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# GENERALIZATION ON PRODUCT DEGREE DISTANCE OF TENSOR PRODUCT OF GRAPHS

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Abstract. In this paper, the exact formulae for the generalized product degree distance, reciprocal product degree distance and product degree distance of tensor product of a connected graph and the complete multipartite graph with partite sets of sizes $m_0$, $m_1$, ..., $m_{r-1}$ are obtained.

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## 1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest $(u, v)$-path in $G$ and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. For two simple graphs $G$ and $H$ their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent whenever $g_1g_2$ is an edge in $G$ and $h_1h_2$ is an edge in $H$. Note that if $G$ and $H$ are connected graphs, then $G \times H$ is connected only if at least one of the graph is nonbipartite. The tensor product of graphs has been extensively studied in relation to the areas such as graph colorings, graph recognition, decompositions of graphs, design theory, see [2, 4, 5, 19, 21].

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [12]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

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Let $G$ be a connected graph. Then *Wiener index* of $G$ is defined as $W(G) = \frac{1}{2} \sum_{u,\, v \,\in\, V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of $G$. This definition can be further generalized in the following way: $W_\lambda(G) = \frac{1}{2} \sum_{u,\, v \,\in\, V(G)} d_G^\lambda(u,v)$, where $d_G^\lambda(u,v) = (d_G(u,v))^\lambda$ and $\lambda$ is a real number [13, 14]. If $\lambda = -1$, then $W_{-1}(G) = H(G)$, where $H(G)$ is Harary index of $G$. In the chemical literature also $W_{\frac{1}{2}}$ [26] as well as the general case $W_\lambda$ were examined [10, 15].

Dobrynin and Kochetova [6] and Gutman [11] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for a connected graph $G$ as $DD(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v)$, where $d_G(u)$ is the degree of the vertex $u$ in $G$. Similarly, the *product degree distance* or *Gutman index* of a connected graph $G$ is defined as $DD_*(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u) d_G(v) d_G(u,v)$. The *additively weighted Harary index*($H_A$) or *reciprocal degree distance*($RDD$) is defined in [3] as $H_A(G) = RDD(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. Similarly, Su et al. [25] introduce the *reciprocal product degree distance* of graphs, which can be seen as a product-degree-weight version of Harary index $RDD_*(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{d_G(u) d_G(v)}{d_G(u,v)}$. In [16], Hamzeh et al. recently introduced generalized degree distance of graphs. Hua and Zhang [18] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. Pattabiraman et al. [22, 23] have obtained the reciprocal degree distance of join, tensor product, strong product and wreath product of two connected graphs in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [3, 20, 24].

The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as $H_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v)$, where $\lambda$ is a real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if $\lambda = -1$, then $H_\lambda(G) = RDD(G)$. Similarly, *generalized product degree distance*, denoted by $H_\lambda^*(G)$, is defined as $H_\lambda^*(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u) d_G(v) d_G^\lambda(u,v)$. If $\lambda = 1$, then $H_\lambda^*(G) = DD_*(G)$ and if $\lambda = -1$, then $H_\lambda^*(G) = RDD_*(G)$. Therefore the study of the above topological indices are important and we try to obtain the results related to these indices. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et al. [16, 17]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. In this paper, the exact formulae for the generalized product degree distance, reciprocal product degree distance and product degree distance of tensor product $G \times K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$, where $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$ is the complete multipartite graph with partite sets of sizes $m_0,\, m_1,\, \ldots,\, m_{r-1}$ are obtained.

The *first Zagreb index* is defined as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2$ and the *second Zagreb index* is defined as $M_2(G) = \sum\limits_{uv \in E(G)} d_G(u) d_G(v)$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices were found to be successful in chemical and physico-chemical applications, especially in QSPR/QSAR studies, see [8, 9].

If $m_0 = m_1 = \ldots = m_{r-1} = s$ in $K_{m_0, m_1, \ldots, m_{r-1}}$ (the complete multipartite graph with partite sets of sizes $m_0, m_1, \ldots, m_{r-1}$), then we denote it by $K_{r(s)}$. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of $G$ induced by $S$. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by $d_G(S,T)$, we mean the sum of the distances in $G$ from each vertex of $S$ to every vertex of $T$, that is, $d_G(S,T) = \sum\limits_{s \in S, t \in T} d_G(s,t)$.

## 2. Generalized product degree distance of tensor product of graphs

Let $G$ be a connected graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, be the complete multiparite graph with partite sets $V_0, V_1, \ldots, V_{r-1}$ with $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \times K_{m_0, m_1, \ldots, m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$. For our convenience, we write

$$
\begin{aligned}
& V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) \\
= & \bigcup_{i=0}^{n-1} \left\{ v_i \times \bigcup_{j=0}^{r-1} V_j \right\} \\
= & \bigcup_{i=0}^{n-1} \left\{ \{v_i \times V_0\} \bigcup \{v_i \times V_1\} \bigcup \ldots \bigcup \{v_i \times V_{r-1}\} \right\} \\
= & \bigcup_{i=0}^{n-1} \left\{ B_{i0} \bigcup B_{i1} \bigcup \ldots \bigcup B_{i(r-1)} \right\}, \text{ where } B_{ij} = v_i \times V_j \\
= & \bigcup_{\substack{i=0 \\ j=0}}^{\substack{r-1 \\ n-1}} B_{ij}.
\end{aligned}
$$

Let $\mathscr{B} = \{B_{ij}\}_{\substack{i = 0,1,\ldots,n-1 \\ j = 0,1,\ldots,r-1}}$. If $v_i v_k \in E(G)$, then the subgraph $\langle B_{ij} \bigcup B_{kp} \rangle$ of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$ is isomorphic to $K_{|V_j|,|V_p|}$ or a totally disconnected graph according to $j \neq p$ or $j = p$. It is used in the proof of the next lemma. The proof of the following lemma follows easily from the structure and properties of $G \times K_{m_0, m_1, \ldots, m_{r-1}}$.

**Lemma 2.1.** *Let $G$ be a connected graph on $n \geq 2$ vertices and let $B_{ij}$, $B_{kp} \in \mathscr{B}$ of the graph $G \times K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 3$.*

(i) *For any two distinct vertices in $B_{ij}$, their distance is 2.*

(ii) *Distance between two distinct vertices one from $B_{ij}$ and another from $B_{ip}$, $j \neq p$ is 2.*
(iii) *Distance between two vertices one from $B_{ij}$ and another from $B_{kj}$, $i \neq k$ is 2 or 3 according as $v_iv_k$ lies on a triangle in $G$ or $v_iv_k \in E(G)$ and $v_iv_k$ does not lies on a triangle in $G$.*
(iv) *If $v_iv_k \in E(G)$, then distance between two vertices one in $B_{ij}$ and the another in $B_{kp}$, $i \neq k$, $j \neq p$ is 1.*
(v) *If $v_iv_k \notin E(G)$, then distance between the vertices one in $B_{ij}$ and another in $B_{kp}$ is $d_G(v_i, v_k)$.*

The proof of the following lemma follows easily from Lemma 2.1 and hence it is left to the reader. The lemma is used in the proof of the main theorem of this section.

**Lemma 2.2.** *Let $G$ be a connected graph on $n \geq 2$ vertices and let $B_{ij}$, $B_{kp} \in \mathscr{B}$ of the graph $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 3$.*
*(i) If $v_iv_k \in E(G)$, then*

$$d^{\lambda}_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_j m_p, & \text{if } j \neq p, \\ 2^{\lambda} m_j^2, & \text{if } j = p \text{ and } v_iv_k \text{ is on a triangle of } G, \\ 3^{\lambda} m_j^2, & \text{if } j = p \text{ and } v_iv_k \text{ is not on a triangle of } G. \end{cases}$$

*(ii) If $v_iv_k \notin E(G)$, then* $d^{\lambda}_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_j m_p \, d^{\lambda}_G(v_i, v_k), & if \ j \neq p, \\ m_j^2 \, d^{\lambda}_G(v_i, v_k), & if \ j = p. \end{cases}$

*(iii)* $d^{\lambda}_{G'}(B_{ij}, B_{ip}) = \begin{cases} 2^{\lambda} m_j(m_j - 1), & if \ j = p, \\ 2^{\lambda} m_j m_p, & if \ j \neq p. \end{cases}$

**Lemma 2.3.** *Let $G$ be a connected graph and let $B_{ij}$ in $G' = G \times K_{m_0, m_1, \ldots, m_{r-1}}$. Then the degree of a vertex $(v_i, u_j) \in B_{ij}$ in $G'$ is $d_{G'}((v_i, u_j)) = d_G(v_i)(n_0 - m_j)$, where $n_0 = \sum\limits_{j=0}^{r-1} m_j$.*

**Lemma 2.4.** *Let $n_0$ and $q$ be the number of vertices and edges of $K_{m_0, m_1, \ldots, m_{r-1}}$. Then the sums $\sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j m_p = 2q$, $\sum\limits_{j=0}^{r-1} m_j^2 = n_0^2 - 2q$, $\sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j^2 m_p = n_0 q - 3t = \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j m_p^2$, $\sum\limits_{j=0}^{r-1} m_j^3 = n_0^3 - 3n_0 q + 3t$ and $\sum\limits_{j=0}^{r-1} m_j^4 = n_0^4 - 4n_0^2 q + 2q^2 + 4n_0 t - 4\tau$, where $t$ and $\tau$ are the number of triangles and $K_4^{'s}$ in $K_{m_0, m_1, \ldots, m_{r-1}}$.*

**Theorem 2.5.** *Let $G$ be a connected graph with $n \geq 2$ vertices and let $E_2$ be the set of edges of $G$ which do not lie on any $C_3$ of it. If $n_0$ and $q$ are the number of vertices and edges of $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, respectively, then* $H^*_{\lambda}(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2 H^*_{\lambda}(G) + 2^{\lambda-1} M_1(G)\Big(4q^2 - n_0 q - 3t\Big) + \Big((2^{\lambda} -$

$1)M_2(G) + (3^\lambda - 2^\lambda)\sum_{v_iv_k \in E_2} d_G(v_i)d_G(v_k)\Big)\Big(2q^2 - 2n_0t - 4\tau\Big)$, *where* $t$ *and* $\tau$ *are the number of triangles and* $K_4^{'s}$ *in* $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$.

*Proof.* Let $G' = G \times K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$. Clearly,

$$\begin{aligned} H_\lambda^*(G') &= \frac{1}{2}\sum_{B_{ij},\, B_{kp} \in \mathscr{B}} d_{G'}(B_{ij})d_{G'}(B_{kp})d_{G'}^\lambda(B_{ij}, B_{kp}) \\ &= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ip})d_{G'}^\lambda(B_{ij}, B_{ip}) \\ &+ \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kj})d_{G'}^\lambda(B_{ij}, B_{kj}) \\ &+ \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kp})d_{G'}^\lambda(B_{ij}, B_{kp}) \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ij})d_{G'}^\lambda(B_{ij}, B_{ij})\Bigg) \\ &= \frac{1}{2}\{S_1 + S_2 + S_3 + S_4\}, \end{aligned} \tag{1}$$

where $S_1$ to $S_4$ are the sums of the above terms, in order.

We shall calculate $S_1$ to $S_4$ of (1) separately.

First we compute $S_1$. By Lemmas 2.2 and 2.3, we obtain:

$$\begin{aligned} &\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ip})d_{G'}^\lambda(B_{ij}, B_{ip}) \\ =\ &\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} \Big((n_0 - m_j)(n_0 - m_p)(d_G(v_i))^2\Big)\, 2^\lambda\, m_jm_p \end{aligned} \tag{2}$$

Summing (2) over $i = 0, 1, \ldots, n-1$, we get:

$$\begin{aligned} &\sum_{i=0}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ip})d_{G'}^\lambda(B_{ij}, B_{ip}) \\ =\ &\sum_{i=0}^{n-1} d_G^2(v_i)\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1} 2^\lambda\Big(n_0^2 - n_om_j - n_0m_p + m_jm_p\Big)m_jm_p. \end{aligned}$$

Now by Lemma 2.4, we have

$$S_1 \quad = \quad 2^\lambda\Big(2q^2 + 2n_0t + 4\tau\Big)M_1(G). \tag{3}$$

Next we compute $S_2$. For this, initially we calculate $S_2' = \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1} d_{G'}(B_{ij})d_{G'}(B_{kj})d^{\lambda}_{G'}(B_{ij},B_{kj})$. Let $E_1 = \{uv \in E(G)\,|\, uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$.

$$\begin{aligned}
S_2' = & \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} d_{G'}(B_{ij})d_{G'}(B_{kj})d^{\lambda}_{G'}(B_{ij},B_{kj}) \\
& + \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E_1}}^{n-1} d_{G'}(B_{ij})d_{G'}(B_{kj})d^{\lambda}_{G'}(B_{ij},B_{kj}) \\
& + \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E_2}}^{n-1} d_{G'}(B_{ij})d_{G'}(B_{kj})d^{\lambda}_{G'}(B_{ij},B_{kj}) \\
= & \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)m_j^2\ d^{\lambda}_G(v_i,v_k) \\
& + \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E_1}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)2^{\lambda}\, m_j^2 \\
& + \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E_2}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)3^{\lambda}\, m_j^2, \text{ by Lemmas 2.2 and 2.3} \\
= & \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)m_j^2\ d^{\lambda}_G(v_i,v_k) \\
& + \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E_1}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)\Big(2^{\lambda}\, m_j^2 + m_j^2 - m_j^2\Big) \\
& + \sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E_2}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)\Big(3^{\lambda}\, m_j^2 + m_j^2 - m_j^2\Big), \\
& \qquad\qquad \text{adding and subtracting } m_j^2 \text{ for both 2nd and 3rd sums.}
\end{aligned}$$

$$
\begin{aligned}
&= \Big( \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \notin E(G)}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)m_j^2\ d_G^{\lambda}(v_i,v_k) \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \in E_1}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)m_j^2\ d_G^{\lambda}(v_i,v_k) \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)m_j^2\ d_G^{\lambda}(v_i,v_k)\Big) \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \in E_1}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)(2^{\lambda}-1)m_j^2 \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)(3^{\lambda}-1)m_j^2,
\end{aligned}
$$

since $d_G^{\lambda}(v_i,v_k)=1$ if $v_iv_k\in E_1$ and $v_iv_k\in E_2$

$$
\begin{aligned}
&= \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)m_j^2\ d_G^{\lambda}(v_i,v_k) \\
&\quad +\Big( \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \in E_1}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)(2^{\lambda}-1)m_j^2 \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)(2^{\lambda}-1)m_j^2\Big) \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k \in E_2}}^{n-1} (n_0-m_j)^2 d_G(v_i)d_G(v_k)(3^{\lambda}-2^{\lambda})m_j^2 \\
&= (n_0-m_j)^2 m_j^2\Big(2H_{\lambda}^{*}(G)+2M_2(G)(2^{\lambda}-1) \\
&\quad +2(3^{\lambda}-2^{\lambda}) \sum_{v_iv_k\in E_2} d_G(v_i)d_G(v_k)\Big), \qquad (4)
\end{aligned}
$$

where $M_2(G)$ is the second Zagreb index of $G$. Note that each edge $v_iv_k$ of $G$ is being counted twice in the sum, namely, $v_iv_k$ and $v_kv_i$.

Now summing (4) over $j = 0, 1, \ldots, r-1$, we get,

$$S_2 = \Big(2H^*_\lambda(G) + 2(2^\lambda - 1)M_2(G) + 2(3^\lambda - 2^\lambda) \sum_{v_iv_k \in E_2} d_G(v_i)d_G(v_k)\Big) \sum_{j=0}^{r-1} \Big(n_0^2 m_j^2 + m_j^4 - 2n_0 m_j^3\Big).$$

Now by Lemma 2.4, we have

$$S_2 = \Big(2H^*_\lambda(G) + 2(2^\lambda - 1)M_2(G) + 2(3^\lambda - 2^\lambda) \sum_{v_iv_k \in E_2}^{n-1} d_G(v_i)d_G(v_k)\Big) \Big(2q^2 - 2n_0t - 4\tau\Big). \tag{5}$$

Next we compute $S_3$. By Lemmas 2.2 and 2.3, we obtain:

$$S_3 = \sum_{\substack{i,\,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,\,p=0 \\ j \neq p}}^{r-1} \Big((n_0 - m_j)d_G(v_i)(n_0 - m_p)d_G(v_k)\Big)m_j m_p \; d^\lambda_G(v_i, v_k)$$

$$= \sum_{\substack{i,\,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,\,p=0 \\ j \neq p}}^{r-1} \Big(n_0^2 m_j m_p - n_0 m_j^2 m_p - n_0 m_j m_p^2 + m_j^2 m_p^2\Big) d_G(v_i)d_G(v_k)d^\lambda_G(v_i, v_k).$$

By Lemma 2.4 and the definition of generalized product degree distance, we have

$$S_3 = 2H^*_\lambda(G)\Big(2q^2 + 2n_0t + 4\tau\Big). \tag{6}$$

Finally, we compute $S_4$. By Lemmas 2.2 and 2.3, we obtain:

$$S_4 = \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} 2^\lambda (n_0 - m_j)^2 d^2_G(v_i) m_j(m_j - 1)$$

$$= \Big(\sum_{i=0}^{n-1} d^2_G(v_i)\Big) \sum_{j=0}^{r-1} 2^\lambda (n_0 - m_j)^2 m_j(m_j - 1).$$

By Lemma 2.4, we have

$$S_4 = 2^\lambda M_1(G)\Big(2q^2 - n_0q - 2n_0t - 3t - 4\tau\Big). \tag{7}$$

Using (1) and the sums $\mathbf{S_1}$,$\mathbf{S_2}$,$\mathbf{S_3}$ and $\mathbf{S_4}$ in (3),(5),(6) and (7), respectively, we have,

$$H^*_\lambda(G') = 4q^2 H^*_\lambda(G) + 2^{\lambda-1}M_1(G)\Big(4q^2 - n_0q - 3t\Big) + \Big(2q^2 - 2n_0t - 4\tau\Big) \Big((2^\lambda - 1)M_2(G) + (3^\lambda - 2^\lambda) \sum_{v_iv_k \in E_2} d_G(v_i)d_G(v_k)\Big).$$

□

Using Theorem 2.5, we have the following corollaries.

**Corollary 2.6.** *Let $G$ be a connected graph with $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $H^*_\lambda(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2 H^*_\lambda(G) + 2^{\lambda-1} M_1(G)\Big(4q^2 - n_0 q - 3t\Big) + (2^\lambda - 1)M_2(G)\Big(2q^2 - 2n_0 t - 4\tau\Big),\ r \geq 3.$

For a triangle free graph, $E_2 = E(G)$ and hence $\sum\limits_{v_i v_k \,\in\, E_2} d_G(v_i) d_G(v_k) = M_2(G)$.

**Corollary 2.7.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $H^*_\lambda(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2 H^*_\lambda(G) + 2^{\lambda-1} M_1(G)\Big(4q^2 - n_0 q - 3t\Big) + (3^\lambda - 1)M_2(G)\Big(2q^2 - 2n_0 t - 4\tau\Big),\ r \geq 3.$

If $m_i = s,\ 0 \leq i \leq r-1$, in Theorem 2.5, Corollaries 2.6 and 2.7, we have the following corollaries.

**Corollary 2.8.** *Let $G$ be a connected graph with $n \geq 2$ vertices. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then* $H^*_\lambda(G \times K_{r(s)}) = r^2(r-1)^2 s^4 H^*_\lambda(G) + 2^{\lambda-1} M_1(G) r s^3 \Big(rs(r-1)^2 - r^2 + 2r - 1\Big) + \Big((2^\lambda - 1)M_2(G) + (3^\lambda - 2^\lambda) \sum\limits_{v_i v_k \,\in\, E_2} d_G(v_i) d_G(v_k)\Big) r(r-1)^2 s^4,\ r \geq 3.$

**Corollary 2.9.** *Let $G$ be a connected graph with $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $H^*_\lambda(G \times K_{r(s)}) = r^2(r-1)^2 s^4 H^*_\lambda(G) + 2^{\lambda-1} M_1(G) r s^3 \Big(rs(r-1)^2 - r^2 + 2r - 1\Big) + (2^\lambda - 1)M_2(G) r(r-1)^2 s^4,\ r \geq 3.$

**Corollary 2.10.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $H^*_\lambda(G \times K_{r(s)}) = r^2(r-1)^2 s^4 H^*_\lambda(G) + 2^{\lambda-1} M_1(G) r s^3 \Big(rs(r-1)^2 - r^2 + 2r - 1\Big) + (3^\lambda - 1)M_2(G) r(r-1)^2 s^4,\ r \geq 3.$

If we consider $s = 1$, in Corollaries 2.8, 2.9 and 2.10, we have the following corollaries.

**Corollary 2.11.** *Let $G$ be a connected graph with $n \geq 2$ vertices. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then* $H^*_\lambda(G \times K_r) = r^2(r-1)^2 H^*_\lambda(G) + 2^{\lambda-1} M_1(G) r(r-1)^3 + \Big((2^\lambda - 1)M_2(G) + (3^\lambda - 2^\lambda) \sum\limits_{v_i v_k \,\in\, E_2} d_G(v_i) d_G(v_k)\Big) r(r-1)^2,\ r \geq 3.$

**Corollary 2.12.** *Let $G$ be a connected graph on $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $H^*_\lambda(G \times K_r) = r^2(r-1)^2 H^*_\lambda(G) + 2^{\lambda-1} M_1(G) r(r-1)^3 + (2^\lambda - 1)M_2(G) r(r-1)^2$, *where $r \geq 3$.*

**Corollary 2.13.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $H^*_\lambda(G \times K_r) = r^2(r-1)^2 H^*_\lambda(G) + 2^{\lambda-1} M_1(G) r(r-1)^3 + (3^\lambda - 1) M_2(G) r(r-1)^2,\ r \geq 3.$

## 3. Reciprocal product degree distance of tensor product of graphs

Using $\lambda = -1$ in Theorem 2.5, we have the reciprocal product degree distance of the graph $G \times K_{m_0, m_1, \ldots, m_{r-1}}$.

**Corollary 3.1.** *Let $G$ be a connected graph with $n \geq 2$ vertices. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then* $RDD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2 RDD_*(G) + \frac{M_1(G)}{4}\Big(4q^2 - n_0 q - 3t\Big) - \Big(\frac{M_2(G)}{2} + \frac{1}{6}\sum\limits_{v_i v_k \in E_2} d_G(v_i) d_G(v_k)\Big)\Big(2q^2 - 2n_0 t - 4\tau\Big).$

Using Corollary 3.1, we have the following corollaries.

**Corollary 3.2.** *Let $G$ be a connected graph with $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $RDD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2 RDD_*(G) + \frac{M_1(G)}{4}\Big(4q^2 - n_0 q - 3t\Big) - \frac{M_2(G)}{2}\Big(2q^2 - 2n_0 t - 4\tau\Big),\ r \geq 3.$

**Corollary 3.3.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $RDD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2 RDD_*(G) + \frac{M_1(G)}{4}\Big(4q^2 - n_0 q - 3t\Big) - \frac{2M_2(G)}{3}\Big(2q^2 - 2n_0 t - 4\tau\Big),\ r \geq 3.$

If $m_i = s,\ 0 \leq i \leq r-1$, in Corollaries 3.1, 3.2 and 3.3, we have the following corollaries:

**Corollary 3.4.** *Let $G$ be a connected graph with $n \geq 2$ vertices. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then* $RDD_*(G \times K_{r(s)}) = r^2(r-1)^2 s^4\ RDD_*(G) + \frac{M_1(G)}{4} rs^3\Big(rs(r-1)^2 - r^2 + 2r - 1\Big) - \Big(\frac{M_2(G)}{2} + \frac{1}{6}\sum\limits_{v_i v_k \in E_2} d_G(v_i) d_G(v_k)\Big) rs^4 (r-1)^2,\ r \geq 3.$

**Corollary 3.5.** *Let $G$ be a connected graph with $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $RDD_*(G \times K_{r(s)}) = r^2(r-1)^2 s^4\ RDD_*(G) + \frac{M_1(G)}{4} rs^3\Big(rs(r-1)^2 - r^2 + 2r - 1\Big) - \frac{M_2(G)}{2} rs^4 (r-1)^2,\ r \geq 3.$

**Corollary 3.6.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $RDD_*(G \times K_{r(s)}) = r^2(r-1)^2 s^4\ RDD_*(G) + \frac{M_1(G)}{4} rs^3\Big(rs(r-1)^2 - r^2 + 2r - 1\Big) - \frac{2M_2(G)}{3} rs^4 (r-1)^2,\ r \geq 3.$

If we consider $s = 1$ in Corollaries 3.4, 3.5, 3.6, we have the following corollaries.

**Corollary 3.7.** *Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then* $RDD_*(G \times K_r) = r(r-1)^2\Big(rRDD_*(G)+\frac{1}{4}(r-1)M_1(G)-\frac{1}{2}M_2(G)-\frac{1}{6}\sum_{v_iv_k\in E_2} d_G(v_i)d_G(v_k)\Big),\ r \geq 3.$

**Corollary 3.8.** *Let $G$ be a connected graph on $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $RDD_*(G \times K_r) = r(r-1)^2\Big(rRDD_*(G) + \frac{1}{4}(r-1)M_1(G) - \frac{1}{2}M_2(G)\Big)$, *where* $r \geq 3$.

**Corollary 3.9.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $RDD_*(G \times K_r) = r(r-1)^2\Big(rRDD_*(G) + \frac{1}{4}(r-1)M_1(G) - \frac{2}{3}M_2(G)\Big),\ r \geq 3.$

By direct calculations we obtain expressions for the values of the Harary indices of $K_n$ and $C_n$. $H(K_n) = \frac{n(n-1)}{2}$ and $H(C_n) = n\Big(\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 1$ when $n$ is even, and $n\Big(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big)$ otherwise. Similarly, $RDD_*(K_n) = \frac{n(n-1)^3}{2}$,$RDD(K_n) = n(n-1)^2$ and $RDD_*(C_n) = RDD(C_n) = 4H(C_n)$.

One can observe that $M_1(C_n) = 4n,\ n \geq 3,\ M_1(P_1) = 0,\ M_1(P_n) = 4n - 6,\ n > 1$ and $M_1(K_n) = n(n-1)^2$. Similarly, $M_2(P_n) = 4(n-2)$, $M_2(C_n) = 4n$, and $M_2(K_n) = \frac{n(n-1)^3}{2}$.

Using Corollaries 3.8 and 3.9, we obtain the reciprocal product degree distance of the graphs $K_n \times K_r$ and $C_n \times K_r$.

**Example 1.** $(i)$ $RDD_*(K_n \times K_r) = \frac{nr}{12}(n-1)^2(r-1)^2(6nr - 4n - 3r + 1)$.

$$(ii)\ RDD_*(C_n \times K_r) = \begin{cases} r(r-1)^2\Big(4rH(C_n) + n(r-3)\Big), & \text{if } n = 3, \\ r(r-1)^2\Big(4rH(C_n) + \frac{n}{3}(3r-11)\Big), & \text{if } n > 3. \end{cases}$$

## 4. Product degree distance of tensor product of graphs

Using $\lambda = 1$ in Theorem 2.5, we have the product degree distance of the graph $G \times K_{m_0, m_1, \ldots, m_{r-1}}$.

**Corollary 4.1.** *Let $G$ be a connected graph with $n \geq 2$ vertices and let $E_2$ be the set of edges of $G$ which do not lie on any $C_3$ of it. If $n_0$ and $q$ are the numbers of vertices and edges of $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 3$, respectively, then* $DD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2\, DD_*(G) + M_1(G)\Big(4q^2 - n_0q - 3t\Big) + \Big(M_2(G) + \sum_{v_iv_k \in E_2} d_G(v_i)d_G(v_k)\Big)\Big(2q^2 - 2n_0t - 4\tau\Big),\ r \geq 3.$

Using Corollary 4.1, we have the following corollaries.

**Corollary 4.2.** *Let $G$ be a connected graph with $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $DD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2\, DD_*(G) + M_1(G)\Big(4q^2 - n_0q - 3t\Big) + M_2(G)\Big(2q^2 - 2n_0t - 4\tau\Big),\ r \geq 3.$

**Corollary 4.3.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $DD_*(G \times K_{m_0, m_1, \ldots, m_{r-1}}) = 4q^2\, DD_*(G) + M_1(G)\Big(4q^2 - n_0 q - 3t\Big) + 2M_2(G)\Big(2q^2 - 2n_0 t - 4\tau\Big),\ r \geq 3.$

If $m_i = s,\ 0 \leq i \leq r-1$, in Corollaries 4.1,4.2 and 4.3, we have the following corollaries.

**Corollary 4.4.** *Let $G$ be a connected graph with $n \geq 2$ vertices. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then* $DD_*(G \times K_{r(s)}) = r^2(r-1)^2 s^4 DD_*(G) + M_1(G) r s^3 \Big(rs(r-1)^2 - r^2 + 2r - 1\Big) + \Big(M_2(G) + \sum\limits_{v_i v_k \,\in\, E_2} d_G(v_i) d_G(v_k)\Big)$ $rs^4(r-1)^2,\ r \geq 3.$

**Corollary 4.5.** *Let $G$ be a connected graph with $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $DD_*(G \times K_{r(s)}) = r^2(r-1)^2 s^4 DD_*(G) + M_1(G) r s^3 \Big(rs(r-1)^2 - r^2 + 2r - 1\Big) + M_2(G) r s^4 (r-1)^2,\ r \geq 3.$

**Corollary 4.6.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $DD_*(G \times K_{r(s)}) = r^2(r-1)^2 s^4 DD_*(G) + M_1(G) r s^3 \Big(rs(r-1)^2 - r^2 + 2r - 1\Big) + 2M_2(G) r s^4 (r-1)^2,\ r \geq 3.$

If we consider $s = 1$, in Corollaries 4.4, 4.5 and 4.6, we have the following corollaries.

**Corollary 4.7.** *Let $G$ be a connected graph with $n \geq 2$ vertices. Let $E_2$ be the set of edges of $G$ which do not lie on a triangle. Then* $DD_*(G \times K_r) = r^2(r-1)^2 DD_*(G) + M_1(G) r (r-1)^3 + \Big(M_2(G) + \sum\limits_{v_i v_k \in E_2} d_G(u_i) d_G(u_k)\Big) r(r-1)^2,\ r \geq 3.$

**Corollary 4.8.** *Let $G$ be a connected graph on $n \geq 2$ vertices. If each edge of $G$ is on a $C_3$, then* $DD_*(G \times K_r) = r^2(r-1)^2 DD_*(G) + M_1(G) r (r-1)^3 + M_2(G) r (r-1)^2,\ r \geq 3.$

**Corollary 4.9.** *If $G$ is a connected triangle free graph on $n \geq 2$ vertices, then* $DD_*(G \times K_r) = r^2(r-1)^2 DD_*(G) + M_1(G) r (r-1)^3 + 2M_2(G) r (r-1)^2,\ r \geq 3.$

One can observe that $DD_*(P_n) = \frac{(n-1)}{3}(2n^2 - 4n + 3),\ n \geq 3,\ DD_*(K_n) = \frac{n(n-1)^3}{2}$ and

$$DD_*(C_n) = \begin{cases} \frac{n^3}{2}, & \text{if } n \text{ is even} \\ \frac{n(n^2-1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Using Corollaries 4.8 and 4.9, we obtain the product degree distance of the following graphs.

**Example 2.** $(i)$ $DD_*(K_n \times K_r) = \frac{nr(n-1)^2(r-1)^2}{2}(nr+n+r-3)$.
$(i)$ $DD_*(P_n \times K_r) = \frac{r(r-1)^2}{3}(2n^3r - 6n^2r + 19nr - 21r + 12n - 30)$.

$$(ii)\ DD_*(C_n \times K_r) = \begin{cases} \frac{nr}{2}(r-1)^2(n^2r+8r+8), & \text{if } n \text{ is even} \\ \frac{nr}{2}(r-1)^2(n^2r+7r+8), & \text{if } n > 3 \text{ is odd} \\ \frac{nr}{2}(r-1)^2(n^2r+7r), & \text{if } n = 3. \end{cases}$$

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