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SOME FAMILIES OF IDEAL-HOMOGENEOUS POSETS

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Abstract. A partially ordered set P is *ideal-homogeneous* provided that for any ideals I and J, if $I \cong_\sigma J$, then there exists an automorphism σ^* such that $\sigma^*|_I = \sigma$. Behrendt [1] characterizes the ideal-homogeneous partially ordered sets of height 1. In this paper, we characterize the ideal-homogeneous partially ordered sets of height 2 and find some families of ideal-homogeneous partially ordered sets.

1. Introduction

Suppose $(P, \leqslant)$ is a finite partially ordered set (simply called a finite poset) with a partial order relation $\leqslant$, which is simply denoted by P for convenience. If $Q \subset P$, we may refer to Q also as a poset, having in mind the subposet $(Q, \leqslant)$ whose order relation is the restriction of $(P, \leqslant)$'s. If P is a finite ordered set and $x \in P$, then the *height* $h(x)$ is the maximal cardinality of a chain in $\{y \in P \mid y < x\}$. The *height* of a poset P, denoted by $ht(P)$, is maximum of all $h(x)$ for $x \in P$. For a poset P and $x \in P$, let $U[x] = \{y \in P \mid y \geqslant x \text{ in } P\}$, say the up-set of x, and $D[x] = \{y \in P \mid y \leqslant x \text{ in } P\}$, say the down-set of x. Also, we let $U[A] = \cup_{x \in A} U[x]$, the up-set of A, and $D[A] = \cup_{x \in A} D[x]$ the down-set of A for a nonempty subposet A of P. A map $f : (P, \leqslant) \to (Q, \leqslant')$ of posets is *order-preserving* if $x \leqslant y$ implies $f(x) \leqslant' f(y)$ in Q for all $x, y \in P$. Two posets $(P, \leqslant)$ and $(Q, \leqslant')$ are *isomorphic* if there exists an order-preserving bijection $f : (P, \leqslant) \to (Q, \leqslant')$ such that f^{-1} is also order-preserving. We denote the set of all automorphisms of a poset P by $\mathrm{Aut}(P)$. An *ideal* I of P is a non-empty subset of P such that if $x \leqslant y$ for $x \in P$ and $y \in I$, then $x \in I$. A poset P is *ideal-homogeneous*, provided that, for any ideals I and J with $I \cong_\sigma J$, there exists an automorphism $\sigma^* \in \mathrm{Aut}\, P$ such that $\sigma^*|_I = \sigma$. A poset P is *weakly ideal-homogeneous*, provided that for each I of P and $\sigma \in \mathrm{Aut}(I)$, there is $\sigma^* \in \mathrm{Aut}(P)$ such that $\sigma^*|_I = \sigma$.

It is very natural to ask whether every isomorphism between finite substructures can be extendable to an automorphism of the whole structure. In 1993,

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some results on the homogeneity for finite partially ordered sets were given by G. Behrendt [1], and they made resume to investigate the relationship between the homogeneity conditions for finite partially ordered sets. The following theorem, due to Behrendt [1], characterizes the (weakly) ideal-homogeneous posets of height 1. For a positive integer n, $[n]$ is the set of positive interger less than or equal to n.

Theorem 1.1 ([1])**.** *Let $(P, \leqslant)$ be a finite partially ordered set of height 1. The followings are equivalent.*

(i) *$(P, \leqslant)$ is ideal-homogeneous.*
(ii) *$(P, \leqslant)$ is weakly ideal-homogeneous.*
(iii) *There exist a positive integer n and a function $f : [n] \to \mathbb{N}$ such that there exists $i \in [n]$ with $f(i) \neq 0$ and $(P, \leqslant)$ is isomorphic to $(X, \leqslant)$, where*

$$X = [n] \cup \{(S, i) \mid \emptyset \neq S \subseteq [n],\ 1 \leq i \leq f(|S|)\}$$

and for $k \in [n]$, $\emptyset \neq S \subseteq [n]$, $1 \leq i \leq f(|S|)$, let

$$k \leqslant (S, i) \quad \textit{if and only if} \quad k \in S.$$

In this paper, we characterize the ideal-homogeneous partially ordered sets of height 2 and find some families of ideal-homogeneous partially ordered sets. The other definitions not written in this paper and general properties of posets follow from [2].

2. Construction

Let $X = [n]$ and $P(X)$ be the power set of X. For all $k = 1, \ldots, n$, let $A_k(X)$ be the set of k-element subsets of X, that is, $A_k(X) = \{S_1, S_2, \ldots, S_{\binom{n}{k}}\}$ where $|S_i| = k$ for $i = 1, \ldots, \binom{n}{k}$. Then $|A_k(X)| = \binom{n}{k}$. Let $M_1(X)$ be a multi-set of nonempty subsets of X with the multiplicities $m_k \geq 0$ for $A_k(X)$ for each $k = 1, \ldots, n$, such that

(1) every element S of $M_1(X)$ is a nonempty subset of X,
(2) if $S(\neq \emptyset) \in M_1(X)$ with $|S| = k$ for some k, then it has multiplicity m_k, that means it appears m_k times in $M_1(X)$.
(3) if $S(\neq \emptyset) \in M_1(X)$ with $|S| = k$ for some k, then $T \in M_1(X)$ for any $T \in A_k(X)$.

Therefore, if $S, T \in M_1(X)$ with $|S| = |T| = k$, then both S and T appear m_k times in $M_1(X)$. Hence for each k, it can be said that m_k is not only the multiplicity of an element S of $A_k(X)$ but also the multiplicity of $A_k(X)$. Thus we may write $M_1(X)$ using $A_k(X)$'s as

$$\begin{aligned} M_1(X) = &(A_1, 1) \cup \cdots \cup (A_1, m_1) \\ &\bigcup (A_2, 1) \cup \cdots \cup (A_2, m_2) \bigcup \cdots \\ &\bigcup (A_n, 1) \cup \cdots \cup (A_n, m_n), \end{aligned} \tag{1}$$

where $(A_k, i) = (A_k(X), i)$ is the i-th copy of $A_k(X)$ for $1 \leq i \leq m_k$ and $m_k \geq 0$. Hence if $S \in A_k(X)$ for some k, then we may notice $(S, i) \in (A_k(X), i)$ for all i and we say (S, i) is the i-th copy of S also, denote $S_i = (S, i)$. Note that if $m_k = 0$ for some k, then $(A_k, i) = \phi$ for all i.

We may define $M_1(Y)$ for any non-empty set Y. Let us consider $M_1(Y)$, where $Y = (A_k(X), i)$ for fixed i and k. For all $t = 1, \ldots, \binom{n}{k}$, let $B_t(Y)$ be the set of t-element subsets of Y, that is, $B_t(Y) = \left\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{\binom{\binom{n}{k}}{t}}\right\}$ where $|\Sigma_i| = t$ for $i = 1, \ldots, \binom{\binom{n}{k}}{t}$. Then $M_1((A_k(X), i))$ is the multi-set of nonempty subsets of $(A_k(X), i)$ with the multiplicities $a_t^k \geq 0$ for $B_t(Y)$ for all $t = 1, \ldots, \binom{n}{k}$, such that

(1) every element Σ_i of $M_1(A_k(X), i)$ is a nonempty subset of $(A_k(X), i)$,
(2) if $\Sigma_i (\neq \emptyset) \in M_1(A_k(X), i)$ with $|\Sigma_i| = t$ for some t, then it has multiplicity a_t^k, that means it appears a_t^k times in $M_1(A_k(X), i)$.

Let $M_1((A_k(X), i)) = M_1(A_k(X), i)$ and $\Omega_{\binom{n}{k}} = \left(a_1^k, a_2^k, \ldots, a_{\binom{n}{k}}^k\right)$ for convenience. Hence $(\Sigma_i, j) \in M_1(A_k, i)$ is defined as the j-th copy of Σ_i in $M_1(A_k, i)$, where $1 \leq j \leq a_{|\Sigma_i|}^k$. We write $(\Sigma_i, j) = \{\Sigma_i\}_j$.

Now we define a (second level) multi-set $M_2(X)$ of X as:

$$M_2(X) = \bigcup_{k=1}^{n} \bigcup_{i=1}^{m_k} \bigcup_{r=1}^{b_k} M_1(A_k(X), i)^r, \tag{2}$$

where $(m_1, m_2, \ldots, m_n)$, $m_k \geq 0$, $\Omega_{\binom{n}{k}} = \left(a_1^k, a_2^k, \ldots, a_{\binom{n}{k}}^k\right)$ with $a_t^k \geq 0$ for $t = 1, \ldots, \binom{n}{k}$, and $b = (b_1, b_2, \cdots, b_n)$, $b_k \geq 1$ are the multiplicities for A_k of $M_1(X)$, the nonempty subsets of $B_t((A_k, i))$ of $M_2(X)$, and $M_1(A_k(X), i)$ for all $1 \leq i \leq m_k$, $k = 1, \ldots, n$, respectively and $1 \leq r \leq b_k$. If $m_k = 0$ for some k, then $M_2(X)$ does not have $M_1(A_k, i)$ for $i = 1, \ldots, m_k$. And if $a_t^k = 0$ for some k and t, then $M_2(X)$ does not have t-elements subset of (A_k, i) for all $i = 1, 2, \ldots, m_k$. The next example shows a construct process of X, $M_1(X)$, and $M_2(X)$ for given n.

Example 1. Let $n = 3$ and hence $X = [3]$. Then we have

$$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}, \tag{3}$$

where $A_1 = \{\{1\}, \{2\}, \{3\}\}$, $A_2 = \{\{1,2\}, \{1,3\}, \{2,3\}\}$, and $A_3 = \{\{1,2,3\}\}$. Suppose $(m_1, m_2, m_3) = (1, 3, 2)$ is the multiplicities of A_k for $k = 1, 2, 3$. Then we have

$$\begin{aligned} (A_1, 1) &= (\{\{1\}, \{2\}, \{3\}\}, 1) = \{(\{1\}, 1), (\{2\}, 1), (\{3\}, 1)\} \\ &= \{\{1\}_1, \{2\}_1, \{3\}_1\}, \\ (A_2, i) &= (\{\{1,2\}, \{1,3\}, \{2,3\}\}, i) \\ &= \{(\{1,2\}, i), (\{1,3\}, i), (\{2,3\}, i)\} \end{aligned} \tag{4}$$

$$= \{\{1,2\}_i, \{1,3\}_i, \{2,3\}_i\} \qquad \text{for } i = 1,2,3,$$
$$(A_3, i) = (\{\{1,2,3\}\}, i) = \{(\{1,2,3\},1), (\{1,2,3\},2)\}$$
$$= \{\{1,2,3\}_1, \{1,2,3\}_2\}.$$

Therefore we can write $M_1(X)$ as:

$$\begin{aligned} M_1(X) = \big\{ &\{1\}_1, \{2\}_1, \{3\}_1, \{1,2\}_1, \{1,3\}_1, \{2,3\}_1, \\ &\{1,2\}_2, \{1,3\}_2, \{2,3\}_2, \{1,2\}_3, \{1,3\}_3, \{2,3\}_3 \\ &\{1,2,3\}_1, \{1,2,3\}_2 \big\}. \end{aligned} \tag{5}$$

Suppose $\Omega_{\binom{3}{1}} = (1,2,2)$, $\Omega_{\binom{3}{2}} = (1,1,3)$, $\Omega_{\binom{3}{3}} = (2)$, $b = (b_1, b_2, b_3) = (2,1,2)$. Then we have

$$\begin{aligned} M_1(A_1,1)^j &= M_1(\{\{1\}_1, \{2\}_1, \{3\}_1\}) \quad \text{with} \quad \Omega_{\binom{3}{1}} = (1,2,2) \\ &= \big\{ (\{\{1\}_1\},1)^j, (\{\{2\}_1\},1)^j, (\{\{3\}_1\},1)^j, \\ &\quad (\{\{1\}_1,\{2\}_1\},1)^j,\ (\{\{1\}_1,\{3\}_1\},1)^j,\ (\{\{2\}_1,\{3\}_1\},1)^j, \\ &\quad (\{\{1\}_1,\{2\}_1\},2)^j,\ (\{\{1\}_1,\{3\}_1\},2)^j,\ (\{\{2\}_1,\{3\}_1\},2)^j, \\ &\quad (\{\{1\}_1,\{2\}_1,\{3\}_1\},1)^j, (\{\{1\}_1,\{2\}_1,\{3\}_1\},2)^j \big\} \end{aligned} \tag{6}$$

or we can write it for convenience

$$\begin{aligned} M_1(A_1,1)^j = \big\{ &\{\{1\}_1\}_1^j, \{\{2\}_1\}_1^j, \{\{3\}_1\}_1^j, \\ &\{\{1\}_1,\{2\}_1\}_1^j,\ \{\{1\}_1,\{3\}_1\}_1^j,\ \{\{2\}_1,\{3\}_1\}_1^j, \\ &\{\{1\}_1,\{2\}_1\}_2^j,\ \{\{1\}_1,\{3\}_1\}_2^j,\ \{\{2\}_1,\{3\}_1\}_2^j, \\ &\{\{1\}_1,\{2\}_1,\{3\}_1\}_1^j, \{\{1\}_1,\{2\}_1,\{3\}_1\}_2^j \big\}, \end{aligned} \tag{7}$$

where $j = 1,2$ which means we have 2 copies of $M_1(A_1,1)$ since $b_1 = 2$,

$$\begin{aligned} &M_1(A_2,i)^1 \\ &= M_1(\{\{1,2\}_i, \{1,3\}_i, \{2,3\}_i\} \quad \text{with} \quad \Omega_{\binom{3}{2}} = (1,1,3) \\ &= \big\{ (\{\{1,2\}_i\},1)^1, (\{\{1,3\}_i\},1)^1, (\{\{2,3\}_i\},1)^1, \\ &\quad (\{\{1,2\}_i,\{1,3\}_i\},1)^1, (\{\{1,2\}_i,\{2,3\}_i\},1)^1, (\{\{1,3\}_i,\{2,3\}_i\},1)^1, \\ &\quad (\{\{1,2\}_i,\{1,3\}_i,\{2,3\}_i\},1)^1, (\{\{1,2\}_i,\{1,3\}_i,\{2,3\}_i\},2)^1, \\ &\quad (\{\{1,2\}_i,\{1,3\}_i,\{2,3\}_i\},3)^1 \big\} \\ &= \big\{ \{\{1,2\}_i\}_1^1, \{\{1,3\}_i\}_1^1, \{\{2,3\}_i\}_1^1, \\ &\quad \{\{1,2\}_i,\{1,3\}_i\}_1^1, \{\{1,2\}_i,\{2,3\}_i\}_1^1, \{\{1,3\}_i,\{2,3\}_i\}_1^1, \\ &\quad \{\{1,2\}_i,\{1,3\}_i,\{2,3\}_i\}_1^1, \{\{1,2\}_i,\{1,3\}_i,\{2,3\}_i\}_2^1, \\ &\quad \{\{1,2\}_i,\{1,3\}_i,\{2,3\}_i\}_3^1 \big\} \end{aligned} \tag{8}$$

so that we have only one copy of $M_1(A_2,i)$ for each for $i=1,2,3$, since $b_2=1$ and

$$
\begin{aligned}
M_1(A_3,i)^j &= M_1(\{1,2,3\}_i)^j \quad \text{with} \quad \Omega_{\binom{3}{3}}=(2) \\
&= \big\{(\{\{1,2,3\}_i\},1)^j,(\{\{1,2,3\}_i\},2)^j\big\} \qquad (9)\\
&= \big\{\{\{1,2,3\}_i\}_1^j,\{\{1,2,3\}_i\}_2^j\big\},
\end{aligned}
$$

where $j=1,2$ which means we have 2 copies of $M_1(A_3,i)$ for each $i=1,2$ since $b_3=2$. Therefore, the second-level multi-set $M_2(X)$, where $(m_1,m_2,m_3)=(1,3,2)$, $\Omega_{\binom{3}{1}}=(1,2,2)$, $\Omega_{\binom{3}{2}}=(1,1,3)$, $\Omega_{\binom{3}{3}}=(2)$, and $b=(b_1,b_2,b_3)=(2,1,2)$ as

$$
\begin{aligned}
(10)\qquad & M_2(X)\\
&= \bigcup_{k=1}^{3}\bigcup_{i=1}^{m_k}\bigcup_{r=1}^{b_k} M_1(A_k,i)^r\\
&= M_1(A_1,1)^1 \cup M_1(A_1,1)^2\\
&\quad \cup M_1(A_2,1)^1 \cup M_1(A_2,2)^1 \cup M_1(A_2,3)^1\\
&\quad \cup M_1(A_3,1)^1 \cup M_1(A_3,1)^2 \cup M_1(A_3,2)^1 \cup M_1(A_3,2)^2\\
&= \big\{\ \{\{1\}_1\}_1^1,\{\{2\}_1\}_1^1,\{\{3\}_1\}_1^1,\\
&\quad \{\{1\}_1,\{2\}_1\}_1^1,\ \{\{1\}_1,\{3\}_1\}_1^1,\ \{\{2\}_1,\{3\}_1\}_1^1,\\
&\quad \{\{1\}_1,\{2\}_1\}_2^1,\ \{\{1\}_1,\{3\}_1\}_2^1,\ \{\{2\}_1,\{3\}_1\}_2^1,\\
&\quad \{\{1\}_1,\{2\}_1,\{3\}_1\}_1^1,\{\{1\}_1,\{2\}_1,\{3\}_1\}_2^1, \qquad (M_1(A_1,1)^1 \text{ part})\\
&\quad \{\{1\}_1\}_1^2,\{\{2\}_1\}_1^2,\{\{3\}_1\}_1^2,\\
&\quad \{\{1\}_1,\{2\}_1\}_1^2,\ \{\{1\}_1,\{3\}_1\}_1^2,\ \{\{2\}_1,\{3\}_1\}_1^2,\\
&\quad \{\{1\}_1,\{2\}_1\}_2^2,\ \{\{1\}_1,\{3\}_1\}_2^2,\ \{\{2\}_1,\{3\}_1\}_2^2,\\
&\quad \{\{1\}_1,\{2\}_1,\{3\}_1\}_1^2,\{\{1\}_1,\{2\}_1,\{3\}_1\}_2^2, \qquad (M_1(A_1,1)^2 \text{ part})\\
&\quad \{\{1,2\}_1\}_1,\{\{1,3\}_1\}_1,\{\{2,3\}_1\}_1,\\
&\quad \{\{1,2\}_1,\{1,3\}_1\}_1,\{\{1,2\}_1,\{2,3\}_1\}_1,\{\{1,3\}_1,\{2,3\}_1\}_1,\\
&\quad \{\{1,2\}_1,\{1,3\}_1,\{2,3\}_1\}_1,\{\{1,2\}_1,\{1,3\}_1,\{2,3\}_1\}_2,\\
&\quad \{\{1,2\}_1,\{1,3\}_1,\{2,3\}_1\}_3, \qquad (M_1(A_2,1)^1 \text{ part})\\
&\quad \{\{1,2\}_2\}_1,\{\{1,3\}_2\}_1,\{\{2,3\}_2\}_1,\\
&\quad \{\{1,2\}_2,\{1,3\}_2\}_1,\{\{1,2\}_2,\{2,3\}_2\}_1,\{\{1,3\}_2,\{2,3\}_2\}_1,\\
&\quad \{\{1,2\}_2,\{1,3\}_2,\{2,3\}_2\}_1,\{\{1,2\}_2,\{1,3\}_2,\{2,3\}_2\}_2,\\
&\quad \{\{1,2\}_2,\{1,3\}_2,\{2,3\}_2\}_3, \qquad (M_1(A_2,2)^1 \text{ part})\\
&\quad \{\{1,2\}_3\}_1,\{\{1,3\}_3\}_1,\{\{2,3\}_3\}_1,\\
&\quad \{\{1,2\}_3,\{1,3\}_3\}_1,\{\{1,2\}_3,\{2,3\}_3\}_1,\{\{1,3\}_3,\{2,3\}_3\}_1,
\end{aligned}
$$

$$\{\{1,2\}_3,\{1,3\}_3,\{2,3\}_3\}_1,\{\{1,2\}_3,\{1,3\}_3,\{2,3\}_3\}_2,$$
$$\{\{1,2\}_3,\{1,3\}_3,\{2,3\}_3\}_3, \qquad (M_1(A_2,3)^1 \text{ part})$$
$$\{\{1,2,3\}_1\}_1^1,\{\{1,2,3\}_1\}_2^1,\ \{\{1,2,3\}_2\}_1^1,\{\{1,2,3\}_2\}_2^1,$$
$$(M_1(A_3,1)^1 \text{ and } M_1(A_3,2)^1 \text{ part})$$
$$\{\{1,2,3\}_1\}_1^2,\{\{1,2,3\}_1\}_2^2,\ \{\{1,2,3\}_2\}_1^2,\{\{1,2,3\}_2\}_2^2\}$$
$$(M_1(A_3,1)^2 \text{ and } M_1(A_3,2)^2 \text{ part}).$$

For a positive integer n, let $X=[n]$. Define a poset $(Z,\leqslant)$, where

$$Z=[n]\cup M_1(X)$$

and for $S\in M_1(X)$ with $|S|=k$(so $S\in A_k$) and $x\in[n]$,

$$x\leqslant(S,i)=S_i \quad \text{if and only if} \quad x\in S_i,$$

where $1\le k\le n$, and $(S,i)=S_i$ is the i-th copy of S in (A_k,i) for all i, $1\le i\le m_k$. Then we can easily find that

$$(Z,\leqslant)\cong(X,\leqslant),$$

where X is the poset defined in Theorem 1.1 by Behrendt [1].

Now a family of ideal-homogeneous partially ordered sets of height 2 is constructed.

Construction of Z^2:
For a positive integer n, let $X=[n]$. Define a poset $(Z^2,\leqslant)$ as

$$Z^2=X\cup M_1(X)\cup M_2(X),$$

where $(m_1,m_2,\ldots,m_n)$, $m_k\ge 0$ is the multiplicity for A_k of $M_1(X)$, $\Omega_{\binom{n}{k}}=\left(a_1^k,a_2^k,\ldots,a_{\binom{n}{k}}^k\right)$ with $a_t^k\ge 0$ for $t=1,\ldots,\binom{n}{k}$ is for $B_t(A_k,i)$ of (A_k,i) of $M_2(X)$, and $b=(b_1,b_2,\ldots,b_n)$, $b_k\ge 1$ is for $M_1(A_k(X),i)$ for all $1\le i\le m_k$, $k=1,\ldots,n$. The order relations on Z^2 are defined as follows:

Order 1: For $S_i\in M_1(X)$ with $|S_i|=k$(so $S_i\in A_k$) and $x\in[n]$,

$$x\leqslant S_i=(S,i) \quad \text{if and only if} \quad x\in S_i,$$

where $1\le i\le m_k$ and $S_i=(S,i)$ is the i-th copy of S in (A_k,i) for all i, $1\le i\le m_k$, $k=1,\ldots,n$.

Order 2: For some k, i, and r, if $\Sigma_i\in M_1(A_k,i)^r\subset M_2(X)$ and $S_i\in(A_k,i)$, then

$$S_i\leqslant\Sigma_i \quad \text{if and only if} \quad S_i\in\Sigma_i$$

for all $\Sigma_i\in M_1(A_k,i)^r$, where $r=1,\ldots,b_k$.

Order 3: For some k, r, i and j with $i\ne j$, if $\Sigma_j\in M_1(A_k,j)^r\subset M_2(X)$ and $S_i\in(A_k,i)$ (Note that for $i\ne j$, (A_k,i) and (A_k,j) are basically identical hence we may say $\Sigma_i=\Sigma_j$ as a set nevertheless $\Sigma_i\in M_1(A_k,i)$ and $\Sigma_j\in M_1(A_k,j)$ and hence we may define order relation between

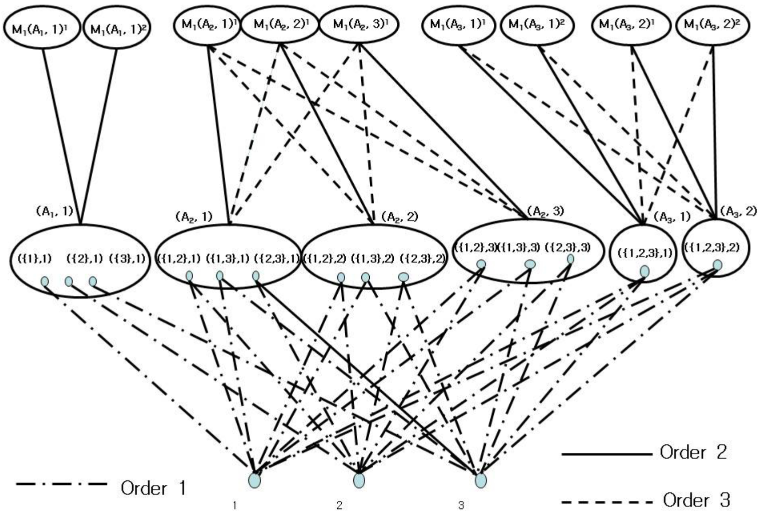

FIGURE 1. A poset of height 2 in Example 1

S_i and $\Sigma_j \in M_1(A_k, j)$ or S_j and $\Sigma_i \in M_1(A_k, i)$ in addition to Order 2 above) then

$$S_i \leqslant \Sigma_j \quad \text{if and only if} \quad S_i \in \Sigma_j$$

for all $\Sigma_j \in M_1(A_k, j)^r$, where $r = 1, \ldots, b_k$.

Consequently, if $x \leqslant S_i$ and $S_i \leqslant \Sigma_j$, then $x \leqslant \Sigma_j$ for every i and j, $1 \leq i, j \leq m_k$, $k = 1, \ldots, n$.

The poset $Z^2 = X \cup M_1(X) \cup M_2(X)$ in Example 1 with $n = 3$ and the order relations defined in Order 1, 2, and 3 above is roughly illustrated in Figure 1. The lines between the circled sets means there are order relations among the elements of them and the transitivity law holds.

3. Main results

Lemma 3.1. *Let $(Z^2, \leqslant)$ be the poset in Construction of Z^2 with order relations Order 1, 2, and 3. Suppose that $b_k = 1$ for all k where $b = (b_1, b_2, \ldots, b_n)$ is the multiplicity $M_1(A_k(X), i)$ for all $1 \leq i \leq m_k$, $k = 1, \ldots, n$. Then $(Z^2, \leqslant)$ is ideal-homogeneous of height* 2.

Proof. Let I_1 and I_2 be ideals of $(Z^2, \leqslant)$ and

$$\alpha : (I_1, \leqslant) \to (I_2, \leqslant)$$

an isomorphism. Assume $I \cap M_2(X) \neq \varnothing$ for all ideals in this proof, if not, it is of height 2. By the construction, it is clear that if $I_1 \cong I_2$, then $I_1 \cap M_1(A_k, \cdot) = \varnothing$

if and only if $I_2 \cap M_1(A_k, \cdot) = \varnothing$ for all $k = 1, 2, \ldots, n$. Hence there are finitely many numbers of k such that $I_1 \cap M_1(A_k, \cdot) \neq \varnothing$ and $I_2 \cap M_1(A_k, \cdot) \neq \varnothing$. Without loss of generality, for finite subset K_1 and J_1^k of $[n]$ and $[m_k]$, respectively, we can assume that

$$I_1 \cap M_2(X) \subset \bigcup_{k \in K_1} \bigcup_{i \in J_1^k} M_1(A_k, i)$$

and for finite subset K_2 and J_2^k of $[n]$ and $[m_k]$, respectively, we can assume that

$$I_2 \cap M_2(X) \subset \bigcup_{k \in K_2} \bigcup_{i \in J_2^k} M_1(A_k, i).$$

By the consideration above, it is clear that $K_1 = K_2$, if not, we conclude that I_1 is not isomophic to I_2. Therefore, without loss of generality (all other cases can be treated in the same way), we can assume that, for some k,

$$I_1 \cap M_2(X) \subset M_1(A_k, 1) \cup M_1(A_k, 2)$$

and

$$I_2 \cap M_2(X) \subset M_1(A_k, 1) \cup M_1(A_k, 3).$$

Let

$$I_1 \cap M_2(X) = B \cup E \quad \text{and} \quad I_2 \cap M_2(X) = C \cup F,$$

where $B = I_1 \cap M_1(A_k, 1)$, $C = I_2 \cap M_1(A_k, 3)$, $E = I_1 \cap M_1(A_k, 2)$, and $F = I_2 \cap M_1(A_k, 1)$. Without loss of generality again, we can assume that $B \cong_\alpha C$ and $E \cong_\alpha F$. If not, we have two cases (i) $B \cong_\alpha C \cup F'$ where $F' \subset M_1(A_k, 1)$ and (ii) $B \cong_\alpha C'$ where $C' \subset C$. For the case (i), there exists $F'' \subset M_1(A_k, 3)$ such that $F' \cong_\alpha F''$. Note that $D[F'] \cong D[F'']$. Hence new C is obtained by replacing F' by F'' to be $B \cong_\alpha C$. We may do similarly for the case (ii).

By restriction, α induces a bijection β_1 between $D[B] \cap (A_k, 1)$ and $D[C] \cap (A_k, 1)$ and bijection β_2 between $D[E] \cap (A_k, 1)$ and $D[F] \cap (A_k, 1)$ which can be extended together to a permutation α_1 on $(A_k, 1)$. That is, $\alpha_1(D[B] \cap (A_k, 1)) = D[C] \cap (A_k, 1)$, $\alpha_1(D[E] \cap (A_k, 1)) = D[F] \cap (A_k, 1)$, and all other elements in $(A_k, 1)$ are fixed. Note that, since $\Omega_{\binom{n}{k}} = \left(a_1^k, a_2^k, \ldots, a_{\binom{n}{k}}^k\right)$ works for all $M_1(A_k, i)$ for all $i = 1, 2, 3$, and by the order relation Order 3, we have $D[B] \cap (A_k, 1) = D[B] \cap (A_k, 2) = D[B] \cap (A_k, 3)$, $D[C] \cap (A_k, 1) = D[C] \cap (A_k, 2) = D[C] \cap (A_k, 3)$, $D[E] \cap (A_k, 1) = D[E] \cap (A_k, 2) = D[E] \cap (A_k, 3)$, and $D[F] \cap (A_k, 1) = D[F] \cap (A_k, 2) = D[F] \cap (A_k, 3)$. Therefore, by the similar way to the case of restriction of α to α_1, α induces a permutation α_2 on $(A_k, 2)$ and permutation α_3 on $(A_k, 3)$, respectively. That is, $\alpha_i(D[B] \cap (A_k, i)) = D[C] \cap (A_k, i)$, $\alpha_i(D[E] \cap (A_k, i)) = D[F] \cap (A_k, i)$, and all other elements in (A_k, i) are fixed for all $i = 2, 3$ which means that α_i are identical on (A_k, i) for all $i = 1, 2, 3$. That means $\alpha_1(S) = \alpha_2(S) = \alpha_3(S)$ for all $S \subset (A_k, \cdot)$. Let $\beta = \alpha_1 = \alpha_2(S) = \alpha_3(S)$.

For each $\Sigma \subset (A_k, i)$ with $|\Sigma| = j$ and $l \in \{1, 2\}$ let

$$U_l(\Sigma) = \{c \,|\, 1 \leq c \leq a_j^k \text{ with } \Sigma_c \in I_l\}.$$

Then a bijection $\beta_\Sigma : U_1(\Sigma) \to U_2(\beta(\Sigma))$ can be defined in order to associate to $\alpha(\Sigma, c) = (\beta(\Sigma), \beta_\Sigma(c))$ for all $i = 1, 2, 3$ and this can be extended to a permutation τ_Σ of $\{c \,|\, 1 \leq c \leq a_j^k\}$. Note that β_Σ and τ_Σ work for all (A_k, i) for all $i = 1, 2, 3$ in the same way. Also β induces a bijection between $I_1 \cap [n]$ and $I_2 \cap [n]$, which can be extended to a permutation τ of $[n]$. Now let $\sigma^*(m) = \tau(m)$ for $m \in [n]$, $\sigma^*(S) = \beta(S)$ for $S \in (A_k, 1)$, $\sigma^*(\Sigma, i) = (\beta(\Sigma), \tau_\Sigma(i))$ for $(\Sigma, i) \in (A_k, i)$ for all $i = 1, 2, 3$, and all other elements of the poset are fixed. Then it is not hard to see that σ^* is an automorphism of $(Z^2, \leqslant)$ such that $\sigma^* | I_1 = \sigma$. □

The following lemma is a generalization of Lemma 3.1. Because there is no restriction to multiplicities $M_1(A_k(X), i)$ for $1 \leq i \leq m_k$, $k = 1, \ldots, n$.

Lemma 3.2. *Let $(Z^2, \leqslant)$ be the poset defined in Construction of Z^2 with order relations Order 1, 2, and 3. Then $(Z^2, \leqslant)$ is ideal-homogeneous of height* 2.

Proof. If $(A_k(X), i) \neq \varnothing$, $M_1(A_k(X), i)^r$, $r = 1, \ldots, b_k$, are r copies of multiset of $(A_k(X), i)$ for fixed k. Then by Order 3, every r copies of $M_1(A_k(X), i)$ are in the up-set of $(A_k(X), i)$ for every $i \in \{1, \ldots, m_k\}$. Then the proof in Lemma 3.1 can be applied to the copies of $M_1(A_k(X), i)$ in exactly the same way. □

The following lemmas are special cases of Lemma 3.1, especially, there are no relations between $M(A_k(X), i)$ and $(A_k(X), j)$ for $i \neq j$ and there are the restrictions on the multiplicities $b = (b_1, b_2, \ldots, b_n)$.

Lemma 3.3. *Let $(Z^2, \leqslant)$ be the poset in Construction of Z^2 with order relations Order 1, 2 only except 3. Suppose that $b_k = 1$ for all k where $b = (b_1, b_2, \ldots, b_n)$ is multiplicities $M_1(A_k(X), i)$ for $1 \leq i \leq m_k$, $k = 1, \ldots, n$. Then $(Z^2, \leqslant)$ is ideal-homogeneous of height* 2.

Proof. This is a corollary of Lemma 3.2. □

Now with no restrictions on the multiplicities $b = (b_1, b_2, \ldots, b_n)$, we have the following lemma.

Lemma 3.4. *Let b_k be a positive integer for each k. Let $(Z^2, \leqslant)$ be the poset in Construction of Z^2 with order relations Order 1, 2 only except 3. Then $(Z^2, \leqslant)$ is ideal-homogeneous.*

Proof. This is a corollary of Lemma 3.3. □

Therefore we conclude our main theorem as follows:

Theorem 3.5. *Let $(P, \leqslant)$ be a finite partially ordered set of height 2. The followings are equivalent.*

(i) $(P, \leqslant)$ *is ideal-homogeneous.*
(ii) $(P, \leqslant)$ *is weakly ideal-homogeneous.*
(iii) $(P, \leqslant)$ *is one of the posets constructed in Lemmas* 3.1-3.4.

Proof. Trivially, (i) implies (ii). Assume that (ii) holds. Let $X = [n]$ be the set of minimal elements of $(P, \leqslant)$. If S_1 and S_2 are k-element subsets of X, where $1 \leq k \leq n$, then there is a permutation α of X mapping S_1 onto S_2. Since α is an automorphism of X, and X is an ideal of $(P, \leqslant)$, α can be extended to an automorphism β of $(P, \leqslant)$ by (ii). Suppose that

$$T_i = \{S \in M_1(X) \mid \text{for } x \in P \text{ we have } x \leqslant S \text{ if and only if } x \in S_i\}$$

for $i \in \{1, 2\}$. Then T_i is the set of points in $M_1(X)$ which is the common up-set of all points of S_i for $i \in \{1, 2\}$, and β has to map T_1 onto T_2. Therefore, for every k-element subset S_i of X there is the same number $m_k = |T_i|$ of elements in $M_1(X)$ which cover all elements of S_i and no others.

Here we want to specialize the abstract set S in T_i. Without loss generality, let us assume that we regard the order relation on $(P, \leqslant)$ as set inclusion. Clearly, for any k-element subset of X, $S_i \in T_i$ since $x \leqslant S_i$ if and only if $x \in S_i$. If $S \in T_i$, then S is the common up-set of all points of S_i and hence S should contain all the elements of S_i and no others. Hence we have $S = S_i$. Since there are $\binom{n}{k}$ k-element subsets of X, let $A_k(X) = \{S_1, S_2, \ldots, S_{\binom{n}{k}}\}$, where S_i are the k-element subset of X for all $k = 1, \ldots, n$. Then $A_k(X) \subset M_1(X)$ for all $k = 1, \ldots, n$. The number m_k means that there are m_k copies of S_i for $1 \leq i \leq \binom{n}{k}$. Hence m_k is the multiplicity of $A_k(X)$ for all $k = 1, \ldots, n$. If $m_k = 0$, then there is no $A_k(X)$ at all for all $k = 1, \ldots, n$.

Since for each k-element subset S_i, $1 \leq i \leq m_k$, there are m_k copies of it, it deduce that there are m_k copies of $A_k = A_k(X)$ after all. Hence we have

$$M_1(X) = \bigcup_{k=1}^{n} \bigcup_{i=1}^{m_k} (A_k, i). \tag{11}$$

If $S_i \in T_i$ and $S_i = \{a_1, a_2, \ldots, a_k\} \subset X = [n]$, then S_i can be denoted by $S_i = a_1 a_2 \cdots a_k$ or $(S_i, j) = (a_1 a_2 \cdots a_k, j)$, where j, $1 \leq j \leq m_k$, means that (S_i, j) is the j-th copy of S_i. Now let Σ_1 and Σ_2 be l-element subsets of $\bigcup_{i=1}^{m_k}(A_k, i)$ that is, Σ_i has S_i's as elements. There are some cases to be considered according to their membership where they belong to:

Case 1: For every $k = 1, 2, \ldots, n$, suppose Σ_1 and Σ_2 are subsets of (A_k, i) for some i, say $(A_k, 1)$. Without loss of generality, let us assume that $\Sigma_1 = \{S_1, S_2\}$ and $\Sigma_2 = \{S_1, S_3\}$ where $S_i \in (A_k, 1)$ for $i = 1, 2, 3$. Then $\beta(T_1 \cup T_2) = T_1 \cup T_3$. Hence β deduces a permutation τ of $(A_k, 1)$ mapping Σ_1 onto Σ_2 by restriction and also τ is a permutation of (A_k, i) for all i since $\tau(T_1 \cup T_2) = T_1 \cup T_3$ and $T_j \cap (A_k, i) \neq \emptyset$ for all i and $j = 1, 2, 3$. Suppose that

$$\mathfrak{T}_j = \{\Sigma \in M_2(X) \mid \text{for } S \in M_1(X) \text{ we have } S \leqslant \Sigma \text{ if and only if } S \in \Sigma_j\}$$

for $j \in \{1, 2\}$. Then $\mathfrak{T}_j$ is the set of points in $M_2(X)$ which is the common up-set of all points of Σ_j for $j \in \{1, 2\}$, and τ has to map $\mathfrak{T}_1$ onto $\mathfrak{T}_2$. Therefore, for every l-element subset Σ of $(A_k, 1)$, there exists the same number a_l^k of elements which cover all elements of Σ and no others. Let $K_l(X)$ be the set of l-element subsets of $(A_k, 1)$, that is, $K_l = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_{\binom{\binom{n}{k}}{l}}\}$ for all $l = 1, \ldots, \binom{n}{k}$. Since for each l-element subset Σ_j, $1 \le j \le \binom{\binom{n}{k}}{l}$, there are a_l^k copies of it, it implies that there are a_l^k copies of K_l for all $l = 1, \ldots, \binom{n}{k}$ after all. In all, we have the multiplicity $\Omega_{\binom{n}{k}} = \left(a_1^k, a_2^k, \ldots, a_{\binom{n}{k}}^k\right)$, where $a_t^k \ge 0$, for all $t = 1, \ldots, \binom{n}{k}$ for $M_1(A_k, 1)$ for all $k = 1, 2, \ldots, n$.

Subcase 1: For all k, suppose $M_1(A_k, 1)$ has multiplicity 1, that is $b_k = 1$ for all k with order relations Order 1 and 2 only. It is then clear that $(P, \leqslant)$ is isomorphic to $(Z^2, \leqslant)$ in Lemma 3.3.

Subcase 2: Suppose $M_1(A_k, 1)$ has multiplicity $b_k \ge 1$ for all k with order relations Order 1 and 2 only. It is then clear that $(P, \leqslant)$ is isomorphic to $(Z^2, \leqslant)$ in Lemma 3.4.

Case 2: For every $k = 1, 2, \ldots, n$, suppose $\Sigma_1 \subset (A_k, 1)$ and $\Sigma_2 \subset (A_k, 2)$, respectively. Without loss of generality, let us assume that $\Sigma_1 = \{S_1, S_2\}$ and $\Sigma_2 = \{S_3, S_4\}$. Then $\beta(T_1 \cup T_2) = T_3 \cup T_4$. Note that $T_j \cap (A_k, i) \ne \emptyset$ for all $i = 1, 2$ and $j = 1, 2, 3, 4$. Let Σ_2' be a copy of Σ_2 in $T_2 \cap (A_k, 1)$ and Σ_1' be a copy of Σ_1 in $T_1 \cap (A_k, 2)$. Hence β deduces a permutation τ of (A_k, i) by restriction for all $i = 1, 2$ and τ should map the commom up-set of all points of $\Sigma_1 \cup \Sigma_2'$ in $M_1(A_k, 1)$ onto the commom up-set of all points of $\Sigma_1' \cup \Sigma_2$ in $M_1(A_k, 2)$. Hence there are order relations between $(A_k, 1)$ and $M_1(A_k, 2)$, and between $(A_k, 2)$ and $M_1(A_k, 1)$. In general, for every l-element subset Σ of (A_k, i), there exists the same number a_l^k of elements which cover all elements of Σ, and no others. Similar to Case 1, we have the multiplicity $\Omega_{\binom{n}{k}} = \left(a_1^k, a_2^k, \ldots, a_{\binom{n}{k}}^k\right)$ where $a_t^k \ge 0$ for all $t = 1, \ldots, \binom{n}{k}$ for $M_1(A_k, i)$ for all $i = 1, \ldots, m_k$ and $k = 1, \ldots, n$.

Subcase 1: $M_1(A_k, i)$ has the multiplicities $b_k = 1$ for $k = 1, \ldots, n$. It is then clear that $(P, \leqslant)$ is isomorphic to $(Z^2, \leqslant)$ in Lemma 3.1.

Subcase 2: $M_1(A_k, i)$ has the multiplicities $b_k > 1$ for $k = 1, \ldots, n$. It is then clear that $(P, \leqslant)$ is isomorphic to $(Z^2, \leqslant)$ in Lemma 3.2. In all (iii) holds. By Lemmas 3.1-3.4, (iii) implies (i). □

4. Applications

For a positive integer n, let $X = [n]$. Define a poset $(P^2, \leqslant)$, where

$$P^2 = X \cup M_1(X) \cup M_2(X)$$

and $(m_1, m_2, \ldots, m_n) = (1, 1, \ldots, 1)$, $\Omega_{\binom{n}{k}} = (1, \ldots, 1)$ and $b_k = 1$ for all k. Then $M_1(X)$ is the power set of X except the empty set, and $M_2(X)$ is the set of power sets (except the empty set) of $A_k(X)$ for all $1 \le k \le n$.

Hence we have the following corollary.

Corollary 4.1. *With the order relations defined Order 1, and 2, $(P^2, \leqslant)$ is ideal-homogeneous.*

For given posets P and Q, $P \oplus Q$ is represented as a poset with a property that $x \leqslant y$ if and only if $x \in P$ and $y \in Q$. Let $\{B_1, B_2, \ldots, B_n\}$ be a set of antichains. Now construct a poset P which is isomorphic to $B_1 \oplus B_2 \oplus \cdots \oplus B_n$. Let $X_1 = [|B_1|]$ and the multiplicities $(m_1, m_2, \ldots, m_{|B_1|}) = (0, 0, \ldots, |B_2|)$. Then for all $i = 1, \ldots, |B_2|$ we have $(A_{|B_1|}, i)$. Let $T_2(X) = \bigcup_{i=1}^{|B_2|}(A_{|B_1|}, i)$. Then $|T_2(X)| = |B_2|$. Now, let $X_2 = [|B_2|]$ and the multiplicities $(m_1, m_2, \ldots, m_{|B_2|}) = (0, 0, \ldots, |B_3|)$. Then let $T_3(X) = \bigcup_{i=1}^{|B_3|}(A_{|B_2|}, i)$. Then $|T_3(X)| = |B_3|$. Likewise, define a poset P, recursively, and at last $X_{n-1} = [|B_{n-1}|]$ and the multiplicities $(m_1, m_2, \ldots, m_{|B_{n-1}|}) = (0, 0, \ldots, |B_n|)$ and $T_n(X) = \bigcup_{i=1}^{|B_n|}(A_{|B_{n-1}|}, i)$. Then $|T_n(X)| = |B_n|$. The order relations between X_i and $T_i(X_i)$ in each steps $i = 1, \ldots, n$ are defined by set inclusion. Then the poset P constructed is isomorphic to $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ of height n. Therefore we have the following:

Theorem 4.2. *Let $\{B_1, B_2, \ldots, B_n\}$ be a set of antichains. Then the poset P constructed above which is isomorphic to $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is ideal-homogeneous of height n.*

Proof. Let I and J be ideals of $P = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ and $\alpha : (I, \leqslant) \to (J, \leqslant)$ is an isomorphism. Let I_m and J_m be the set of maximal elements in I and J, respectively. Then for some k, I_m, $J_m \subset B_k$ and by construction we have $I = D[I_m]$ and $J = D[J_m]$ and hence $D[I_m] \cong D[J_m]$. Therefore, by restriction, α induces a bijection between $I_m \cap J_m$, which can be extended to a permutation on B_k. Also it can be extended to an automorphism σ^* of P such that $\sigma^*|I = \sigma$. □

We find some family of posets which are ideal homogeneous, but it is just partial solutions for the following Behrendt's problem [1].

Problem 2. Give a classification of all finite (weakly) ideal-homogeneous ordered sets.

References

[1] G. Behrendt, *Homogeneity in Finite Ordered Sets*, Order **10** (1993), no. 1, 65–75.

[2] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Second edition, Cambridge University Press, 2002.

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