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STABILITY OF $MAP/PH/c/K$ QUEUE WITH CUSTOMER RETRIALS AND SERVER VACATIONS

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Abstract. We consider the $MAP/PH/c/K$ queue in which blocked customers retry to get service and servers may take vacations. The time interval between retrials and vacation times are of phase type (PH) distributions. Using the method of mean drift, a sufficient condition of ergodicity is provided. A condition for the system to be unstable is also given by the stochastic comparison method.

1. Introduction

Consider the queueing system that consists of a virtual space, called orbit of infinite size and a service facility with multiple servers and a waiting space of finite capacity. If an arriving customer finds an available space in service facility, the customer enters the service facility. Otherwise, the customer joins orbit and repeats its request after random amount of time, called retrial time until the customer gets into the service facility. The servers are allowed to take a vacation after service. The queueing system with retrials is called retrial queue and the system in which the servers may take vacation is called vacation queue.

Retrial queues and vacation queues have been studied separately for last several decades. The readers can refer the monographs [2, 8] for retrial queues and the monographs [21, 22] for vacation queues. The literature for the retrial queues with vacations is rapidly increasing. Many papers have been devoted to performance analysis of the system in steady state e.g. [1, 6, 11, 14] for the system with constant retrial rate in which only one customer in orbit can retry and see [5] for linear retrial policy.

The first step to be showed in the steady state performance analysis is the stability condition of the system under which the Markov process describing

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the system behavior is ergodic, that is, it is irreducible and positive recurrent. This paper concerns the stability of the multi-server finite buffer queueing system with retrials and vacations. Many papers provide stability conditions of the multi-server retrial queues with the help of the criteria in [23] based on the mean drifts e.g. see [7, 8, 10]. He et al. [10] derive a sufficient condition for the $BMAP/PH/c/K$ retrial queue with PH-retrial time to be positive recurrent using mean drifts and a necessary condition is given by the sample path approach. Breuer et al. [4] present a stability condition for $BMAP/PH/c$ retrial queue by using the matrix generating function of embedded Markov chain at the epochs of transitions. Morozov [17] provides a stability condition for $GI/G/c$ retrial queue with exponential retrial time by applying the renewal technique. Kim [12] derives a necessary and sufficient condition for a retrial queueing network with different classes of customers and several servers by using fluid limit technique. See Kim and Kim [13] for more details about stability of retrial queues.

In this paper, a sufficient condition of ergodicity for the $MAP/PH/c/K$ queues with retrials and vacations in which the retrial times and vacation times are of phase type (PH) distributions is given. The model considered in this paper is very general and contains many models arising in literature of retrial queues and vacation queues as special cases. The mean drift method is used for sufficient condition and stochastic comparison approach is used to show the condition under which the system is unstable. The model is described in Section 2. The stability condition for the system without specifying the vacation policy is given in Section 3. Section 4 is devoted to the stability of the systems with specific vacation policies. Nonstability is treated in Section 5.

2. The model

Consider the $MAP/PH/c/K$ queue which consists of an orbit with infinite capacity and a service facility with c identical servers in parallel and $K-c$ waiting positions. Customers arrive from outside according to a Markovian arrival process (MAP) with representation (D_0, D_1). The matrices D_0 and D_1 are square matrices of size l with negative diagonal elements $[D_0]_{ii} < 0$, nonnegative off diagonal elements $[D_0]_{ij} \geq 0$, $i \neq j$ and $D_1 \geq 0$ satisfying $(D_0 + D_1)\boldsymbol{e} = 0$, where $\boldsymbol{e}$ is the column vector of appropriate size whose components are all one. For detailed description of MAP, see [15, 16]. Let $\boldsymbol{\pi}_a$ be the stationary distribution of $D = D_0 + D_1$ and denote the arrival rate by $\lambda = \boldsymbol{\pi}_a D_1 \boldsymbol{e}$. If the number of customers in service facility is less than K upon arrival, the arriving customer enters the service facility and leaves the system after service. On the other hand, if an arriving customer finds that there are K customers in service facility, the customer joins orbit and repeats its request after random amount of time until the customer gets into the service facility. The customers in orbit retry independently and the retrial times of each customer are independent. We assume that the retrial time distribution of a customer in orbit is of phase type $PH(\boldsymbol{\theta},\boldsymbol{U})$, where $\boldsymbol{\theta} = (\theta_1,\dots,\theta_g)$ with $\boldsymbol{\theta e} = 1$ and $\boldsymbol{U} = (u_{ij})$ is a nonsingular $g \times g$ matrix with $u_{ii} = -u_i < 0$, $1 \le i \le g$. Let $\boldsymbol{u} = (u_1,\dots,u_g)$, $\boldsymbol{\gamma} = -\boldsymbol{Ue} = (\gamma_1,\dots,\gamma_g)^T$, where $\boldsymbol{x}^T$ is the transpose of the vector $\boldsymbol{x}$. It can be seen that $\boldsymbol{\pi}_r = \frac{1}{m_r}\boldsymbol{\theta}(-U)^{-1}$ is a stationary distribution of $U^* = U + \boldsymbol{\gamma\theta}$, where $m_r = \boldsymbol{\theta}(-\boldsymbol{U})^{-1}\boldsymbol{e}$ is the mean retrial time. For detailed description of the PH-distribution and PH-renewal process, see [18, Chapter 2]. The service time distribution of a customer is of phase type $PH(\boldsymbol{\beta},\boldsymbol{S})$, where $\boldsymbol{\beta} = (\beta_1,\dots,\beta_m)$ with $\boldsymbol{\beta e} = 1$ and $\boldsymbol{S} = (s_{ij})$ is a nonsingular $m \times m$ matrix with $s_{ii} = -s_i < 0$, $1 \le i \le m$ and $\boldsymbol{s} = (s_1,\dots,s_m)$. Let $\boldsymbol{\pi}_s$ be the stationary distribution of $S^* = \boldsymbol{S} + \boldsymbol{S}^0\boldsymbol{\beta}$, where $\boldsymbol{S}^0 = -\boldsymbol{Se}$ and denote the service rate by $\mu = \boldsymbol{\pi}_s\boldsymbol{S}^0$. Here, we assume that the servers may take vacations. The specific vacation rules will be described in Section 4.

For later use, we introduce some notation for vectors and matrices. Define $|\boldsymbol{x}| = \sum_{i=1}^n x_i$ and $\boldsymbol{x}\cdot\boldsymbol{y} = \sum_{i=1}^n x_i y_i$ and $\boldsymbol{x} \ge 0$ means that $x_i \ge 0$, $1 \le i \le n$ for vectors $\boldsymbol{x} = (x_1,\dots,x_n)$ and $\boldsymbol{y} = (y_1,\dots,y_n)$ and let $\mathbb{Z}_+ = \{0,1,2,\dots\}$ and denote by I_n the identity matrix of size n. The Kronecker product$A \otimes B$ and Kronecker sum $A \oplus B$ of two square matrices $A = (a_{ij})$ of size m and $B = (b_{ij})$ of size n are defined by

$$A \otimes B = (a_{ij}B),\ A \oplus B = A \otimes I_n + I_m \otimes B$$

and let $A^{\otimes k} = A \otimes A^{\otimes(k-1)}$, $A^{\oplus k} = A \oplus A^{\oplus(k-1)}$ with $(A^{\otimes 0}) \otimes B = B \otimes (A^{\otimes 0}) = B$. For more details for Kronecker product and Kronecker sum, see [9].

3. Preliminaries

Let $X_i(t)$ be the number of customers in orbit whose retrial phase is of i, $1 \le i \le g$ and $Y_0(t)$ be the number of customers in service facility at time t. Let $\boldsymbol{Z}(t)$ be the state of service facility that includes arrival state, the number of customers in service facility and the the phase of service times of busy servers, the server state (in vacation or in service facility) at time t. The state space of $\boldsymbol{Z}(t)$ may depend not only on the number of customers in the service facility but also the service disciplines such as vacation rule, the distribution of vacation time. Then $\boldsymbol{\Psi} = \{(\boldsymbol{X}(t), Y_0(t), \boldsymbol{Z}(t)), t \ge 0\}$ with $\boldsymbol{X}(t) = (X_1(t),\dots,X_g(t))$ is a continuous time Markov chain on the state space $\mathcal{S} = \cup_{\boldsymbol{n}\in\mathbb{Z}_+^g}\mathcal{S}(\boldsymbol{n})$, where $\mathcal{S}(\boldsymbol{n}) = \cup_{k=0}^K \mathcal{S}_k(\boldsymbol{n})$ with $\mathcal{S}_k(\boldsymbol{n}) = \{(\boldsymbol{n},k,j) : 1 \le j \le z_k\}$, $0 \le k \le K$. Denote the number of elements of $\mathcal{S}_k(\boldsymbol{n})$ by $|\mathcal{S}_k(\boldsymbol{n})| = z_k$ and let $M = \sum_{k=0}^K z_k$. The matrix $Q(\boldsymbol{n},\boldsymbol{n}')$ corresponding to the transition rates between levels $\mathcal{S}(\boldsymbol{n})$ and $\mathcal{S}(\boldsymbol{n}')$ is given by

$$Q(\boldsymbol{n},\boldsymbol{n}') = \begin{cases} \theta_i Q_0, & \boldsymbol{n}' = \boldsymbol{n} + \boldsymbol{e}_i,\ 1 \le i \le g, \\ Q_1 - (\boldsymbol{n}\cdot\boldsymbol{u})I_M + \left(\sum_{i=1}^g n_i\gamma_i\theta_i\right) I_*, & \boldsymbol{n}' = \boldsymbol{n}, \\ n_i u_{ij} I_M + n_i\gamma_i\theta_j I_*, & \boldsymbol{n}' = \boldsymbol{n} - \boldsymbol{e}_i + \boldsymbol{e}_j,\ i \ne j, \\ n_i\gamma_i Q_2, & \boldsymbol{n}' = \boldsymbol{n} - \boldsymbol{e}_i,\ 1 \le i \le g, \\ 0, & \text{otherwise}, \end{cases}$$

where $\boldsymbol{e}_i$ is the vector whose ith component is 1 and others are all zero and

$$I_* = \begin{pmatrix} O & \\ & I_{z_K} \end{pmatrix}.$$

The $M \times M$ matrices Q_0 and Q_2 are for an arrival to orbit and a departure from orbit, respectively and Q_i, $i = 0, 1, 2$ are of the form

$$Q_1 = \begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & \\ & \ddots & \ddots & \ddots & \\ & & A_2^{(K-1)} & A_1^{(K-1)} & A_0^{(K-1)} \\ & & & A_2^{(K)} & A_1^{(K)} \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} O & & & \\ & \ddots & & \\ & & O & \\ & & & A_0^{(K)} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} O & H^{(0)} & & \\ & \ddots & \ddots & \\ & & O & H^{(K-1)} \\ & & & O \end{pmatrix},$$

where $H^{(k)}\boldsymbol{e} = \boldsymbol{e}$, $k = 0, 1, \ldots, K-1$ and the block matrices $A_i^{(k)}$ corresponding to the transitions in service facility when there are k customers in service facility are determined by the specific vacation and $(Q_0 + Q_1)\boldsymbol{e} = 0$.

Proposition 3.1. *Let*

$$A(t) = A_0^{(K)} + tA_1^{(K)} + t^2 A_2^{(K)} H^{(K-1)}, \ 0 \le t \le 1.$$

If there exist a constant $0 < t_0 < 1$ and a positive vector $\boldsymbol{x}_K$ such that $A(t_0)\boldsymbol{x}_K < 0$, then the Markov chain $\boldsymbol{\Psi}$ is positive recurrent.

Proof. See Appendix. □

Theorem 3.2. *Assume that $\boldsymbol{\Psi}$ is irreducible. Let R be the minimal nonnegative solution of the equation*

$$A_0^{(K)} + RA_1^{(K)} + R^2 A_2^{(K)} H^{(K-1)} = 0.$$

Let $\boldsymbol{\pi}$ be the stationary distribution of $A(1)$ and $\xi = sp(R)$ be the spectral radius of R. If $\boldsymbol{\pi} A_0^{(K)}\boldsymbol{e} < \boldsymbol{\pi} A_2^{(K)} H^{(K-1)}\boldsymbol{e}$ or equivalently $\xi < 1$, then $\boldsymbol{\Psi}$ is ergodic.

Proof. It follows from [18, Theorem 3.1.1] that $\xi = sp(R) < 1$ if and only if $\boldsymbol{\pi} A_0^{(K)}\boldsymbol{e} < \boldsymbol{\pi}(A_2^{(K)} H^{(K-1)})\boldsymbol{e}$. Let τ be the maximal elements of the diagonal entries of the matrix $-A_1^{(K)}$ and

$$B_0 = \frac{1}{\tau} A_0^{(K)}, \ B_1 = \frac{1}{\tau} A_1^{(K)} + I, \ B_2 = \frac{1}{\tau} A_2^{(K)} H^{(K-1)},$$

where I is the identity matrix of size z_K. Applying the arguments in the proof of Lemma 1.3.4 in [18] to the matrix $B(t) = B_0 + tB_1 + t^2 B_2$, it can be seen that if $\xi < 1$, then $\chi(t) = sp(B(t)) < t$, $\xi < t < 1$. Thus for each t with $\xi < t < 1$, there exist a positive vector $\boldsymbol{y}(t)$ such that

$$\left(\frac{1}{\tau}A(t) + tI\right)\boldsymbol{y}(t) = B(t)\boldsymbol{y}(t) = \chi(t)\boldsymbol{y}(t),$$

and hence

$$A(t)\boldsymbol{y}(t) = -\tau(t - \chi(t))\boldsymbol{y}(t) < 0, \ \xi < t < 1.$$

The theorem is immediate from Proposition 3.1. □

4. Sufficient condition

Now we consider the $MAP/PH/c/K$ retrial queue with server vacations. Vacation policies are determined by vacation startup rules and vacation termination rules. Readers can refer the monograph [22] for various vacation policies in multi-server system. Here, we consider the two vacation startup rules (a, b)-vacation policy introduced by [24] and asynchronous vacation. Under the (a, b)-vacation rule, if any a $(1 \le a < c)$ or more servers are idle at a service completion, that is, the number of customers at the service facility is less than or equal to $a^* = c - a$ upon a service completion, then b $(b \le a)$ servers among idle servers take a vacation at the same time and the remaining $b^* = c - b$ servers are available. The maximum number of servers on vacation is b even though there are idle servers more than $2b$. The servers in vacation return at the same time when the vacation period ends. The vacation time distribution is assumed to be of phase type $PH(\boldsymbol{\delta}, V)$, where $\boldsymbol{\delta} = (\delta_1, \dots, \delta_w)$ with $\boldsymbol{\delta e} = 1$ and $\boldsymbol{V} = (v_{ij})$ is a nonsingular $w \times w$ matrix with $v_{ii} = -v_i < 0$, $1 \le i \le w$. Let $\boldsymbol{V}^0 = -\boldsymbol{Ve} = (v_1^0, \dots, v_w^0)^T$ and $\boldsymbol{\pi}_v$ be the stationary distribution of $\boldsymbol{V}^* = \boldsymbol{V} + \boldsymbol{V}^0\boldsymbol{\delta}$, $\nu^* = \boldsymbol{\pi}_v\boldsymbol{V}^0$ the vacation rate.

Under the asynchronous vacation policy, a server in service facility starts a vacation independently if the server finds no waiting customer in the system at his or her service completion instant. We consider two kinds of vacation termination rules, single vacation policy and multiple vacation policy. Under the single vacation policy, the servers take only one vacation and after the vacation the servers either serve the waiting customer in service facility if any or stay idle. A multiple vacation policy requires the servers to keep taking vacation until they find the system that is not in the vacation startup condition at the end of each vacation.

1. $MAP/PH/c/K$ retrial queue with vacation under (a, b)-vacation start up rule and single vacation policy. Let $J_a(t)$ be the phase of arrival process and the server state $J_s(t)$ at time t is defined by

$$J_s(t) = \begin{cases} 0, & c \text{ servers are available} \\ j, & \text{the phase of vacation time is of } j,\ 1 \le j \le w. \end{cases}$$

Let $Y_i(t)$ be the phase of service time of the ith working server at time t and $\boldsymbol{X}(t) = (X_1(t), \dots, X_g(t))$. Note that the stability condition given in

Proposition 3.1 depends only on the matrices $A_0^{(K)}$, $A_1^{(K)}$, $A_2^{(K)}$ and $H^{(K-1)}$. So, we consider these matrices corresponding to the Markov chain $\mathbf{\Psi}$ for two cases (i) $K > c$ or $a > 1$ and (ii) $K = c$, $a = b = 1$ separately.

Case (i): *$K > c$ or $a > 1$.* Let $b_0(t) = \min(Y_0(t), c)$ be the number of busy servers when no servers are in vacation and $b_1(t) = \min(Y_0(t), b^*)$ the number of busy servers when b servers are in vacation and $\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_{b_j(t)}(t))$, where $j = 0$ for $J(t) = 0$ and $j = 1$ for $J(t) > 0$. Let

$$J_I(t) = \begin{cases} 0, & \text{all the } c \text{ servers are busy,} \\ i, & \text{the server } i \text{ is idle and is not in vacation, } 1 \le i \le c. \end{cases}$$

Then $\mathbf{\Psi} = \{(\boldsymbol{X}(t), Y_0(t), Z(t)), t \ge 0\}$ with $Z(t) = (J_I(t), J_s(t), J_a(t), \boldsymbol{Y}(t))$ is a continuous time Markov chain. It can be easily seen that the block matrices $A_i^{(K)}$ are given as follows:

$$A_0^{(K)} = \begin{pmatrix} A_{K0}^V & 0 \\ 0 & A_{K0} \end{pmatrix},\ A_1^{(K)} = \begin{pmatrix} A_{K1}^V & \tilde{A}_{K1}^V \\ 0 & A_{K1} \end{pmatrix},$$
$$A_2^{(K)} = \begin{pmatrix} A_{K2}^V & 0 \\ 0 & A_{K2} \end{pmatrix},\ H^{(K-1)} = \begin{pmatrix} H_{K-1}^V & 0 \\ 0 & H_{K-1} \end{pmatrix},$$

where A_{Ki}^V, $i = 0, 1, 2$ are corresponding to the transitions rates in server vacations and A_{Ki}, $i = 0, 1, 2$ are to the transitions rates when no servers are in vacations and $\tilde{A}_{Ki}$ are to the transitions rates when server states are changed from no vacations to vacation. The block matrices of $A_i^{(K)}$ and $H^{(K-1)}$ are given as follows: for $K > c$,

$$\begin{aligned} A_{K0}^V &= I_w \otimes D_1 \otimes I_m^{\otimes b^*}, & A_{K0} &= D_1 \otimes I_m^{\otimes c}, \\ A_{K1}^V &= V \oplus D_0 \oplus S^{\oplus b^*}, & A_{K1} &= D_0 \oplus \boldsymbol{S}^{\oplus c}, \\ \tilde{A}_{K1}^V &= \boldsymbol{V}^0 \otimes I_l \otimes I_m^{\otimes b^*} \otimes \boldsymbol{\beta}^{\otimes(c-b^*)}, & & \\ A_{K2}^V &= I_w \otimes I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus b^*}, & A_{K2} &= I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c}, \\ H_{K-1}^V &= I_w \otimes I_l \otimes I_m^{\otimes b^*}, & H_{K-1} &= I_l \otimes I_m^{\otimes c}, \end{aligned}$$

and for $K = c$ and $a > 1$,

$$A_{c2} = (I_w \otimes I_l \otimes \boldsymbol{S}^0 \otimes I_m^{\otimes c-1}, I_w \otimes I_l \otimes I_m \otimes \boldsymbol{S}^0 \otimes I_m^{\otimes c-2}, \ldots, I_w \otimes I_l \otimes I_m^{\otimes c-1} \otimes \boldsymbol{S}^0)$$

and

$$H_{c-1} = \begin{pmatrix} I_l \otimes \boldsymbol{\beta} \otimes I_m^{\otimes c-1} \\ I_l \otimes I_m \otimes \boldsymbol{\beta} \otimes I_m^{\otimes c-2} \\ \vdots \\ I_l \otimes I_m^{\otimes c-1} \otimes \boldsymbol{\beta} \end{pmatrix}$$

and the remaining block matrices are the same as those for the cases of $K > c$. Note that

$$A_2^{(c)} H^{(c-1)} = \begin{pmatrix} I_w \otimes I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus b^*} & 0 \\ 0 & I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c} \end{pmatrix}$$

which is the same as $A_2^{(K)} H^{(K-1)}$ for $K > c$.

Case (ii): $K = c$ *and* $a = b = 1$. Denote by $J_V(t)$ the server in vacation at time t and let $\boldsymbol{Y}^*(t) = (Y_1(t), \ldots, Y_{i-1}(t), Y_{i+1}(t), \ldots, Y_c(t))$ when $J_V(t) = i$ and $\boldsymbol{Y}^*(t) = (Y_1(t), \ldots, Y_c(t))$ when there are no servers in vacation. Then $\boldsymbol{\Psi}^* = \{(\boldsymbol{X}(t), Y_0(t), Z^*(t)), t \geq 0\}$ with $Z(t) = (J_V(t), J_s(t), J_a(t), \boldsymbol{Y}^*(t))$ is a continuous time Markov chain. It can be easily seen that the block matrices $A_i^{(c)}$ are of the form as follows:

$$A_0^{(c)} = \begin{pmatrix} A_{c0}^V & 0 \\ 0 & A_{c0} \end{pmatrix}, \; A_1^{(c)} = \begin{pmatrix} A_{c1}^V & \tilde{A}_{c1}^V \\ 0 & A_{c1} \end{pmatrix},$$
$$A_2^{(c)} = \begin{pmatrix} A_{c2}^V & 0 \\ \tilde{A}_{c2} & 0 \end{pmatrix}, \; H^{(c-1)} = \begin{pmatrix} H_{c-1}^V & 0 \\ 0 & H_{c-1} \end{pmatrix}.$$

The block matrix components for $A_i^{(c)}$ are given as follows

$$\begin{aligned} A_{c0}^V &= I_c \otimes I_w \otimes D_1 \otimes I_m^{\otimes c-1}, & A_{c0} &= D_1 \otimes I_m^{\otimes c}, \\ A_{c1}^V &= I_c \otimes (V \oplus D_0 \oplus S^{\oplus c-1}), & A_{c1} &= D_0 \oplus S^{\oplus c}, \\ A_{c2}^V &= I_c \otimes I_w \otimes I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c-1} \end{aligned}$$

and

$$\tilde{A}_{c1}^V = \begin{pmatrix} V_0 \otimes I_l \otimes \boldsymbol{\beta} \otimes I_m^{\otimes c-1} \\ V_0 \otimes I_l \otimes I_m \otimes \boldsymbol{\beta} \otimes I_m^{\otimes c-2} \\ \vdots \\ V_0 \otimes I_l \otimes I_m^{\otimes c-1} \otimes \boldsymbol{\beta} \end{pmatrix},$$

$$\tilde{A}_{c2} = (\boldsymbol{\delta} \otimes I_l \otimes \boldsymbol{S}^0 \otimes I_m^{\otimes c-1}, \boldsymbol{\delta} \otimes I_l \otimes I_m \otimes \boldsymbol{S}^0 \otimes I_m^{\otimes c-2}, \ldots, \boldsymbol{\delta} \otimes I_m^{\otimes c-1} \otimes \boldsymbol{S}^0).$$

Note that $H_{c-1}^V = I_c \otimes I_w \otimes I_l \otimes I_m^{\otimes c-1}$ and

$$A_2^{(c)} H^{(c-1)} = \begin{pmatrix} A_{c2}^V & 0 \\ \tilde{A}_{c2} & 0 \end{pmatrix}.$$

Proposition 4.1. *A sufficient condition for $MAP/PH/c/K$ retrial queue with vacation under (a, b)-vacation start up rule and single vacation policy to be positive recurrent is*

$$\rho = \frac{\lambda}{c\mu} < 1 \quad \textit{for } K > c \textit{ or } a > 1, \tag{1}$$

$$\rho = \frac{\lambda}{c\mu} < \frac{(c-1)\mu + \nu^*}{c\mu + \nu^*} \quad \textit{for } K = c \textit{ and } a = b = 1. \tag{2}$$

Proof. We consider the two cases separately.

Case (i): $K > c$ *or* $a > 1$. It can be easily seen that

$$A(t) = A_0^{(K)} + tA_1^{(K)} + t^2 A_2^{(K)} H^{(K-1)} = \begin{pmatrix} A_{00}(t) & A_{01}(t) \\ 0 & A_{11}(t) \end{pmatrix}$$

with

$$\begin{aligned} A_{00}(t) &= I_w \otimes D \otimes I_m^{\otimes b^*} + t\boldsymbol{V} \oplus D_0 \oplus \boldsymbol{S}^{\oplus b^*} + t^2 I_w \otimes I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus b^*}, \\ A_{01}(t) &= t\boldsymbol{V}^0 \otimes I_l \otimes I_m^{\otimes b^*} \otimes \boldsymbol{\beta}^{\oplus (c-b^*)}, \end{aligned}$$

$$A_{11}(t) = D_1 \otimes I_m^{\otimes c} + tD_0 \oplus \boldsymbol{S}^{\oplus c} + t^2 I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c}.$$

Let R be the minimal nonnegative solution of the equation

$$A_0^{(K)} + RA_1^{(K)} + R^2 A_2^{(K)} H^{(K-1)} = 0.$$

It can be seen from the formula of $A(1)$ that R is of the form

$$R = \begin{pmatrix} R_{00} & R_{01} \\ 0 & R_{11} \end{pmatrix}.$$

It is sufficient to show that $sp(R) = \max(sp(R_{00}), sp(R_{11})) < 1$.

Note that R_{00} is the minimal nonnegative solution of the equation

$$I_w \otimes D \otimes I_m^{\otimes b^*} + R_{00}(\boldsymbol{V} \oplus D_0 \oplus \boldsymbol{S}^{\oplus b^*}) + R_{00}^2(I_w \otimes I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus b^*}) = 0.$$

Since $A_{00}(1)\boldsymbol{e} \leq 0$ and $A_{00}(1)\boldsymbol{e} \neq 0$, the spectral radius $\xi_0 = sp(R_{00})$ of R_{00} is $\xi_0 < 1$, see [18, Corollary 1.3.1].

Note that R_{11} is the minimal nonnegative solution of the equation

$$D_1 \otimes I_m^{\otimes c} + R_{11}(D_0 \oplus \boldsymbol{S}^{\oplus c}) + R_{11}^2(I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c}) = 0$$

and $\boldsymbol{\pi} = \boldsymbol{\pi}_a \otimes (\boldsymbol{\pi}_s)^{\otimes c}$ is the stationary distribution of $A_{11}(1)$. It follows from [18, Theorem 3.1.1] that $\xi_1 = sp(R_{11}) < 1$ if and only if $\lambda = \boldsymbol{\pi} A_0^{(K)}\boldsymbol{e} < \boldsymbol{\pi}(A_2^{(K)} H^{(K-1)})\boldsymbol{e} = c\mu$. Thus if $\rho < 1$, then $sp(R) = \max(\xi_0, \xi_1) < 1$, and hence it follows from Theorem 3.2 that the condition (1) is a sufficient condition for $\boldsymbol{\Psi}$ to be positive recurrent.

Case (2): *$K = c$ and $a = b = 1$.* Write

$$A(t) = A_0^{(c)} + tA_1^{(c)} + t^2 A_2^{(c)} H^{(c-1)} = \begin{pmatrix} A_{00}^*(t) & A_{01}^*(t) \\ A_{10}^*(t) & A_{11}^*(t) \end{pmatrix}.$$

The block matrices $A_{ij}^*(t)$, $i, j = 0, 1$ are given as follows:

$$\begin{aligned} A_{00}^*(t) &= A_{c0}^V + tA_{c1}^V + t^2 A_{c2}^V, & A_{01}^*(t) &= t\tilde{A}_{c1}^V, \\ A_{10}^*(t) &= t^2 \tilde{A}_{c2}, & A_{11}^*(t) &= A_{c0} + tA_{c1}. \end{aligned}$$

Let $\boldsymbol{\pi}^* = \frac{1}{c\mu+\nu^*}(\boldsymbol{\pi}_1^*, \boldsymbol{\pi}_2^*)$, where

$$\begin{aligned} \boldsymbol{\pi}_1^* &= \mu\boldsymbol{e} \otimes \boldsymbol{\pi}_v \otimes \boldsymbol{\pi}_a \otimes \boldsymbol{\pi}_s^{\otimes c-1}, \\ \boldsymbol{\pi}_2^* &= \nu^* \boldsymbol{\pi}_a \otimes \boldsymbol{\pi}_s^{\otimes c} \end{aligned}$$

with $\boldsymbol{e}$ is c-dimensional row vector whose components are all 1. It can be easily seen that $\boldsymbol{\pi}^*$ is the stationary distribution of

$$A(1) = \begin{pmatrix} I_c \otimes (V \oplus D \oplus S^{*\oplus c-1}) & \tilde{A}_{c1}^V \\ \tilde{A}_{c2} & D \oplus S^{\oplus c} \end{pmatrix}$$

and

$$\begin{aligned} \boldsymbol{\pi}^* A_0^{(c)}\boldsymbol{e} &= \lambda, \\ \boldsymbol{\pi}^* A_2^{(c)} H^{(c-1)}\boldsymbol{e} &= \frac{c\mu((c-1)\mu + \nu^*)}{c\mu + \nu^*}. \end{aligned}$$

Thus (2) is a sufficient condition for the system to be positive recurrent. □

Remarks. 1. The condition (1) in Proposition 4.1 does not depend on the vacation time and the value of b. If $b = 0$, then the system becomes retrial queue without vacations. The stability condition (1) is consistent with that of $MAP/PH/c/K$ retrial queue in [10]. Breuer et al. [4] indicated that the proof of [10] is not complete for $K = c$. However, Proposition 4.1 shows that the result in [10] is correct.

2. Note that the condition (2) depends on the mean vacation time. This result can be expected from the single server system with no waiting space and $a = b = 1$ that arriving or retrial customers cannot enter the service facility unless there is an available server. Considering the vacation time as an extra service time, the system becomes a retrial queue with super service time (actual service time plus vacation time) whose stability condition is $\lambda < \left(\frac{1}{\mu} + \frac{1}{\nu^*}\right)^{-1} = \frac{\mu\nu^*}{\nu^*+\mu}$, see [8].

2. $MAP/PH/c/K$ retrial queue with vacation under (a, b)-vacation start up rule and multiple vacation rule. The multiple vacation policy with (a, b)-vacation setup rule requires the servers to keep taking vacation until they find at least $a^* + 1$ customers waiting in service facility. In this case, if $K > c$ or $a > 1$, then we can see that $A_0^{(K)}$, $A_1^{(K)}$ and $A_2^{(K)}H^{(K-1)}$ are the same as those of the case of single vacation and the condition (1) is a sufficient condition for the system to be positive recurrent. For $K = c$ and $a = b = 1$, the event $\{Y_0(t) = c - 1, J(t) = 0\}$ does not occur and $A_2^{(c)}$ and $H^{(c-1)}$ are given as follows:

$$A_2^{(c)} = \begin{pmatrix} A_{c2}^V \\ \tilde{A}_{c2}^V \end{pmatrix}, \quad H^{(c-1)} = \begin{pmatrix} H_{c-1}^V & 0 \end{pmatrix}$$

and the remaining matrices $A_0^{(c)}$ and $A_1^{(c)}$ are the same as those of single vacation system. Since

$$A_2^{(c)}H^{(c-1)} = \begin{pmatrix} A_{c2}^V & 0 \\ \tilde{A}_{c2}^V & 0 \end{pmatrix},$$

the condition (2) is a sufficient condition for the system with $K = c$ and $a = b = 1$ to be positive recurrent. Summarizing the results above, we have the following.

Proposition 4.2. *The conditions* (1) *for $K > c$ or $a > 1$, and* (2) *for $K = c$ and $a = b = 1$ are sufficient condition for $MAP/PH/c/K$ retrial queue with vacation under (a, b)-vacation start up rule and multiple vacation policy to be positive recurrent.*

3. $MAP/PH/c/K$ retrial queue with vacation under asynchronous and single vacation rule. As in the previous model with (a, b)-vacation rule, denote the number of customers in orbit, service facility and arrival phase by

$\boldsymbol{X}(t) = (X_1(t), \ldots, X_g(t))$, $Y_0(t)$ and $J_a(t)$, respectively. Let $b^*(t)$ be the number of available servers and $J_V(t)$ be the servers in vacation at time t. Let $Y_i(t)$ be the service phase of the server i in service and $V_j(t)$ be the phase of vacation time of the server j in vacation and denote $\boldsymbol{V}(t) = (V_j(t), j \in J_V(t))$ and $\boldsymbol{Y}(t) = (Y_i(t), i \notin J_V(t))$, where $j \in J_V(t) = (j_1, j_2, \ldots, j_m)$ means that the server $j = j_l$ for some $1 \le l \le m$ is in vacation at time t. Then $\boldsymbol{\Psi} = \{(\boldsymbol{X}(t), Y_0(t), \boldsymbol{Z}(t)), t \ge 0\}$ with $\boldsymbol{Z}(t) = (b^*(t), J_V(t), \boldsymbol{V}(t), J_a(t), \boldsymbol{Y}(t)$ is a continuous time Markov chain. In this case, the formulae for the matrices $A_i^{(K)}$ of Q_j, $j = 0, 1, 2$ are as follows:

$$A_0^{(K)} = \begin{pmatrix} A_{0,00}^{(K)} & & & \\ & A_{0,11}^{(K)} & & \\ & & \ddots & \\ & & & A_{0,cc}^{(K)} \end{pmatrix},$$

$$A_1^{(K)} = \begin{pmatrix} A_{1,00}^{(K)} & A_{1,01}^{(K)} & & \\ & \ddots & \ddots & \\ & & A_{1,c-1,c-1}^{(K)} & A_{1,c-1,c}^{(K)} \\ & & & A_{1,cc}^{(K)} \end{pmatrix},$$

$$A_2^{(K)} = \begin{pmatrix} 0 & & & & \\ 0 & A_{2,11}^{(K)} & & & \\ & \ddots & \ddots & & \\ & & 0 & A_{1,c-1,c-1}^{(K)} & \\ & & & A_{2,c,c-1}^{(K)} & A_{2,cc}^{(K)} \end{pmatrix},$$

where $A_{i,jj'}^{(K)}$ is corresponding to the transition rates when the number of customers in service facility is K and the number of available servers in service facility is j. The matrices $A_{i,jj'}^{(K)}$ for $0 \le j \le c-2$ have complex structure. However, we need the exact formulae only for $A_{i,jj'}^{(K)}$, $j, j' = c-1, c$ for ergodic condition and they are given as follows:

$$A_{0,jj}^{(K)} = \begin{cases} I_c \otimes (I_w \otimes D_1 \otimes I_m^{\otimes c-1}), & j = c-1 \\ D_1 \otimes I_m^{\otimes c}, & j = c, \end{cases}$$

$$A_{1,jj}^{(K)} = \begin{cases} I_c \otimes (\boldsymbol{V} \oplus D_0 \oplus \boldsymbol{S}^{\oplus c-1}), & j = c-1 \\ D_0 \oplus \boldsymbol{S}^{\oplus c}, & j = c, \end{cases}$$

$$A_{1,c-1,c}^{(K)} = \begin{pmatrix} \boldsymbol{V}^0 \otimes I_l \otimes \boldsymbol{\beta} \otimes I_m^{\otimes c-1} \\ \boldsymbol{V}^0 \otimes I_l \otimes I_m \otimes \boldsymbol{\beta} \otimes I_m^{\otimes c-2} \\ \vdots \\ \boldsymbol{V}^0 \otimes I_l \otimes I_m^{\otimes c-1} \otimes \boldsymbol{\beta} \end{pmatrix},$$

$$A^{(K)}_{2,c-1,c-1} = I_c \otimes I_w \otimes I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c-1}.$$

The matrix $A^{(K)}_{2,cj}$ is given by $A^{(K)}_{2,c,c-1} = 0$, $K > c$ and for $K = c$,

$$A^{(c)}_{2,c,c-1} = (\boldsymbol{\delta} \otimes I_l \otimes \boldsymbol{S}^0 \otimes I_m^{\otimes c-1}, \boldsymbol{\delta} \otimes I_l \otimes I_m \otimes \boldsymbol{S}^0 \otimes I_m^{\otimes c-2}, \ldots, \boldsymbol{\delta} \otimes I_m^{\otimes c-1} \otimes \boldsymbol{S}^0),$$

$$A^{(K)}_{2,cc} = \begin{cases} I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c}, & K > c \\ 0, & K = c. \end{cases}$$

The matrix $H^{(K-1)}$ is given by

$$H^{(K-1)} = \begin{pmatrix} H_0^{(K-1)} & & \\ & \ddots & \\ & & H_c^{(K-1)} \end{pmatrix},$$

where $H_k^{(K-1)} = I$, $k = 0, 1, \ldots, c-1$ are the identity matrices of appropriate size and $H_c^{(K-1)} = I_c \otimes I_w \otimes I_l \otimes I_m^{\otimes c}$ for $K > c$ and the matrix $H_c^{(c-1)}$ for $K = c$ is not necessary for ergodic condition.

We consider two cases (i) $K > c$ and (ii) $K = c$ separately.

Case (i): $K > c$. It can be easily seen that $A(t) = A_0^{(K)} + tA_1^{(K)} + t^2 A_2^{(K)} H^{(K-1)}$ is of the form

$$A(t) = \begin{pmatrix} A_{00}(t) & A_{01}(t) \\ 0 & A_{11}(t) \end{pmatrix},$$

where

$$A_{01}(t) = \begin{pmatrix} 0 \\ tA^{(K)}_{1,c-1,c} \end{pmatrix},$$

$$A_{11}(t) = D_1 \otimes I_{m^c} + tD_0 \oplus \boldsymbol{S}^{\oplus c} + t^2 I_l \otimes (\boldsymbol{S}^0\boldsymbol{\beta})^{\oplus c}.$$

Since $A_{00}(1)\boldsymbol{e} \le 0$, $A_{00}(1)\boldsymbol{e} \ne 0$ and $A_{11}(t)$ is the same as that of the case of Proposition 4.1, $\rho < 1$ is a sufficient condition for this system with $K > c$ to be positive recurrent.

Case (ii): $K = c$. In this case, write $A(t)$ by

$$A(t) = \begin{pmatrix} A^*_{00}(t) & A^*_{01}(t) \\ 0 & A^*_{11}(t) \end{pmatrix},$$

where

$$A^*_{01}(t) = \begin{pmatrix} 0 & 0 \\ tA^{(K)}_{1,c-2,c-1} & 0 \end{pmatrix}, \quad A^*_{11}(t) = \begin{pmatrix} B_0(t) & tA^{(c)}_{1,c-1,c} \\ t^2 A^{(c)}_{2,c,c-1} & B_1(t) \end{pmatrix}$$

and

$$\begin{aligned} B_0(t) &= A^{(c)}_{0,c-1,c-1} + tA^{(c)}_{1,c-1,c-1} + t^2 A^{(c)}_{2,c-1,c-1} \\ &= I_c \otimes (I_w \otimes D_1 \otimes I_m^{\otimes c-1}) + tI_c \otimes (\boldsymbol{V} \oplus D_0 \oplus \boldsymbol{S}^{\oplus c-1}) + t^2 A^{(c)}_{2,c-1,c-1}, \\ B_1(t) &= D_1 \otimes I_m^{\otimes c} + tD_0 \oplus \boldsymbol{S}^{\oplus c}. \end{aligned}$$

Since $A^*_{00}(1)\boldsymbol{e} \leq 0$, $A^*_{00}(1)\boldsymbol{e} \neq 0$ and $A^*_{11}(t)$ is the same as $A(t)$ of the case $K = c$, $a = b = 1$ in Proposition 4.1, (2) is a sufficient condition for this system with $K = c$ to be positive recurrent. Summarizing the results above, we have the following.

Proposition 4.3. *The conditions* (1) *for* $K > c$ *and* (2) *for* $K = c$ *are sufficient conditions for* $MAP/PH/c/K$ *retrial queue with vacation under asynchronous vacation start up rule and single vacation policy to be positive recurrent.*

4. $MAP/PH/c/K$ retrial queue with vacation under asynchronous and multiple vacation rule. Note that under the asynchronous and multiple vacation rule, the number of working servers cannot be greater than the number of customers in service facility. However, $A(t)$ is the same as that of the case with single vacation rule and the following holds.

Proposition 4.4. *The conditions* (1) *for* $K > c$ *and* (2) *for* $K = c$ *are sufficient conditions for* $MAP/PH/c/K$ *retrial queue with vacation under asynchronous vacation start up rule and multiple vacation policy to be positive recurrent.*

5. Nonergodicity condition

Now we derive the stochastic order relation used for the proof of nonergodicity condition. For convenience, we introduce some notation and define a partial order for vectors. Let $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n)$ and $\boldsymbol{z} = (z-1, \dots, z_m)$. Define a partial order $\boldsymbol{x} \leq \boldsymbol{y}$ by $x_i \leq y_i$ for all $1 \leq i \leq n$ and $(\boldsymbol{x}-\boldsymbol{y})^+ = ((x_1-y_1)^+, \dots, (x_n-y_n)^+)$, where $x^+ = \max(x, 0)$ and $[\boldsymbol{x} : \boldsymbol{z}] = (x_1, \dots, x_n, z_1, \dots, z_m)$ the concatenation of $\boldsymbol{x}$ and $\boldsymbol{z}$. For $E \subset \{1, 2, \dots, k\}$, denote $\boldsymbol{e}_k(E) = (1(j \in E), j = 1, 2, \dots, k)$, where $1(\cdot)$ is the indicator function, the k-dimensional vector whose jth component is 1 if $j \in E$ and 0 if $j \notin E$.

Proposition 5.1. *Denote by* Σ^1 *and* Σ^2*, the ordinary* $G/G/c$ *queueing system with infinite capacity and* $G/G/c/K$ *retrial queue with general retrial time and the servers may not be fully available due to a vacation of servers, respectively. Once entering the service facility, the customer is served in a First-Come-First-Served* (*FCFS*) *fashion and leaves the system upon service completion. We assume that* Σ^1 *and* Σ^2 *have the same arrival process and the same distribution of service times. Let* $Z^i(t)$ *be the number of customers in the system* Σ^i*,* $i = 1, 2$*. Then* $\boldsymbol{Z}^1 = \{Z^1(t),\, t \geq 0\} \leq_{st} \boldsymbol{Z}^2 = \{Z^2(t),\, t \geq 0\}$*, where* $\leq_{st}$ *denotes the usual stochastic ordering of stochastic processes.*

Proof. It is sufficient to show that one can construct two stochastic processes $\hat{\boldsymbol{Z}}^i = \{\hat{Z}^i(t),\, t \geq 0\}$, $i = 1, 2$ on a common probability space such that $\hat{\boldsymbol{Z}}^i \stackrel{d}{=} \boldsymbol{Z}^i$, where $X \stackrel{d}{=} Y$ means that X and Y have the same distribution and $\hat{Z}^1(t) \leq \hat{Z}^2(t)$ for all $t \geq 0$, e.g. see [20]. The construction procedure is similar to that of [19] and we sketch the proof.

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which the independent stochastic processes $\hat{\mathcal{A}} = \{T_n^a, n = 0, 1, \ldots\}$, $\hat{\mathcal{S}} = \{S_n, n = 0, 1, \ldots\}$, $\hat{\mathcal{V}} = \{V_n, n = 0, 1, \ldots\}$, $\hat{\mathcal{R}} = \{\hat{R}_n, n = 0, 1, \ldots\}$ are defined. Here $\hat{\mathcal{A}}$ and $\hat{\mathcal{S}}$ are for the common arrival process and common service sequence in both systems Σ^1 and Σ^2, respectively, $\hat{\mathcal{V}}$ is for the sequence of vacation times and $\hat{\mathcal{R}}$ is for retrials in the system Σ^2. Let $\hat{\boldsymbol{Z}}^i$ be the stochastic process corresponding to $\boldsymbol{Z}^i$ of the system Σ^i with common arrival process $\hat{\mathcal{A}}$, common service sequence $\hat{\mathcal{S}}$ for $i = 1, 2$ and the sequence $\hat{\mathcal{V}}$ of vacation time and retrial process $\hat{\mathcal{R}}$ for $i = 2$. We assume that $\hat{Z}^1(0) = \hat{Z}^2(0) = 0$, that is, both systems start with empty state. Since the system state can be changed by the events of arrivals, service completions, retrials, vacations and returning from vacation, it suffices to observe the system only when these events occur. Let T_n be the time when the nth event of any of the types in any system takes place. Denote by C_k^i, $k \geq 1$ be the kth customer to enter the service facility of Σ^i, $i = 1, 2$. Note that since Σ^1 has infinite size of buffer, C_k^1 is the kth customer arriving to Σ^1. Let $\xi_k^i(t)$ be the remaining service time of C_k^i at time t. If C_k^i completes its service time at time s, then $\xi_k^i(t) = 0$ for $t \geq s$ and if C_k^i is in queue of service facility at time t, then $\xi_k^i(t) = S_k$, the service time of C_k^i.

Let $\hat{Z}_n^i = \hat{Z}^i(T_n)$ and $\xi_{k,n}^i = \xi_k^i(T_n)$. Let A_n be the number of arrivals to the system by T_n and $\Xi_n^i = (\xi_{k,n}^i, k = 1, 2, \ldots, A_n)$, $i = 1, 2$. We shall show the following by induction on n:

$$\text{(3)} \qquad \text{(i) } \hat{Z}_n^1 \leq \hat{Z}_n^2; \quad \text{(ii) } \Xi_n^1 \leq \Xi_n^2.$$

It can be easily seen that the inequalities (i) and (ii) of (3) hold until a service completion occurs and we assume the inequalities (3) hold for $k = 1, 2, \ldots, n$. Let $T_k^{b,i}$ be the epoch of the kth arrival to the service facility of Σ^i, and $T_k^{s,i}$ and $T_k^{c,i}$ the epoch of starting service and service completion time of the customer C_k^i, respectively. Since Σ^1 has a buffer of infinite size in service facility and the customers C_j^1 and C_j^2, $1 \leq j \leq k$ are served in FIFO rule with the same length of service time S_j and the number of available servers in Σ^1 is always greater than or equal to that of Σ^2, we have that

$$\text{(4)} \qquad T_k^{b,1} = T_k^{a,1} \leq T_k^{b,2}, \quad T_k^{s,1} \leq T_k^{s,2}, \quad T_k^{c,1} \leq T_k^{c,2}.$$

We classify the customers that arrive to the system by time T_n as follows: for $i = 1, 2$,

$$I_{0,n}^i = \{k \geq 1 : C_k^i \text{ completed its service by } T_n\},$$
$$I_{1,n}^i = \{k \geq 1 : C_k^i \text{ is being served at } T_n\},$$
$$I_{2,n}^i = \{k \geq 1 : C_k^i \text{ is in queue of service facility at } T_n\},$$
$$I_{3,n}^2 = \{k \geq 1 : C_k^2 \text{ is in orbit at } T_n\}.$$

It follows from the induction hypothesis and the relations (4) that

$$I_{2,n}^1 \subset I_{2,n}^2 \cup I_{3,n}^2, \; I_{1,n}^2 \subset I_{0,n}^1 \cup I_{1,n}^1.$$

Now we show the inequalities of (3) hold for $n+1$ in each class of the events of external arrival, retrial and service completion, end of a vacation. Let $\tau_n = T_{n+1} - T_n$.

Case 1. External arrival. In this case, $\hat{Z}^i_{n+1} = \hat{Z}^i_n + 1$, $i = 1, 2$ and

$$\Xi^i_{n+1} = [(\Xi^i_n - \tau_n \boldsymbol{e}_{A_n}(I^i_{1,n}))^+ : S_{n+1}]),\ i = 1, 2$$

and it can be easily seen that (3) holds for $n+1$.

Case 2. Service completion. Let

$$\begin{aligned} G_1 &= (I^1_{1,n} - I^2_{1,n}) \cup \{k \in I^1_{1,n} \cap I^2_{1,n} : \xi^1_{k,n} < \xi^2_{k,n}\}, \\ G_2 &= I^1_{0,n} \cap I^2_{1,n}, \\ G_3 &= \{k \in I^1_{1,n} \cap I^2_{1,n} : \xi^1_{k,n} = \xi^2_{k,n}\} \end{aligned}$$

and

$$\tau(G_j) = \min\{\xi^i_{k,n} \,:\, k \in G_j,\ i = 1, 2\},\ j = 1, 2, 3.$$

If a service completion occurs at T_{n+1}, then $\tau_n = \min(\tau(G_1), \tau(G_2), \tau(G_3))$. Since $\xi^1_{k,n} \le \xi^2_{k,n}$, $1 \le k \le A_n$ and $\xi^1_{k,n} = 0$ for $k \in G_2$, we have that

$$\begin{aligned} \xi^1_{k,n+1} &= 0 \le (\xi^2_{k,n} - \tau_n)^+ = \xi^2_{k,n+1},\ k \in G_2, \\ \xi^1_{k,n+1} &= (\xi^1_{k,n} - \tau_n)^+ \le (\xi^2_{k,n} - \tau_n)^+ = \xi^2_{k,n+1},\ k \notin G_2 \end{aligned}$$

and hence $\Xi^1_{n+1} \le \Xi^2_{n+1}$. It remains to show that $\hat{Z}^1_{n+1} \le \hat{Z}^2_{n+1}$.

(i) If $\tau_n = \tau(G_1)$ or $\tau_n = \tau(G_3)$, then by induction hypothesis we have that

$$\hat{Z}^1_{n+1} = Z^1_n - |G_j| \le Z^2_n - |G_j| \le \hat{Z}^2_{n+1},\ j = 1, 3$$

where $|A|$ is the number of elements in the set A.

(ii) If $\tau_n = \tau(G_2)$, then $|G_2| \ge 1$ and $Z^1_n \le Z^2_n - |G_2|$ and hence we have that

$$\hat{Z}^1_{n+1} = Z^1_n \le Z^2_n - |G_2| = \hat{Z}^2_{n+1}.$$

Case 3. Retrial in $\boldsymbol{\Sigma^2}$. In this case, $\hat{Z}^1_{n+1} = Z^1_n \le Z^2_n = \hat{Z}^2_{n+1}$. If a customer at orbit enters the service facility of Σ^2 at time T_{n+1} and the customer is C^2_k, then the customer C^2_k enters the service facility at T_{n+1}, then $k \in I^2_{3,n} \subset \cup^2_{j=0} I^1_j$ and the service time of C^2_k is already determined by the service time S_k of C^1_k and hence $\xi^1_{k,n+1} = (\xi^1_{k,n} - \tau_n)^+ \le S_k = \xi^2_{k,n+1}$. Thus by induction hypothesis $\Xi^1_{n+1} \le \Xi^2_{n+1}$.

Case 4. Returning from vacation in $\boldsymbol{\Sigma^2}$. In this case, $\hat{Z}^1_{n+1} = Z^1_n \le Z^2_n = \hat{Z}^2_{n+1}$. If $k \in I^2_{k,n} \subset \cup^2_{j=0} I^1_j$, the service time of C^2_k is already determined by the service time S_k of C^1_k and hence $\xi^1_{k,n+1} = (\xi^1_{k,n} - \tau_n)^+ \le S_k = \xi^2_{k,n+1}$. Thus by induction hypothesis $\Xi^1_{n+1} \le \Xi^2_{n+1}$. □

Remark. The relation (4) is a key result for the proof of Proposition 5.1. If Σ^1 is a $G/G/c$ vacation queue with (a, b) vacation policy, then the relation $T^{s,1}_k \le T^{s,2}_k$ may not hold because the number of available servers in Σ^1 may be less than that of Σ^2 by a vacation.

In the proof of Proposition 5.1, we do not use the specific vacation policy. Thus Proposition 5.1 holds for the system under any vacation policy. It is well known that if $\rho = \frac{\lambda}{c\mu} > 1$, where λ and μ are arrival rate and service rate of each server respectively, then the ordinary $G/G/c$ is not stable in the sense that there exists no stationary workload process e.g. see [3]. We have from Proposition 5.1 the following.

Proposition 5.2. *If $\rho > 1$, then the $G/G/c/K$ retrial queues in which servers may take a vacation is not stable.*

Appendix. Proof of Proposition 3.1

We prove the ergodicity condition of the Markov chain $\boldsymbol{\Psi}$ based on the following mean drift method, see [23] or [8, Statement 8].

Theorem A.3. *Let $\{X(t), t \geq 0\}$ be a Markov process with discrete state space $\mathcal{S}$ and transition rates q_{sp}, $\sum_{p\in\mathcal{S}} q_{sp} = 0$. Assume that there exists a test function (or called Lyapunov function) $\varphi(s)$ on $\mathcal{S}$ that is bounded from below such that $y_s = \sum_{p\in\mathcal{S}} \varphi(p)q_{sp} < \infty$ for all $s \in \mathcal{S}$ and for some $\epsilon > 0$, $y_s < -\epsilon$ for all $s \in \mathcal{S}$ except perhaps a finite number of states. Then $\{X(t), t \geq 0\}$ is regular and ergodic.*

Proof of Proposition 3.1. We introduce subsets of the states $\mathcal{G} = \{1, 2, \ldots, g\}$ of $PH(\boldsymbol{\theta}, \boldsymbol{U})$ distribution of retrial times. Let $\mathcal{G}(1) = \{i \in \mathcal{G} : \gamma_i > 0\}$ and for $2 \leq k \leq k_0$

$$\mathcal{G}(k) = \{i \in \mathcal{G} : i \notin \cup_{k=1}^{k-1}\mathcal{G}(k) \text{ and } u_{ij} > 0 \text{ for some } j \in \mathcal{G}(k-1)\},$$

where k_0 is a positive integer such that $\mathcal{G}(k_0)$ is nonempty and $\cup_{h=1}^{k_0}\mathcal{G}(h) = \mathcal{G}$. If $\gamma_i > 0$ for all $i \in \mathcal{G}$, then $k_0 = 1$ and $\mathcal{G}(1) = \mathcal{G}$. It is clear that $\mathcal{G}(1)$ is nonempty and the customer whose retrial phase is $i \in \mathcal{G}(k)$ can retry after (at least) k transitions. Let $\mathcal{R}(k) = \cup_{h=k}^{k_0}\mathcal{G}(h)$ and $|\boldsymbol{n}(k)| = \sum_{i\in\mathcal{R}(k)} n_i$, $k = 1, 2, \ldots, k_0$ for $\boldsymbol{n} = (n_1, \ldots, n_g)$.

Let $z = \frac{1}{t_0} > 1$ and set

$$\boldsymbol{x}_k = \frac{1}{z^{K-k}} H^{(k)} H^{(k+1)} \cdots H^{(K-1)} \boldsymbol{x}_K, \ k = 0, 1, \ldots, K-1,$$

then it can be easily seen that the M-dimensional vector

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_0 \\ \vdots \\ \boldsymbol{x}_K \end{pmatrix}$$

satisfies $(Q_2 - zI^*)\boldsymbol{x} = 0$, where $I^* = I_M - I_*$. Let $\varphi(\boldsymbol{n}) = \sum_{k=0}^{k_0} \varphi_k(\boldsymbol{n})$, where

$$\varphi_0(\boldsymbol{n}) = z^{|\boldsymbol{n}|}\boldsymbol{x},$$
$$\varphi_k(\boldsymbol{n}) = a z^{|\boldsymbol{n}(k)|}\boldsymbol{e}, \ k = 1, 2, \ldots, k_0,$$

and $a > 0$ will be determined later. It is sufficient from Theorem A.3 to show that there exists a positive number $\epsilon > 0$ such that

$$\psi(\boldsymbol{n}) = \sum_{\boldsymbol{n}' \in \mathbb{Z}_+^g} Q(\boldsymbol{n}, \boldsymbol{n}')\varphi(\boldsymbol{n}') = \sum_{k=0}^{k_0} \Phi_k(\boldsymbol{n}) \le -\epsilon \boldsymbol{e} \tag{A.1}$$

for all but a finite number of $\boldsymbol{n} \in \mathbb{Z}_+^g$, where

$$\Phi_k(\boldsymbol{n}) = \sum_{\boldsymbol{n}' \in \mathbb{Z}_+^g} Q(\boldsymbol{n}, \boldsymbol{n}')\varphi_k(\boldsymbol{n}'),\ k = 0, 1, \dots, k_0.$$

Some algebra yields that

$$\begin{aligned}\Phi_0(\boldsymbol{n}) &= z^{|\boldsymbol{n}|-1}[z(Q_1 + zQ_0)\boldsymbol{x} + (\boldsymbol{n} \cdot \boldsymbol{\gamma})(Q_2 - zI^*)\boldsymbol{x}] \\ &= z^{|\boldsymbol{n}|-1} z(Q_1 + zQ_0)\boldsymbol{x},\end{aligned}$$

where $(Q_2 - zI^*)\boldsymbol{x} = 0$ is used. Similarly, using $\mathcal{R}(1) = \mathcal{G}$, $Q_1\boldsymbol{e} = -Q_0\boldsymbol{e}$ and $Q_2\boldsymbol{e} = I^*\boldsymbol{e}$ we have that

$$\Phi_1(\boldsymbol{n}) = az^{|\boldsymbol{n}|-1}(z - 1)(zQ_0\boldsymbol{e} - (\boldsymbol{n} \cdot \boldsymbol{\gamma})I^*\boldsymbol{e}).$$

Let

$$\begin{aligned}\psi_1(\boldsymbol{n}) &= z^{-(|\boldsymbol{n}|-1)}(\Phi_0(\boldsymbol{n}) + \Phi_1(\boldsymbol{n})) \\ &= z(Q_1 + zQ_0)\boldsymbol{x} + a(z - 1)zQ_0\boldsymbol{e} - a(z - 1)(\boldsymbol{n} \cdot \boldsymbol{\gamma})I^*\boldsymbol{e}.\end{aligned}$$

Noting from the construction of $\boldsymbol{x}$ that the last z_K elements of $z(Q_1 + zQ_0)\boldsymbol{x}$ is

$$[z(Q_1 + zQ_0)\boldsymbol{x}]_K = z^2 A(t_0)\boldsymbol{x}_K < 0$$

and $a(z - 1)(\boldsymbol{n} \cdot \boldsymbol{\gamma})I^*\boldsymbol{e}$ tends to infinite, we can take $a > 0$ so small that $[z(Q_1 + zQ_0)\boldsymbol{x}]_K + a(z - 1)zQ_0\boldsymbol{e} < 0$. Since the first $M - z_K$ elements of $\psi_1(\boldsymbol{n})$ tends to $-\infty$ as n_i $(i \in \mathcal{G}(1))$ goes to infinite, there exist an $\epsilon_1 > 0$ and an integer $K_1 > 0$ such that $\psi_1(\boldsymbol{n}) < -\epsilon_1\boldsymbol{e}$ if $n_i > K_1$ for some $i \in \mathcal{G}(1)$.

Since $\gamma_i = 0$ for $i \in \mathcal{R}(k)$, $k \ge 2$ and $Q_2\boldsymbol{e} = \boldsymbol{e} - I_*\boldsymbol{e}$, we can easily obtain the followings:

$$\sum_{i=1}^{g} Q(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_i)\varphi_k(\boldsymbol{n} - \boldsymbol{e}_i) = az^{|\boldsymbol{n}(k)|}(\boldsymbol{n} \cdot \boldsymbol{\gamma})Q_2\boldsymbol{e} = az^{|\boldsymbol{n}(k)|}(\boldsymbol{n} \cdot \boldsymbol{\gamma})(\boldsymbol{e} - I_*\boldsymbol{e}),$$

$$\sum_{i=1}^{g} Q(\boldsymbol{n}, \boldsymbol{n})\varphi_k(\boldsymbol{n}) = az^{|\boldsymbol{n}(k)|}\left[Q_1\boldsymbol{e} - (\boldsymbol{n} \cdot \boldsymbol{u})\boldsymbol{e} + \left(\sum_{i=1}^{g} n_i\gamma_i\theta_i\right) I_*\boldsymbol{e}\right],$$

$$\sum_{i=1}^{g} Q(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i)\varphi_k(\boldsymbol{n} + \boldsymbol{e}_i) = az^{|\boldsymbol{n}(k)|}\left((z - 1)\sum_{i \in \mathcal{R}(k)} \theta_i + 1\right) Q_0\boldsymbol{e}.$$

Furthermore,

$$\sum_{i=1}^{g} \sum_{j=1, j\ne i}^{g} Q(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_i + \boldsymbol{e}_j)\varphi_k(\boldsymbol{n} - \boldsymbol{e}_i)$$

$$= a\sum_{i=1}^{g} n_i \sum_{j=1,j\neq i}^{g} u_{ij} z^{|(\boldsymbol{n}-\boldsymbol{e}_i+\boldsymbol{e}_j)(k)|} I_M \boldsymbol{e} + a\sum_{i=1}^{g} n_i\gamma_i \sum_{j=1,j\neq i}^{g} \theta_j z^{|(\boldsymbol{n}-\boldsymbol{e}_i+\boldsymbol{e}_j)(k)|} I_* \boldsymbol{e}$$

$$= az^{|\boldsymbol{n}(k)|}\left((\boldsymbol{n}\cdot\boldsymbol{u}) - (\boldsymbol{n}\cdot\boldsymbol{\gamma}) - \frac{1}{z}(z-1)\sum_{i\in\mathcal{R}(k)} n_i \sum_{j\notin\mathcal{R}(k)} u_{ij} \right.$$

$$\left. +(z-1)\sum_{i\notin\mathcal{R}(k)} n_i \sum_{j\in\mathcal{R}(k)} u_{ij} \right)\boldsymbol{e},$$

$$+ az^{|\boldsymbol{n}(k)|}\sum_{i\notin\mathcal{R}(k)} n_i\gamma_i \left((z-1)\sum_{j\in\mathcal{R}(k)} \theta_j + 1 \right) I_*\boldsymbol{e}.$$

Summarizing the results above with $Q_1\boldsymbol{e} = -Q_0\boldsymbol{e}$ yields that for $2 \le k \le k_0$,

$$\Phi_k(\boldsymbol{n}) = az^{|\boldsymbol{n}(k)|}(z-1)\left(\sum_{i\in\mathcal{R}(k)} \theta_i\right) Q_0\boldsymbol{e}$$

$$+ az^{|\boldsymbol{n}(k)|}\sum_{i\in\mathcal{G}(1)} n_i\gamma_i \left((z-1)\sum_{j\in\mathcal{R}(k)} \theta_j + \theta_i \right) I_*\boldsymbol{e}$$

$$+ az^{|\boldsymbol{n}(k)|-1}(z-1)\left(z\sum_{i\notin\mathcal{R}(k)} n_i \sum_{j\in\mathcal{R}(k)} u_{ij}\boldsymbol{e} - \sum_{i\in\mathcal{R}(k)} n_i \sum_{j\notin\mathcal{R}(k)} u_{ij}\boldsymbol{e} \right).$$

Since $u_{ij} = 0$ for $i \in \mathcal{G}(k)$, $j \in \mathcal{G}(l)$, $l \le k-2$, we have

$$\sum_{i\in\mathcal{R}(k)} n_i \sum_{j\notin\mathcal{R}(k)} u_{ij} = \sum_{i\in\mathcal{R}(k)} n_i \sum_{j\in\mathcal{G}(1)\cup\cdots\mathcal{G}(k-1)} u_{ij} = \sum_{i\in\mathcal{G}(k)} n_i \sum_{j\in\mathcal{G}(k-1)} u_{ij}.$$

Note that

$$\sum_{k=2}^{k_0} z^{|\boldsymbol{n}(k)|}\sum_{i\notin\mathcal{R}(k)} n_i \sum_{j\in\mathcal{R}(k)} u_{ij}\boldsymbol{e} = \sum_{k=2}^{k_0} z^{|\boldsymbol{n}(k)|}\sum_{l=1}^{k-1}\sum_{i\in\mathcal{G}(l)} n_i \sum_{j\in\mathcal{R}(k)} u_{ij}\boldsymbol{e}$$

$$= \sum_{l=1}^{k_0-1}\sum_{i\in\mathcal{G}(l)} n_i \sum_{k=l+1}^{k_0} z^{|\boldsymbol{n}(k)|}\sum_{j\in\mathcal{R}(k)} u_{ij}\boldsymbol{e}$$

$$= \sum_{i\in\mathcal{G}(1)} n_i \sum_{k=2}^{k_0} z^{|\boldsymbol{n}(k)|}\sum_{j\in\mathcal{R}(k)} u_{ij}\boldsymbol{e} + \sum_{l=2}^{k_0}\sum_{i\in\mathcal{G}(l)} n_i \sum_{k=l+1}^{k_0} z^{|\boldsymbol{n}(k)|}\sum_{j\in\mathcal{R}(k)} u_{ij}\boldsymbol{e}.$$

Thus

$$\psi(\boldsymbol{n}) = z^{|\boldsymbol{n}|-1}\sum_{m=1}^{6}\psi_m(\boldsymbol{n}),$$

where

$$\psi_2(\boldsymbol{n}) = az(z-1)\sum_{k=2}^{k_0}\frac{1}{z^{|\boldsymbol{n}|-|\boldsymbol{n}(k)|}}\left(\sum_{i\in\mathcal{R}(k)}\theta_i\right)Q_0\boldsymbol{e},$$

$$\psi_3(\boldsymbol{n}) = az\sum_{i\in\mathcal{G}(1)}\sum_{k=2}^{k_0}\frac{n_i\gamma_i}{z^{|\boldsymbol{n}|-|\boldsymbol{n}(k)|}}\left((z-1)\sum_{j\in\mathcal{R}(k)}\theta_j+\theta_i\right)I_*\boldsymbol{e},$$

$$\psi_4(\boldsymbol{n}) = az(z-1)\sum_{i\in\mathcal{G}(1)}\sum_{k=2}^{k_0}\frac{n_i}{z^{|\boldsymbol{n}|-|\boldsymbol{n}(k)|}}\sum_{j\in\mathcal{R}(k)}u_{ij}\boldsymbol{e},$$

$$\psi_5(\boldsymbol{n}) = az(z-1)\sum_{k=2}^{k_0}\sum_{i\in\mathcal{G}(k)}\sum_{l=k+1}^{k_0}\frac{n_i}{z^{|\boldsymbol{n}|-|\boldsymbol{n}(l)|}}\sum_{j\in\mathcal{R}(l)}u_{ij}\boldsymbol{e},$$

$$\psi_6(\boldsymbol{n}) = -a(z-1)\sum_{k=2}^{k_0}\sum_{i\in\mathcal{G}(k)}\frac{n_i}{z^{|\boldsymbol{n}|-|\boldsymbol{n}(k)|}}\sum_{j\in\mathcal{G}(k-1)}u_{ij}\boldsymbol{e}.$$

Since

$$|\boldsymbol{n}|-|\boldsymbol{n}(k)| = \sum_{i\in\mathcal{G}(1)\cup\cdots\cup\mathcal{G}(k-1)} n_i,\ \ 2\le k\le k_0$$

and $\lim_{n\to\infty}\frac{n}{z^n}=0$, $\psi_l(\boldsymbol{n})$, $l=2,\ldots,6$ tend to zero vector if for some $i\in\mathcal{G}(1)$, n_i goes to infinity. Thus there exist a positive number $\epsilon>0$ and a positive integer N_1 such that $\psi(\boldsymbol{n})<-z^{|\boldsymbol{n}|-1}\epsilon\boldsymbol{e}$ if $n_i>N_1$ for some $i\in\mathcal{G}(1)$.

Let $\mathcal{F}(1)=\{\boldsymbol{n}\in\mathbb{Z}_+^g\ :\ n_i\le N_1 \text{ for all } i\in\mathcal{G}(1)\}$. Noting that for $\boldsymbol{n}\in\mathcal{F}(1)$,

$$|\boldsymbol{n}|-|\boldsymbol{n}(2)| = \sum_{i\in\mathcal{G}(1)} n_i \le N_1 g,\ \ \boldsymbol{n}\cdot\boldsymbol{\gamma} = \sum_{i\in\mathcal{G}(1)} n_i\gamma_i \le N_1\boldsymbol{\gamma}\boldsymbol{e}$$

and for $2\le k<l\le k_0$,

$$|\boldsymbol{n}(k)|-|\boldsymbol{n}(l)| = \sum_{i\in\mathcal{G}(k)\cup\cdots\cup\mathcal{G}(l-1)} n_i \ge 0,$$

we can see that $z^{|\boldsymbol{n}|-|\boldsymbol{n}(2)|}\psi_m(\boldsymbol{n})$, $m=2,3,4,5$ are bounded from above on $\mathcal{F}(1)$. Since

$$\begin{aligned} z^{|\boldsymbol{n}|-|\boldsymbol{n}(2)|}\psi_6(\boldsymbol{n}) =&\ -a(z-1)\sum_{i\in\mathcal{G}(2)} n_i\sum_{j\in\mathcal{G}(1)}u_{ij}\boldsymbol{e} \\ &\ -a(z-1)\sum_{k=3}^{k_0}\sum_{i\in\mathcal{G}(k)}\frac{n_i}{z^{\sum_{t\in\mathcal{G}(2)\cup\cdots\cup\mathcal{G}(k)} n_t}}\sum_{j\in\mathcal{G}(k-1)}u_{ij}\boldsymbol{e} \\ \le&\ -a(z-1)\sum_{i\in\mathcal{G}(2)} n_i\sum_{j\in\mathcal{G}(1)}u_{ij}\boldsymbol{e} \end{aligned}$$

and if $i\in\mathcal{G}(2)$, then $u_{ij}>0$ for some $j\in\mathcal{G}(1)$, it can be seen that each component of $z^{|\boldsymbol{n}|-|\boldsymbol{n}(2)|}\psi_6(\boldsymbol{n})$ tends to $-\infty$ as $n_i\to\infty$ for some $i\in\mathcal{G}(2)$.

Therefore there exist a positive number $\epsilon_2 > 0$ and a positive integer N_2 such that $\psi(\boldsymbol{n}) \le -z^{|\boldsymbol{n}(2)|-1}\epsilon_2 \boldsymbol{e}$ for $\boldsymbol{n} \in \mathcal{F}(1)$ with $n_i > N_2$ for some $i \in \mathcal{G}(2)$.

Repeating this procedure, we can see that there exist a positive number $\epsilon_k > 0$ and a positive integer N_k such that $\psi(\boldsymbol{n}) \le -z^{|\boldsymbol{n}(k)|-1}\epsilon_k \boldsymbol{e}$ for $\boldsymbol{n} \in \mathcal{F}(k-1) = \{\boldsymbol{n} \in \mathbb{Z}_+^g \ : \ n_i \le N_l \text{ for all } i \in \mathcal{G}(l), l = 1, 2, \ldots, k-1\}$ with $n_i > N_k$ for some $i \in \mathcal{G}(k)$, $k = 2, \ldots, k_0$. Thus $(A.1)$ holds possibly except for $\boldsymbol{n} \in \mathbb{Z}_+^g$ with $n_i \le \max(N_1, \ldots, N_{k_0})$, $i = 1, 2, \ldots, g$. □

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