), \mathcal{Z} and \mathcal{Y} are adjacent. Hence $d(\mathcal{X}, \mathcal{Y}) \leq 2$. Now if X_1 and Y_1 are zero, then, by Remark

2.1(iii), there exists a vertex $\mathcal{K} = (K_1, K_2, \dots, K_n)$ with $K_1 \neq 0$, such that $\mathcal{X} \sim \mathcal{K} \sim \mathcal{Y}$. Hence $d(\mathcal{X}, \mathcal{Y}) \leq 2$. Clearly $(Y_1, 0, \dots, 0)$ and $(0, X_2, 0, \dots, 0)$ are two non-adjacent vertices in Γ_T^n . Hence $\text{diam}(\Gamma_T^n) = 2$. \square

Proposition 2.8. *In the graph Γ_T^n , we have the following inequality*

$$\gamma(\Gamma_T^n) \leq |T|^{n-1}(|T| - 1).$$

Proof. We show that the set $D = \{(X_1, X_2, \dots, X_n); X_1 \neq 0\}$ forms a dominating set for Γ_T^n . Let (Y_1, Y_2, \dots, Y_n) be a vertex in Γ_T^n such that

$$(Y_1, Y_2, \dots, Y_n) \notin D.$$

Hence $Y_1 = 0$. Consider two matrices A and B such that $A \in M_{n \times n}(R)$ and $0 \neq B \in T_n(R)$, and $AB = 0$. Then we have

$$\begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \end{bmatrix} \begin{bmatrix} B \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 0 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix},$$

which means that (B, Y_2, \dots, Y_n) and $(0, Y_2, \dots, Y_n)$ are adjacent and

$$(B, Y_2, \dots, Y_n) \in D.$$

Now since D is a dominating set of size $|T|^{n-1}(|T| - 1)$, the result holds. \square

Lemma 2.9. *Assume that T contains a nonidentity unit matrix. Then Γ_T^n is not End-regular.*

Proof. Suppose that $\mathcal{I} = (I_1, 0, \dots, 0)$, $\mathcal{U} = (U_1, 0, \dots, 0)$ and $\mathcal{X} = (X_1, 0, \dots, 0)$ are three vertices of Γ_T^n such that I_1 is the identity matrix, U_1 is a nonidentity unit matrix and $X_1 \neq 0$. Let $\mathcal{Y} \in N(\mathcal{U})$. Then there exists $n \times n$ lower

triangular matrix $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$ such that $A\mathcal{Y}^t = \mathcal{U}^t$ or $A\mathcal{U}^t = \mathcal{Y}^t$.

First suppose that $A\mathcal{Y}^t = \mathcal{U}^t$, and consider the $n \times n$ lower triangular matrix

$A' = \begin{pmatrix} a_{11}U_1^{-1} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$. Then $A'\mathcal{Y}^t = \mathcal{I}^t$, and so $\mathcal{Y} \in N(\mathcal{I})$. Hence $N(\mathcal{U}) \subseteq N(\mathcal{I})$. Now, let $A\mathcal{U}^t = \mathcal{Y}^t$, and consider the $n \times n$ lower triangular

matrix $A'' = \begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ Y_2 & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ Y_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$. Then $A''T^t = \mathcal{Y}^t$, and so $\mathcal{Y} \in N(\mathcal{I})$.

Therefore $N(\mathcal{U}) \subseteq N(\mathcal{I})$. Now, let $\mathcal{W} \in N(\mathcal{X})$. Then one can easily check that the first component of \mathcal{W} is non-zero. Hence, by Remark 2.1(i), we have $N(\mathcal{X}) \subseteq N(\mathcal{U})$. We show that $N(\mathcal{X}) \subsetneq N(\mathcal{U})$. Since T has at least two maximal ideals, there exist non-zero matrices X_1 and Z_1 such that $X_1 \notin \langle Z_1 \rangle$ and $Z_1 \notin \langle X_1 \rangle$, where for a matrix $A \in T$, $\langle A \rangle$ is the ideal generated by A . Now, it is easy to see that $(Z_1, 0, \dots, 0)$ is adjacent to \mathcal{U} but it is not adjacent to \mathcal{X} . Therefore Γ_T^n is not End-regular. \square

3. The structure of $\Gamma_{T_2(R)}^1$

In this section, we restrict $T = T_2(R)$ to be the 2×2 matrices over an arbitrary ring R with identity. We describe the graph $\Gamma_{T_2(R)}^1$ in detail. Let \mathcal{X} and \mathcal{Y} be two vertices of $\Gamma_{T_2(R)}^1$. Then \mathcal{X} and \mathcal{Y} are adjacent if and only if there exists a non-zero arbitrary 2×2 matrix A over R such that $A\mathcal{X} = \mathcal{Y}$ or $A\mathcal{Y} = \mathcal{X}$.

Denote $ZD = ZD(R)$ and $U = U(R)$. To describe the graph $\Gamma_{T_2(R)}^1$ for an arbitrary ring R , we first divide the vertices of $T = T_2(R)$ into the following disjoint subsets:

$$\begin{aligned} X^{(1)} &= \begin{bmatrix} U & R \\ 0 & U \end{bmatrix}, & X^{(2)} &= \begin{bmatrix} U & R \\ 0 & ZD \end{bmatrix}, \\ X^{(3)} &= \begin{bmatrix} ZD & R \\ 0 & U \end{bmatrix}, & X^{(4)} &= \begin{bmatrix} ZD & R \\ 0 & ZD \end{bmatrix}. \end{aligned}$$

That is, $V(\Gamma_{T_2(R)}^1) = X^{(1)} \cup X^{(2)} \cup X^{(3)} \cup X^{(4)}$ is the disjoint union of the four sets $X^{(i)}$'s for $i = 1, 2, 3, 4$. Also, $\Gamma_{T(X^i)}^1$ is the induced subgraph of $\Gamma_{T_2(R)}^1$, and $V(\Gamma_{T(X^i)}^1) = X^{(i)}$ for $i = 1, 2, 3, 4$. In the following, let E_{ij} be a subgraph of $\Gamma_{T_2(R)}^1$ with vertex set $X^{(i)} \cup X^{(j)}$, and consists of all edges from vertices $X^{(i)}$ to $X^{(j)}$ for $i, j = 1, 2, 3, 4$.

Proposition 3.1. *The subgraph E_{11} is a complete graph.*

Proof. Let

$$\mathcal{X} = \begin{bmatrix} u_1 & r_2 \\ 0 & u_3 \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} u'_1 & r'_2 \\ 0 & u'_3 \end{bmatrix} \in X^{(1)},$$

where $u_1, u'_1, u_3, u'_3 \in U$, and $r_2, r'_2 \in R$. In this case, there exists a non-zero arbitrary 2×2 matrix of the form

$$A = \begin{bmatrix} u_1^{-1}u'_1 & r'_2u_3^{-1} - u_1^{-1}u'_1r_2u_3^{-1} \\ 0 & u'_3u_3^{-1} \end{bmatrix},$$

such that $A\mathcal{X} = \mathcal{Y}$. Then \mathcal{X} and \mathcal{Y} are adjacent, and so E_{11} is complete. \square

Proposition 3.2. *For $j = 2, 3, 4$, the subgraph E_{1j} is a complete graph.*

Proof. Similar to the proof of Proposition 3.1, the results hold. \square

The following corollary follows from Propositions 3.1 and 3.2.

Corollary 3.3. *In the graph $\Gamma_{T_2(R)}^1$, we have $\text{diam}(\Gamma_{T_2(R)}^1) = 2$.*

Theorem 3.4. *Let R be a finite arbitrary ring with identity. Let $|R| = k$ and $|ZD(R)| = z$. Then the number of edges of $\Gamma_{T_2(R)}^1$ is at least equal to*

$$\frac{1}{2}k(k-z)^2(k^3 + 2k^2z - kz^2 - 1).$$

Proof. By Propositions 3.1 and 3.2, the number of edges in $\Gamma_{T_2(R)}^1$ is at least equal to the sum of the number of edges of the complete graph E_{11} and E_{1j} , for $j = 2, 3, 4$. So,

$$\begin{aligned} |E(\Gamma_{T_2(R)}^1)| &\geq \frac{[k(k-z)^2][k(k-z)^2 - 1]}{2} \\ &\quad + k(k-z)^2[kz(k-z) + kz(k-z) + kz^2] \\ &= \frac{1}{2}k(k-z)^2(k^3 + 2k^2z - kz^2 - 1). \end{aligned} \quad \square$$

In the rest of this section, we assume that R is a finite field with k elements. Clearly $|U(R)| = k - 1$. In this case

$$\begin{aligned} X^{(1)} &= \begin{bmatrix} U & R \\ 0 & U \end{bmatrix}, & X^{(2)} &= \begin{bmatrix} U & R \\ 0 & 0 \end{bmatrix}, \\ X^{(3)} &= \begin{bmatrix} 0 & R \\ 0 & U \end{bmatrix}, & X^{(4)} &= \begin{bmatrix} 0 & R^* \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Recall that $|X^{(1)}| = k(k-1)^2$, $|X^{(2)}| = |X^{(3)}| = k(k-1)$ and $|X^{(4)}| = k-1$. So the number of vertices of $\Gamma_{T_2(R)}^1$ is equal to $(k-1)(k^2 + k + 1)$.

Lemma 3.5. *Let R be a finite field. Then the following statements hold.*

- (i) *For each $i = 1, 3, 4$, the subgraph E_{ii} is complete.*
- (ii) *For each $j = 2, 3, 4$, the subgraph E_{1j} is complete.*
- (iii) *The subgraph E_{34} is complete.*

Proof. (i) We only give the proof for the case that $i = 4$, the other situations are similar.

Let

$$\mathcal{X} = \begin{bmatrix} 0 & r_2 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} 0 & r'_2 \\ 0 & 0 \end{bmatrix} \in X^{(4)},$$

where $r_2, r'_2 \in R \setminus \{0\}$. In this situation, there exists a non-zero arbitrary 2×2 matrix of the form

$$A = \begin{bmatrix} r_2^{-1}r'_2 & a_2 \\ 0 & a_4 \end{bmatrix},$$

such that $a_2, a_4 \in R$ and $A\mathcal{X} = \mathcal{Y}$. Then \mathcal{X} and \mathcal{Y} are adjacent and the subgraph E_{44} is complete.

(ii) We only give the proof for the case that $j = 3$, and the proof of other situations is obtained similarly. Let

$$\mathcal{X} = \begin{bmatrix} u_1 & r_2 \\ 0 & u_3 \end{bmatrix} \in X^{(1)}, \quad \mathcal{Y} = \begin{bmatrix} 0 & r'_2 \\ 0 & u'_3 \end{bmatrix} \in X^{(3)},$$

where $u_1, u_3, u'_3 \in U$ and $r_2, r'_2 \in R$. \mathcal{X} and \mathcal{Y} are adjacent, since there exists a 2×2 matrix $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ such that $a_1u_1 = 0, a_2u_3 = r'_2, a_3u_1 = 0$ and $a_4u_3 = u'_3$, by considering $a_1 = a_3 = 0, a_2 = u_3^{-1}r'_2$ and $a_4 = u_3^{-1}u'_3$. So the subgraph E_{13} is complete.

(iii) The proof of this part is similar to the proof of previous parts. □

Lemma 3.6. *Let R be a finite field with n elements. Then the subgraph E_{22} is isomorphic to the union of n complete graphs K_{n-1} .*

Proof. For each pair of distinct vertices in $X^{(2)}$, one of the following situations happens:

(i) Let $\mathcal{X} = \begin{bmatrix} u_1 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} u'_1 & 0 \\ 0 & 0 \end{bmatrix} \in X^{(2)}$, where $u_1, u'_1 \in U$. Then there exists a 2×2 matrix $A = \begin{bmatrix} u_1^{-1}u'_1 & a_2 \\ 0 & a_4 \end{bmatrix}$ such that $a_2, a_4 \in R$. Hence $A\mathcal{X} = \mathcal{Y}$, and in this case we have one clique with $n - 1$ vertices.

(ii) Let $\mathcal{X} = \begin{bmatrix} u & u \\ 0 & 0 \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} u' & u' \\ 0 & 0 \end{bmatrix} \in X^{(2)}$, where $u, u' \in U$. Then there exists a 2×2 matrix $A = \begin{bmatrix} u^{-1}u' & u^{-1}u' \\ 0 & a_4 \end{bmatrix}$ such that $a_4 \in R$. Hence $A\mathcal{X} = \mathcal{Y}$, and in this case we have one clique with $n - 1$ vertices.

(iii) Let $\mathcal{X} = \begin{bmatrix} u_1 & r_2 \\ 0 & 0 \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} u'_1 & r'_2 \\ 0 & 0 \end{bmatrix} \in X^{(2)}$, where $u_1, u'_1 \in U$ and $r_2, r'_2 \in R \setminus \{0\}$. Then there exists a non-zero 2×2 matrix $A = \begin{bmatrix} u_1^{-1}u'_1 & a_2 \\ 0 & a_4 \end{bmatrix}$ such that $a_2, a_4 \in R$. Hence $A\mathcal{X} = \mathcal{Y}$, if $r'_2 = u_1^{-1}u'_1r_2$. Since the number of vertices of this part is $(n - 2)(n - 1)$ and $X \cup N_{X^{(2)}}(X)$ forms the complete graph K_{n-1} , we have $(n - 2)$ cliques of size $(n - 1)$. Now, by considering the above cases the result holds. □

Remark 3.7. The subgraphs E_{23} and E_{24} are empty graphs.

In Figure 1, $\Gamma_{T_2(R)}^1$ is pictured by using the sets $X^{(i)}$'s, for a finite field R .

Example 3.8. Let $R = \mathbb{Z}_2$. Then the vertices of $\Gamma_{T_2(\mathbb{Z}_2)}^1$ are

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

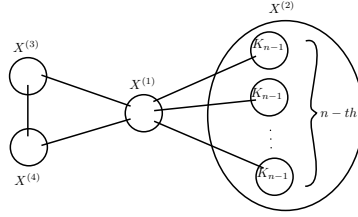


Figure 1

$$X_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad X_7 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and $\Gamma_{T_2(\mathbb{Z}_2)}^1$ is pictured in Figure 2.

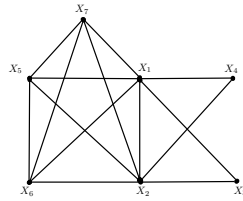


Figure 2

Lemma 3.9. *Let R be a finite field with $|R| = k$. Then the number of edges of $\Gamma_{T_2(R)}^1$ is equal to $\frac{k-1}{2}(k^5 + k^4 - 2k^3 - k - 2)$.*

Proof. By Lemma 3.5, the induced subgraph with vertex set $X^{(1)} \cup X^{(3)} \cup X^{(4)}$ forms the complete graph K_n with $\frac{[(k-1)(k^2+1)][(k-1)(k^2+1)-1]}{2}$ edges, where $n = (k-1)(k^2+1)$. Also, by Lemma 3.6, the number of edges of E_{22} is equal to $\frac{k(k-1)(k-2)}{2}$. Now, by Remark 3.7, it is enough to determine the number of edges of E_{12} . Since all vertices in $X^{(1)}$ are adjacent to all vertices in $X^{(2)}$, the number of edges of E_{12} is equal to $k^2(k-1)^3$. Therefore,

$$\begin{aligned} |E(\Gamma_{T_2(R)}^1)| &= \frac{[(k-1)(k^2+1)][(k-1)(k^2+1)-1]}{2} \\ &\quad + \frac{k(k-1)(k-2)}{2} + k^2(k-1)^3 \\ &= \left(\frac{k-1}{2}\right) [(k-1)(k^2+1)^2 - (k^2+1) + k(k-2) + 2k^2(k-1)^2] \\ &= \left(\frac{k-1}{2}\right)(k^5 + k^4 - 2k^3 - k - 2). \quad \square \end{aligned}$$

The following corollary follows from the proof of Lemma 3.9.

Corollary 3.10. *Let R be a finite field with $|R| = k$. Then the following statements hold.*

- (i) $\omega(\Gamma_{T_2(R)}^1) = (k-1)(k^2+1)$.
- (ii) *The vertex chromatic number of $\Gamma_{T_2(R)}^1$ is equal to $(k-1)(k^2+1)$.*

The following lemma is obtained from Figure 1 by using a simple counting.

Lemma 3.11. *If R be a finite field with $|R| = k$, then*

- (i) $\deg_{X^{(1)}}(a) = k^3 - 2$.
- (ii) $\deg_{X^{(2)}}(a) = (k-2) + k(k-1)^2$.
- (iii) $\deg_{X^{(3)}}(a) = (k-1)(k^2+1) - 1$.
- (iv) $\deg_{X^{(4)}}(a) = (k-1)(k^2+1) - 1$.

Lemma 3.12. *If R is a finite field with $|R| = k$, then the independence number of $\Gamma_{T_2(R)}^1$ is $k+1$.*

Proof. By Lemma 3.6, there are k vertices in $X^{(2)}$ that form an independent set, say I , in $\Gamma_{T_2(R)}^1$. Now, in view of Remark 3.7, we can add one vertex from $X^{(3)}$ or $X^{(4)}$ to I , say a , that $I \cup \{a\}$ is an independent set. So $\gamma(\Gamma_{T_2(R)}^1) \leq k+1$. In view of Figure 1, one can easily see that $\alpha(\Gamma_{T_2(R)}^1) = k+1$. \square

Lemma 3.13. *If R is a finite field with $|R| = k$, then the edge chromatic number of $\Gamma_{T_2(R)}^1$ is*

$$(k-1)(2k^2 - k + 1) - 1 \leq \chi'(\Gamma_{T_2(R)}^1) < (k-1)(2k^2 + 1) - 1.$$

Proof. Since $E_{ii} \cup E_{ij}$, for $i, j \in \{1, 3, 4\}$, forms a clique in $\Gamma_{T_2(R)}^1$, we can color these edges with $(k-1)(k^2+1)$ colors. Also we can color the edges of all cliques of $X^{(2)}$ with the colors that are used to color the edges of $X^{(4)}$. Since each vertex in $X^{(1)}$ is adjacent to all vertices in $X^{(2)}$, there are $k(k-1)^2$ edges between each vertex in $X^{(1)}$ and the vertices in $X^{(2)}$. Now, by considering the first vertex of $X^{(1)}$, we can color the edges between this vertex and the vertices of $X^{(2)}$ by using $k(k-1)^2$ colors. One can easily see that for coloring the edges between the next vertex of $X^{(1)}$ and the vertices of $X^{(2)}$, we can use the $k(k-1)^2$ previous colors with one new color. By continuing this method, we can color the edges of E_{12} with $k(k-1)^2 + k(k-1) - 1$ colors. Therefore $\chi'(\Gamma_{T_2(R)}^1) \leq (k-1)(2k^2+1) - 1$. Now, as we have mentioned before, we can color $E_{ii} \cup E_{ij}$, for $i, j \in \{1, 3, 4\}$ with $(k-1)(k^2+1)$ colors. Also, we can color $E_{12} \cup E_{22}$ with $k(k-1)^2 + k(k-1) - 1 + (k-1)$ colors. On the other hand, we can color $E_{33} \cup E_{44} \cup E_{34}$ with $k(k-1) + (k-1)$ colors. Since $E_{33} \cup E_{44} \cup E_{34}$ is disjoint from $E_{12} \cup E_{22}$, we can use the colors of the edges in $E_{33} \cup E_{44} \cup E_{34}$ for coloring the edges of $E_{12} \cup E_{22}$, if it is possible. Therefore we need at least $(k-1)(k^2+1) + k(k-1)^2 + k(k-1) + (k-2) - k(k-1) - (k-1)$ colors for coloring $\Gamma_{T_2(R)}^1$. Hence $\chi'(\Gamma_{T_2(R)}^1) \geq (k-1)(2k^2 - k + 1)$. \square

The *wiener index*, $W(G)$, is equal to the count of all shortest distances in a graph (cf. [16]). In other words, $W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v)$. The *hyper-wiener index*, is defined for all connected graphs, as a generalization of the wiener index. It is $WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d^2(u, v)$.

Wiener number was defined in 1947 by an American chemist H. Wiener. He used this index to estimate the boiling point of Alkans. There are many situations in communication, facility location, cryptology, architecture etc. where the Wiener index of the corresponding graph or the average distance is of great interest. One of these problems, for example, is to find a spanning tree with minimum average distance.

In the rest of the paper, we investigate the wiener index and hyper-wiener index of $\Gamma_{T_2(R)}^1$, where R is a finite field.

Theorem 3.14. *If R is a finite field with $|R| = k$, then*

$$W(\Gamma_{T_2(R)}^1) = \frac{k-1}{2}(k^5 + k^4 + 4k^3 - 4k^2 - 3k - 2).$$

Proof. Each vertex in $X^{(1)}$ is adjacent to all vertices in $\Gamma_{T_2(R)}^1$. So

$$\sum_{a \in X^{(1)}} \sum_{b \in V(\Gamma_{T_2(R)}^1)} d(a, b) = k(k-1)^2[(k^3 - 1) - 1] = k(k-1)^2(k^3 - 2).$$

Also,

$$\begin{aligned} & \sum_{a \in X^{(2)}} \sum_{b \in V(\Gamma_{T_2(R)}^1) \setminus X^{(2)}} d(a, b) \\ &= k(k-1) \cdot k(k-1)^2 + k(k-1) \cdot 2k(k-1) + k(k-1) \cdot 2(k-1) \\ &= k(k-1)^2(k^2 + k + 2), \end{aligned}$$

and,

$$\sum_{a \in X^{(2)}} \sum_{\substack{b \in X^{(2)} \\ a \neq b}} d(a, b) = k(k-1)[(k-2) + 2(k-1)(k-1)].$$

Hence,

$$\begin{aligned} & \sum_{a \in X^{(2)}} \sum_{b \in V(\Gamma_{T_2(R)}^1)} d(a, b) \\ &= k(k-1)^2(k^2 + k + 2) + k(k-1)[(k-2) + 2(k-1)^2] \\ &= k(k-1)(k^3 + 2k^2 - 2k - 2). \end{aligned}$$

Similarly,

$$\begin{aligned} & \sum_{a \in X^{(3)}} \sum_{b \in V(\Gamma_{T_2(R)}^1)} d(a, b) \\ &= k(k-1)[k(k-1)^2 + 2k(k-1) + (k(k-1) - 1) + (k-1)] \end{aligned}$$

$$= k(k - 1)(k^3 + k^2 - k - 2),$$

and

$$\begin{aligned} & \sum_{a \in X^{(4)}} \sum_{b \in V(\Gamma_{T_2(R)}^1)} d(a, b) \\ &= (k - 1)[k(k - 1)^2 + 2k(k - 1) + k(k - 1) + ((k - 1) - 1)] \\ &= (k - 1)(k^3 + k^2 - k - 2). \end{aligned}$$

Therefore

$$\begin{aligned} W(\Gamma_{T_2(R)}^1) &= \frac{1}{2}[k(k - 1)^2(k^3 - 2) + k(k - 1)(k^3 + 2k^2 - 2k - 2) \\ &\quad + k(k - 1)(k^3 + k^2 - k - 2) + (k - 1)(k^3 + k^2 - k - 2)] \\ &= \frac{k - 1}{2}(k^5 + k^4 + 4k^3 - 4k^2 - 3k - 2). \end{aligned} \quad \square$$

Theorem 3.15. *If R is a finite field with $|R| = k$, then*

$$\begin{aligned} WW(\Gamma_{T_2(R)}^1) &= \frac{1}{2} \left(\sum_{j=1}^{k-1} k(k - 2) - \frac{(k - 2)(k + 1)}{2} + 4(k - 1)^2(k - j) \right) \\ &\quad + \frac{1}{4}(2k^6 + 8k^4 - 15k^3 - 4k^2 + 3k + 6). \end{aligned}$$

Proof. For each $i = 1, 2, 3, 4$, if $a \in X^{(1)}$ and $b \in X^{(i)}$, then

$$\begin{aligned} \sum_{\{a,b\} \subseteq V(\Gamma_{T_2(R)}^1)} d^2(a, b) &= k(k - 1)^2(k^3 - 1) - (1 + 2 + 3 + \dots + k(k - 1)^2) \\ &= \frac{k(k - 1)^2}{2}(k^3 + 2k^2 - k - 3). \end{aligned}$$

The sum of the square of the distances from each vertex of the first clique of $X^{(2)}$ from other vertices $X^{(2)}$ is equal to $(k - 2) + 2^2(k - 1)(k - 1)$. So the sum of the square of the distance from vertices of the first clique of $X^{(2)}$ from other vertices $X^{(2)}$ is equal to

$$\begin{aligned} & \underbrace{[(k - 2) + 2^2(k - 1)^2] + [(k - 3) + 2^2(k - 1)^2] + \dots + [k - (k - 1)] + 2^2(k - 1)^2}_{(k-2)-th} \\ &+ 2^2(k - 1)(k - 1) \\ &= k(k - 2) - (2 + 3 + \dots + (k - 1)) + (k - 2) \times 2^2(k - 1)^2 + 2^2(k - 1)^2 \\ &= k(k - 2) - \left(\frac{(k - 1)(k - 1 + 1)}{2} - 1 \right) + 4(k - 1)(k - 1)^2 \\ &= k(k - 2) - \frac{(k - 2)(k + 1)}{2} + 4(k - 1)(k - 1)^2. \end{aligned}$$

Also, the sum of the square of the distance from each vertex of the second clique of $X^{(2)}$ from other vertices of $(k - 2)$ other cliques is equal to $(k - 2) +$

$2^2(k-1)(k-2)$. So the sum of the square of the distance from vertices of the second clique of $X^{(2)}$ except vertices of the first clique is equal to

$$\begin{aligned} & \underbrace{[(k-2) + 2^2(k-1)(k-2)] + [(k-3) + 2^2(k-1)(k-2)] + \cdots + [k - (k-1)] + 2^2(k-1)(k-2)}_{(k-2)-th} \\ & + 2^2(k-1)(k-2) \\ & = (k-2)k - (2 + 3 + \cdots + (k-1)) + (k-1) \cdot 2^2(k-1)(k-2) \\ & = k(k-2) - \left(\frac{(k-1)(k-1+1)}{2} - 1 \right) + 4(k-1)^2(k-2) \\ & = k(k-2) - \frac{(k-2)(k+1)}{2} + 4(k-1)^2(k-2). \end{aligned}$$

Also, the sum of the square of the distance from each vertex of $X^{(2)}$ from other vertices $X^{(2)}$ is equal to

$$\begin{aligned} & \left[\sum_{j=1}^{k-1} k(k-2) - \frac{(k-2)(k+1)}{2} + 4(k-1)^2(k-j) \right] \\ & + \left[(k-2) + (k-3) + \cdots + (k - (k-1)) \right] \\ & = \left[\sum_{j=1}^{k-1} k(k-2) - \frac{(k-2)(k+1)}{2} + 4(k-1)^2(k-j) \right] \\ & + \left[(k-2) \cdot k - \frac{(k-2)(k+1)}{2} \right]. \end{aligned}$$

Hence, for each $i = 2, 3, 4$, if $a \in X^{(2)}$ and $b \in X^{(i)}$,

$$\begin{aligned} & \sum_{\{a,b\} \subseteq V(\Gamma_{T_2(\mathbb{R})}^1)} d^2(a, b) \\ & = \left[\sum_{j=1}^{k-1} k(k-2) - \frac{(k-2)(k+1)}{2} + 4(k-1)^2(k-j) \right] \\ & + \left[(k-2) \cdot k - \frac{(k-2)(k+1)}{2} \right] + [k(k-1) \cdot 2^2k(k-1)] \\ & + [k(k-1) \cdot 2^2(k-1)] \\ & = \left[\sum_{j=1}^{k-1} k(k-2) - \frac{(k-2)(k+1)}{2} + 4(k-1)^2(k-j) \right] \\ & + (4k^4 - 4k^3 - \frac{7}{2}k^2 + \frac{5}{2}k + 1). \end{aligned}$$

Also, the subgraph $X^{(3)} \cup X^{(4)}$ is a complete graph. Therefore for each $i = 3, 4$, and $a, b \in X^{(i)}$,

$$\begin{aligned} \sum_{\{a,b\} \subseteq V(\Gamma_{T_2(R)}^1)} d^2(a,b) &= \underbrace{(k^2 - 2) + (k^2 - 3) + \cdots + [k^2 - (k^2 - 1)]}_{(k^2-2)-th} \\ &= (k^2 - 2) \cdot k^2 - (2 + 3 + \cdots + (k^2 - 1)) \\ &= k^2 \cdot (k^2 - 2) - \left[\frac{(k^2 - 1)(k^2 - 1 + 1)}{2} - 1 \right] \\ &= k^2(k^2 - 2) - \frac{k^2(k^2 - 1)}{2} + 1. \end{aligned}$$

Hence,

$$\begin{aligned} & WW(\Gamma_{T_2(R)}^1) \\ &= \frac{1}{2} W(\Gamma_{T_2(R)}^1) + \frac{1}{2} \sum_{\{a,b\} \subseteq V(\Gamma_{T_2(R)}^1)} d^2(a,b) \\ &= \frac{k-1}{4} (k^5 + k^4 + 4k^3 - 4k^2 - 3k - 2) + \frac{k(k-1)^2}{4} (k^3 + 2k^2 - k - 3) \\ &\quad + \frac{1}{2} \left[\sum_{j=1}^{k-1} k(k-2) - \frac{(k-2)(k+1)}{2} + 4(k-1)^2(k-j) \right] \\ &\quad + \frac{1}{2} (4k^4 - 4k^3 - \frac{7}{2}k^2 + \frac{5}{2}k + 1) \\ &\quad + \frac{1}{2} \left(k^2(k^2 - 2) - \frac{k^2(k^2 - 1)}{2} + 1 \right) \\ &= \frac{1}{2} \left[\sum_{j=1}^{k-1} k(k-2) - \frac{(k-2)(k+1)}{2} + 4(k-1)^2(k-j) \right] \\ &\quad + \frac{1}{4} (2k^6 + 8k^4 - 15k^3 - 4k^2 + 3k + 6). \quad \square \end{aligned}$$

Example 3.16. Let $R = \mathbb{Z}_2$, then $\chi(\Gamma_{T_2(\mathbb{Z}_2)}^1) = 5$, $\chi'(\Gamma_{T_2(\mathbb{Z}_2)}^1) = 6$, $W(\Gamma_{T_2(\mathbb{Z}_2)}^1) = 28$, and $WW(\Gamma_{T_2(\mathbb{Z}_2)}^1) = 35$.

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