Bull. Korean Math. Soc. **53** (2016), No. 4, pp. 1005–1015
http://dx.doi.org/10.4134/BKMS.b150402
pISSN: 1015-8634 / eISSN: 2234-3016

BOUNDED PARTIAL QUOTIENTS OF SOME CUBIC POWER SERIES WITH BINARY COEFFICIENTS

Khalil Ayadi, Salah Beldi, and Kwankyu Lee

Abstract. It is a surprising but now well-known fact that there exist algebraic power series of degree higher than two with partial quotients of bounded degrees in their continued fraction expansions, while there is no single algebraic real number known with bounded partial quotients. However, it seems that these special algebraic power series are quite rare and it is hard to determine their continued fraction expansions explicitly. To the short list of known examples, we add a new family of cubic power series with bounded partial quotients.

1. Introduction

Khinchin [4] conjectured that no algebraic real number of degree higher than two has bounded partial quotients in its continued fraction expansion. Then betraying the innocent expectation that the same would be true for algebraic power series, Baum and Sweet [1] found an example of cubic power series over $\mathbb{F}_2$ with partial quotients of bounded degree. Then it was realised, first by Mills and Robbins [11], that irrational power series with coefficients in a finite field $\mathbb{F}$ with characteristic p that are roots of equations of the form

$$x = \frac{Ax^{p^r} + B}{Cx^{p^r} + D}, \quad r \geq 0, \quad A, B, C, D \in \mathbb{F}[X]$$

loosely correspond to the quadratic real numbers, and tends to have regular patterns in their continued fraction expansions. These are now called hyperquadratic power series. Hyperquadratic power series are fixed points under the composition of a linear fractional transformation and a Frobenius map of $\mathbb{F}$.

It is known [2, 15] that if a hyperquadratic power series does not have bounded partial quotients, then it has partial quotients of rapidly increasing degrees. So it is either badly approximable or very well approximable by rational functions. Hence the cubic power series of Baum and Sweet belongs to the class of badly approximable hyperquadratic power series over the field $\mathbb{F}_2$.

Received May 27, 2015; Revised April 25, 2016.
2010 *Mathematics Subject Classification.* 11J61, 11J70.
Key words and phrases. power series, continued fraction, finite fields.
This work was supported by research fund from Chosun University, 2015.

Furthermore Mills and Robbins [11] found hyperquadratic power series only with partial quotients of degree one over $\mathbb{F}_p$ for any odd prime p. Following these pioneering works, much efforts have been put to classify hyperquadratic power series in terms of their approximableness by rational functions [5, 14].

Through several works [3, 6, 7, 8, 9], Lasjaunias and his coauthors have showed that there exist large families of hyperquadratic power series with all partial quotients of degree 1, defined for finite fields of any characteristic. For finite fields of characteristic 2, Thakur [14] constructed some non-quadratic power series of even degree with bounded partial quotients. However, the classification problem of hyperquadratic power series is far from complete and more examples are yet to be explored.

In this paper, we provide another interesting family of cubic power series of bounded partial quotients over $\mathbb{F}_2$, defined by equation (9) in Theorem 6. Although this is a big family of power series satisfying a cubic equation of a special form, we prove that they have bounded partial quotients using similar methods to those used by Baum and Sweet for their single cubic equation $Xx^3 + x + X = 0$. In this sense, our family of power series seems akin to the Baum-Sweet cubic. Another peculiarity of the family is that the power series satisfying a "degenerate" equation of (9) has actually unbounded partial quotients. See Corollary 8. Thus in this "compactified" family of cubic power series, we observe that the corner case of power series with bounded partial quotients is actually a power series with unbounded partial quotients.

In Section 2, we review basic facts and notations of the theory of continued fraction expansions of power series. Here we collect several results from Baum and Sweet [1], and Mkaouar [12], which we rely on in the following sections. In Section 3, we present our main results. Lemma 4 is an easy consequence of Lasjaunias' previous result in [5], recalled as Theorem 2 in this paper. We put most efforts to prove Theorem 5. Then our main result, Theorem 6, follows combining Lemma 4 and Theorem 5. In Section 4, we show that the algebraic power series deliberately excluded from Theorem 5 and Theorem 6 have actually unbounded partial quotients in their continued fraction expansions. In Section 5, we exhibit the continued fraction expansions of two simplest algebraic power series from Theorem 6.

2. Preliminaries

Let $\mathbb{F}$ be a finite field. Let $\mathbb{F}((X^{-1}))$ denote the field of formal power series over $\mathbb{F}$. For a nonzero power series

$$\alpha = \sum_{i \leq n_0} c_i X^i \in \mathbb{F}((X^{-1})), \quad n_0 \in \mathbb{Z}, \quad c_{n_0} \neq 0,$$

we define

$$\deg(\alpha) = n_0, \quad |\alpha| = |X|^{n_0}, \quad \lfloor \alpha \rfloor = \sum_{0 \leq i} c_i X^i,$$

where $|X|$ is a fixed real number greater than 1. Let $\deg(0) = -\infty$ and $|0| = 0$. Recall that $|\alpha|$ for power series α defines a non-Archimedean absolute value on $\mathbb{F}((X^{-1}))$ and $\lfloor\alpha\rfloor$ is called the polynomial part of α. Note that $\lfloor\alpha\rfloor$ is characterised as the unique polynomial $f \in \mathbb{F}[X]$ such that $|\alpha - f| < 1$.

The general theory of continued fractions for power series were expounded by Schmidt in [13]. Here we briefly review basic facts and establish some notations. The continued fraction expansion for power series α is defined as the unique expression

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \ldots],$$

where $a_n \in \mathbb{F}[X]$ for $n \geq 0$ and $\deg(a_n) > 0$ for $n > 0$. Let $P_{-1} = 1, Q_{-1} = 0$ and define for $n \geq 0$,

$$\begin{bmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix},$$

and nth complete quotient $\alpha_n = [a_n, a_{n+1}, a_{n+2}, \ldots]$. Then for $n \geq 0$, we have

$$\alpha = \frac{P_n \alpha_{n+1} + P_{n-1}}{Q_n \alpha_{n+1} + Q_{n-1}}. \tag{1}$$

The quotient P_n/Q_n is called the nth convergent of α, and is a rational approximation to α satisfying

$$\left| \alpha - \frac{P_n}{Q_n} \right| = \frac{1}{|a_{n+1}||Q_n|^2}.$$

Thus, if $\deg a_{n+1} = s$, the convergent P_n/Q_n is said to be of accuracy s. Baum and Sweet proved the following in [1].

Theorem 1. (a) *If* $|Q\alpha - P| = |X|^{-s}|Q|^{-1}$ *for* $s > 0$ *and* $\gcd(P, Q) = 1$*, then* $P = P_n$ *and* $Q = Q_n$ *for some* $n \geq 0$*. Hence* $s = \deg a_{n+1}$.

(b) *If* $|Q\alpha - P| = |Q|^{-1}$ *and* $\gcd(P, Q) = 1$*, then* $P = P_n + P_{n-1}$ *and* $Q = Q_n + Q_{n-1}$ *for some* $n \geq 0$.

In the same paper, Baum and Sweet proved that two power series α and β related by a linear fractional transformation

$$\alpha = \frac{A\beta + B}{C\beta + D}, \quad AD - BC \neq 0, \quad A, B, C, D \in \mathbb{F}[X]$$

have the same behaviour regarding the boundedness of their partial quotients. That is, α has bounded partial quotients if and only if so does β.

Our Lemma 4 in the following section is a consequence of the following result by Lasjaunias [5].

Theorem 2. *Let $\alpha \in \mathbb{F}_2((X^{-1}))$ be an irrational power series which is a root of the equation*

$$Xy^3 + Dy + X^l = 0 \tag{2}$$

with $l \geq 1$ and $D \in \mathbb{F}_2[X]$ such that $D(0) = 1$. Then α has bounded partial quotients if and only if $|\alpha| \geq |X|^{-(l+1)}$.

Lastly, we will rely on the following result by Mkaouar [12].

Theorem 3. *Let $P(x) = \sum_{0 \leq i \leq n} A_i x^i$ with $A_i \in \mathbb{F}[X]$ and $n \geq 1$. Suppose that $\deg A_i < \deg A_{n-1}$ for all $0 \leq i \leq n$ and $i \neq n-1$. Then there exists a unique power series α with positive degree satisfying $P(\alpha) = 0$. Moreover $\lfloor \alpha \rfloor = -\lfloor A_{n-1}/A_n \rfloor$.*

Indeed, all equations we deal with in the following sections satisfy Mkaouar's condition. Therefore it is guaranteed that each of these equations possess a unique power series of positive degree as a root. Moreover the first partial quotient is easily computed from the two coefficients of highest degrees.

3. Bounded partial quotients

Lemma 4. *Let $\alpha \in \mathbb{F}_2((X^{-1}))$ be the unique power series of positive degree that is a root of the equation*

$$Cx^3 + Ax^2 + 1 = 0, \tag{3}$$

with $l \geq 1$, $C = X^{2l+1}$, $A = D^2$, and $D \in \mathbb{F}_2[X]$ such that $l \leq \deg D \leq 2l+1$, $D(0) = 1$. Then α has bounded partial quotients.

Proof. We may assume that α is irrational. Let

$$\beta = \frac{X^l \alpha}{D\alpha + 1}.$$

Then we can check that β is a root of (2). Now by Theorem 2, β has bounded partial quotients, and so does α. $\square$

Theorem 5. *Let $\alpha \in \mathbb{F}_2((X^{-1}))$ be the unique power series of positive degree that is a root of the equation*

$$Cx^3 + Ax^2 + 1 = 0 \tag{4}$$

with $C = X^l$, $A = X^{2l} + \sum_{i=2}^{2l-1} e_i X^i + X + 1$ $(e_i \in \mathbb{F}_2)$. Suppose $l \geq 2$. Then α has bounded partial quotients.

Proof. First we see that $|\alpha| = |X|^l$ by Theorem 3. Then since α is a root of (4), we have

$$\alpha = \frac{A\alpha^2 + 1}{C\alpha^2} = \frac{A}{C} + \frac{1}{C\alpha^2}, \tag{5}$$

which shows that

$$|C\alpha - A| = \frac{1}{|C|^2}.$$

Thus the quotient A/C is a convergent to α of accuracy l. It is also clear that all convergents prior to A/C has accuracy smaller than l. Suppose $\alpha = [a_0, a_1, a_2, \dots]$ is the continued fraction expansion of α. Let $P = P_n$, $Q = Q_n$, $a(P,Q) = a_{n+1}$. Then $\gcd(P,Q) = 1$ and

$$|Q\alpha - P| = \frac{1}{|a(P,Q)||Q|}. \tag{6}$$

On the other hand, by (5),

$$|Q\alpha - P| = \left|\frac{Q(A\alpha^2+1)}{C\alpha^2} - P\right| = \frac{\left|(QA-PC)\alpha^2 + Q\right|}{|C|^3}$$

and hence

$$\left|(QA-PC)\alpha^2 + Q\right| = \frac{|C|^3}{|a(P,Q)||Q|}.$$

Let us assume that P/Q is a convergent that comes after A/C. Thus we have $|C||Q\alpha - P| < |Q||C\alpha - A|$, and

$$|QA - PC| = |QA - QC\alpha + QC\alpha - PC| = |Q||C\alpha - A| = \frac{|Q|}{|C|^2}.$$

Hence we can write

$$\left|(QA-PC)\alpha^2 + Q\right| = \frac{|C|}{|a(P,Q)||QA-PC|}.$$

Let $D = \gcd(Q, QA-PC) = \gcd(Q, PC) = \gcd(Q,C)$. Then let $Q = P'D$, $QA - PC = Q'D$. We now have $\gcd(P', Q') = 1$ and

$$\left|Q'\alpha^2 - P'\right| = \frac{|C|}{|a(P,Q)||D|^2|Q'|}. \tag{7}$$

Recall that $\alpha^2 = [a_0^2, a_1^2, \dots]$ by the Frobenius map of $\mathbb{F}_2$. So all convergents of α^2 are squares of polynomials in $\mathbb{F}_2[X]$. By Theorem 1, this implies that we have

$$|a(P,Q)| < \frac{|C|}{|D|^2} \tag{8}$$

unless P' and Q' are both squares. This fact is our essential tool in the following.

As D divides $C = X^l$, there are three cases: $D = X^i, X^{l-1}, X^l$ where $0 \le i \le l-2$. For each case, we will show that $|a(P,Q)| \le |X|^l$.

First case $D = X^i$, $0 \le i \le l-2$: Note that P' has constant term 1. We observe that if P' is a square, then

$$Q' = AP' - X^{l-i}P = (\cdots + X + 1)(\cdots + X^2 + 1) - X^{l-i}P = \cdots + X + 1$$

is not a square. As P' and Q' cannot be both squares, we have

$$|a(P,Q)| < \frac{|C|}{|D|^2}$$

and therefore $|a(P,Q)| \le |X|^{l-1}$.

Second case $D = X^{l-1}$: Again P' has constant term 1 and $Q' = AP' - XP$. If either P' or Q' is not a square, then we can conclude as in the first case. So suppose P' and Q' are both squares. Then $P' = RX^2 + 1$ for some $R \in \mathbb{F}[X]$ and $Q' = (\cdots + X + 1)(RX^2 + 1) - XP = (\cdots + X + 1) - XP$. Thus Q' must have constant term 1. Let $Q' = V^2$ and $P' = U^2$. Since $\gcd(Q', P') = 1$, we also have $\gcd(U, V) = 1$. From (7),

$$|V\alpha - U| = \frac{1}{\sqrt{|a(P,Q)||X^{l-2}|}|V|},$$

by which we see that U/V is a convergent to α with $|a(U,V)| = \sqrt{|a(P,Q)||X^{l-2}|}$. If U/V is a convergent prior to the convergent A/C, then we immediately have $|a(U,V)| \le |X|^{l-1}$, and

$$|a(P,Q)| \le \frac{|X|^{2l-2}}{|X^{l-2}|} = |X|^l.$$

Since V has constant term 1, it is not possible that U/V is equal to A/C. So we can assume that U/V is a convergent that comes after A/C. This means that we can apply the same argument that we applied to (6) now to the convergent U/V of α. Let $U' = V$ and $V' = AV + CU$. Since V has constant term 1, we have $\gcd(U', V') = 1$. Thus we get

$$\left|V'\alpha^2 - U'\right| = \frac{|C|}{|a(U,V)||V'|}.$$

If U' is a square, we can show that V' is not a square by a similar argument as in the first case. Hence U'/V' cannot be a convergent of α^2, which implies $|a(U,V)| < |C|$. So

$$|a(P,Q)| \le \frac{|X|^{2l-2}}{|X^{l-2}|} = |X|^l.$$

We now come to the third case $D = X^l$. We will show that assuming $|a(P,Q)| > |X|^l$ leads to a contradiction. So let us assume that P/Q is a convergent to α with $a(P,Q)$ of smallest degree greater than l. Then P' and Q' must be squares since otherwise we have (8). Let $Q' = V^2$, $P' = U^2$. We have $\gcd(U,V) = 1$ since $\gcd(P', Q') = 1$. Now from (7),

$$|V\alpha - U| = \frac{1}{\sqrt{|a(P,Q)||X^l|}|V|}.$$

Thus U/V is a convergent to α such that $|a(U,V)| = \sqrt{|a(P,Q)||X^l|}$. Then since $\deg a(P,Q) > l$, we have

$$|a(U,V)| > \sqrt{|X|^l |X^l|} = |X|^l,$$

that is $\deg a(U,V) > l$, and moreover

$$\deg a(U,V) = \frac{\deg a(P,Q) + l}{2} < \deg a(P,Q).$$

This contradicts our assumption that the convergent P/Q has $a(P,Q)$ of smallest degree greater than l. □

Theorem 6. *Let $\alpha \in \mathbb{F}_2((X^{-1}))$ be the unique power series of positive degree that is a root of the equation*

$$x^3 + (X^l + B)x^2 + x + B = 0, \tag{9}$$

where $l \geq 3$ and $B \in \mathbb{F}_2[X]$ is such that $\deg(B) < l$, $B(0) = 0$, and $\frac{B}{X}(0) = 1$. Then α has bounded partial quotients.

Proof. Let β be the power series defined by

$$\alpha = \frac{(X^l + B)\beta + 1}{\beta}.$$

Then we see that β is the unique power series of positive degree that is a root of

$$X^l y^3 + (X^{2l} + B^2 + 1)y^2 + 1 = 0.$$

Now suppose l is odd. Let $l = 2l' + 1$. Then the previous equation can be written as

$$X^{2l'+1}y^3 + D^2y^2 + 1 = 0$$

with $\deg(D) = 2l' + 1$, $D(0) = 1$. So by Theorem 4, β has bounded partial quotients, and so does α.

Suppose l is even. Let $l = 2l'$. Let $Y = X^2$. Then

$$Y^{l'}y^3 + (Y^{2l'} + \cdots + Y + 1)y^2 + 1 = 0$$

which has a unique power series root $\gamma(Y)$ of positive degree with bounded partial quotients by Theorem 5. Then clearly $\beta = \gamma(X^2)$ has also bounded partial quotients, and so does α. □

4. Unbounded partial quotients

Observe that in Theorem 5, we deliberately excluded the case when $l = 1$. The next result shows that this provision was necessary.

Theorem 7. *Let α be the unique power series over $\mathbb{F}_2$ of positive degree that is a root of (4) with $l = 1$, that is $C = X$, $A = X^2 + X + 1$. Then the continued fraction expansion of α is $[a_0, a_1, a_2, \dots]$ where $a_0 = X + 1$, and for all $k \geq 1$,*

$$a_{2k-1} = X^{2^k - 1}, \quad a_{2k} = X.$$

In particular, the partial quotients are unbounded.

Proof. By Theorem 3 and direct calculations, we immediately see that $a_0 = X + 1$ and $a_1 = X$. Thus by (1), we have

$$\alpha = \frac{(X^2 + X + 1)\alpha_2 + X + 1}{X\alpha_2 + 1}. \tag{10}$$

On the other hand, by the Frobenius map,

$$\alpha^2 = \left(X + 1 + \frac{1}{\alpha_1}\right)^2 = X^2 + 1 + \frac{1}{\alpha_1^2} \tag{11}$$

and also by the equation (4) of which α is a root,

$$\alpha^2 = \frac{1}{X\alpha + X^2 + X + 1}. \tag{12}$$

Combining (10), (11) and (12), we get

$$\alpha_2 = X + \frac{1}{X\alpha_1^2},$$

which implies $a_2 = X$ and $\alpha_3 = X\alpha_1^2$. Since $\alpha_1 = X + \frac{1}{\alpha_2}$, we have

$$\alpha_3 = X\left(X^2 + \frac{1}{\alpha_2^2}\right) = X^3 + \frac{X}{\alpha_2^2},$$

which implies $a_3 = X^3$ and $\alpha_4 = X^{-1}\alpha_2^2$. We now claim that for all $k \geq 1$,

$$a_{2k} = X,\ a_{2k+1} = X^{2^{k+1}-1},\ \alpha_{2k+1} = X\alpha_{2k-1}^2,\ \alpha_{2k+2} = X^{-1}\alpha_{2k}^2. \tag{13}$$

Clearly (13) is true for $k = 1$. So we assume (13) for $k = l \geq 1$. Then

$$\alpha_{2l+2} = X^{-1}\left(X^2 + \frac{1}{\alpha_{2l+1}^2}\right) = X + \frac{1}{X\alpha_{2l+1}^2},$$

which implies $a_{2l+2} = X$ and $\alpha_{2l+3} = X\alpha_{2l+1}^2$. Then

$$\alpha_{2l+3} = X\left((X^{2^{l+1}-1})^2 + \frac{1}{\alpha_{2l+2}^2}\right) = X^{2^{l+2}-1} + \frac{X}{\alpha_{2l+2}^2},$$

which implies $a_{2l+3} = X^{2^{l+2}-1}$ and $\alpha_{2l+4} = X^{-1}\alpha_{2l+2}^2$. Thus (13) is also true for $k = l + 1$. By induction, we see that (13) holds for all $k \geq 1$. □

We now consider the corner case excluded in Theorem 6.

Corollary 8. *Let α be the unique power series over $\mathbb{F}_2$ of positive degree that is a root of* (9) *with $l = 2$, namely*

$$x^3 + (X^2 + X)x^2 + x + X = 0.$$

Then the partial quotients of α are unbounded.

Proof. Let β be the power series defined by $\alpha = X^2 + X + \frac{1}{\beta}$. Then β is the unique power series of positive degree that is a root of

$$X^2y^3 + (X^4 + X^2 + 1)y^2 + 1 = 0.$$

So $\beta = \gamma(X^2)$ if $\gamma(Y)$ is the unique power series root of positive degree of the equation

$$Yy^3 + (Y^2 + Y + 1)y^2 + 1 = 0.$$

By Theorem 7, we know that $\gamma(Y)$ has unbounded partial quotients. Hence α has also unbounded partial quotients. □

5. Examples

In [10], an algorithm for computing the continue fraction expansion of a linear fractional transformation of a power series of known continued fraction expansion was presented. We now briefly recall how the algorithm can be used to determine the continued fraction expansions of hyperquadratic power series.

If a power series $\alpha = [a_0, a_1, a_2, \dots]$ over $\mathbb{F}_2$ satisfies

$$\alpha = \frac{A\alpha^2 + B}{C\alpha^2 + D}, \quad A, B, C, D \in \mathbb{F}_2[X],$$

then we express the above relation by formally writing

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} a_0^2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1^2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2^2 & 1 \\ 1 & 0 \end{bmatrix} \cdots .$$

Applied to the right-side of the above equation with the known a_0, the algorithm repeatedly transforms it by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a_n^2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_m & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{m+1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{m+r} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \tag{14}$$

revealing progressively all partial quotients $a_1, a_2, \dots$.

We applied the algorithm to the unique power series root α of positive degree of the equation

$$x^3 + (X^3 + X)x^2 + x + X = 0 \quad \text{or} \quad x = \frac{(X^3 + X)x^2 + X}{x^2 + 1}. \tag{15}$$

Note that this is the simplest case of (9) when $l = 3$ and $B = X$. The finite-state machine in Figure 1 summarizes the behaviour of the algorithm. In the figure, we express (14) as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \xrightarrow{a_n | a_m, a_{m+1}, \dots, a_{m+r}} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$$

and also note that initially

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} X^3 + X & X \\ 1 & 1 \end{bmatrix}^*, \quad a_0 = X^3 + X$$

by (15) and Theorem 3. Now we can read the continued fraction expansion of α from the finite-state machine as follows

$$\alpha = [X^3 + X, X^3, X; X, X, X^3, X; X; X^3; X; X^3 + X, X^3; X^3 + X; X; \dots].$$

Our second example is the unique power series root β of positive degree of the equation (9) when $l = 4$ and $B = X$, that is,

$$x^3 + (X^4 + X)x^2 + x + X = 0 \quad \text{or} \quad x = \frac{(X^4 + X)x^2 + X}{x^2 + 1}.$$

The behaviour of the algorithm applied to β is shown in Figure 2. It is interesting that the continued fraction expansion of β are much more complicated than that of α, though they are roots of very similar equations.

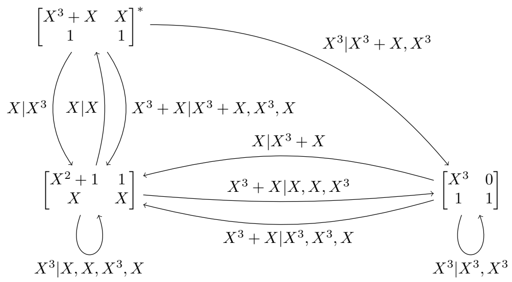

FIGURE 1. Finite-state machine generating partial quotients of α

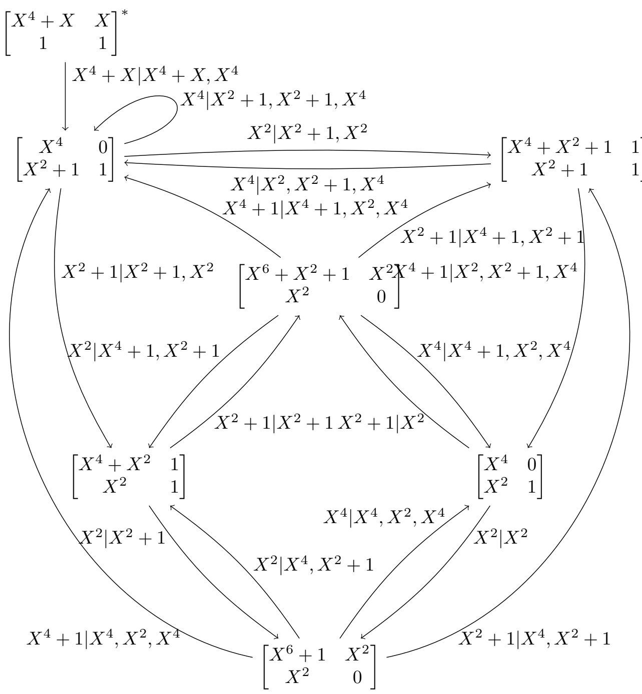

FIGURE 2. Finite-state machine generating partial quotients of β

Finally we appreciate the anonymous reviewer's suggestions that helped improve the exposition.

References

[1] L. E. Baum and M. M. Sweet, *Continued fractions of algebraic power series in characteristic* 2, Ann. of Math. (2) **103** (1976), no. 3, 593–610.

[2] B. de Mathan, *Approximation exponents for algebraic functions in positive characteristic*, Acta Arith. **60** (1992), no. 4, 359–370.

[3] D. Gómez-Pérez and A. Lasjaunias, *Hyperquadratic power series in* $\mathbb{F}_3((T^{-1}))$ *with partial quotients of degree* 1, Ramanujan J. **33** (2014), no. 2, 219–226.

[4] A. I. Khinchin, *Continued Fractions*, University of Chicago Press, 1964.

[5] A. Lasjaunias, *Continued fractions for algebraic formal power series over a finite base field*, Finite Fields Appl. **5** (1999), no. 1, 46–56.

[6] ______, *Quartic power series in* $\mathbb{F}_3((T^{-1}))$ *with bounded partial quotients*, Acta Arith. **95** (2000), no. 1, 49–59.

[7] A. Lasjaunias and J.-J. Ruch, *Algebraic and badly approximable power series over a finite field*, Finite Fields Appl. **8** (2002), no. 1, 91–107.

[8] ______, *Flat power series over a finite field*, J. Number Theory **95** (2002), no. 2, 268–288.

[9] A. Lasjaunias and J.-Y. Yao, *Hyperquadratic continued fractions in odd characteristic with partial quotients of degree one*, J. Number Theory **149** (2015), 259–284.

[10] K. Lee, *Continued fractions for linear fractional transformations of power series*, Finite Fields Appl. **11** (2005), no. 1, 45–55.

[11] W. H. Mills and D. P. Robbins, *Continued fractions for certain algebraic power series*, J. Number Theory **23** (1986), no. 3, 388–404.

[12] M. Mkaouar, *Sur les fractions continues des series formelles quadratiques sur* $f_q(x)$, Acta Arith. **97** (2001), no. 3, 241–251.

[13] W. M. Schmidt, *On continued fractions and diophantine approximation in power series fields*, Acta Arith. **95** (2000), no. 2, 139–166.

[14] D. S. Thakur, *Diophantine approximation exponents and continued fractions for algebraic power series*, J. Number Theory **79** (1999), no. 2, 284–291.

[15] J. F. Voloch, *Diophantine approximation in positive characteristic*, Period. Math. Hungar. **19** (1988), no. 3, 217–225.

Khalil Ayadi
Department of Mathematics
Faculty of Sciences
University of Sfax
Tunisia
E-mail address: ayedikhalil@yahoo.fr

Salah Beldi
Department of Mathematics
Faculty of Sciences
University of Sfax
Tunisia
E-mail address: beldi.salah@gmail.com

Kwankyu Lee
Department of Mathematics Education
Chosun University
Gwangju 61452, Korea
E-mail address: kwankyu@chosun.ac.kr