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CHARACTERIZATION OF GLOBALLY-UNIQUELY-SOLVABLE PROPERTY OF A CONE-PRESERVING Z-TRANSFORMATION ON EUCLIDEAN JORDAN ALGEBRAS

YOON J. SONG

ABSTRACT. Let V be a *Euclidean Jordan algebra* with a *symmetric cone* K. We show that for a **Z**-transformation L with the additional property $L(K) \subseteq K$ (which we will call 'cone-preserving'),

$$\textbf{GUS} \iff \text{strictly copositive on } K \iff \text{monotone} + \textbf{P}.$$

Specializing the result to the Stein transformation $S_A(X) := X - AXA^T$ on the space of real symmetric matrices with the property $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$, we deduce that

$$S_A \ \textbf{GUS} \iff I \pm A \text{ positive definite.}$$

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1. Introduction

Consider a real finite dimensional Hilbert space V and let K be a proper cone in V, i.e., K is a closed convex cone in V with $K \cap (-K) = \{0\}$ and $K - K = V$. Let the dual of K be defined by

$$K^* := \{x \in V : \langle x, y\rangle \geq 0 \quad \text{for all} \quad y \in K\}.$$

Under this setting, the *cone linear complementarity problem*, which is denoted by LCP(L, K, q), is to find $x \in V$ such that

$$x \in K,\ y := L(x) + q \in K^*, \quad \text{and} \quad \langle x, y\rangle = 0. \tag{1}$$

There were many attempts in characterizing the transformation L so that the LCP(L, K, q) has a unique solution for any given $q \in V$, the so-called Globally-Uniquely-Solvable property (**GUS**-property, for short).

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Restricting V to a *Euclidean Jordan algebra* and K to a *symmetric cone* in V, Gowda and Sznajder in 2007 [6] characterized **GUS**-property of an algebra automorphism L (an invertible mapping such that $L(x \circ y) = L(x) \circ L(y)$ (p.469 of [6] where '$\circ$' denotes the Jordan product). They showed that for such transformation, $\mathbf{GUS} = \mathbf{P}$ (Theorem 5.1 of [6]).

In 2013, Yang and Yuan [14] characterized **GUS**-property when K is the second-order cone (also known as the Lorentz cone). They provided sufficient and necessary conditions on the matrix M so that M has the **GUS**-property (Theorem 2 of [14]).

In 2015, Balaji characterized the **GUS**-property of L when $(V, \circ, \langle \cdot, \cdot \rangle)$ is the Jordan spin algebra. He [1] showed that
$\mathbf{GUS} = \mathbf{P}+$ L positive semidefinite on the boundary of K.

In this paper, we focus our attention to a *symmetric cone* K in a *Euclidean Jordan algebra* V where L is a **Z**-transformation with resepect to K such that $L(K) \subseteq K$. The term, **Z**-transformation, is defined by Gowda and J. Tao [8] to designate a linear transformation $L : V \to V$ is such that

$$x \in K, \quad y \in K^*, \quad \text{and} \quad \langle x, y \rangle = 0 \quad \Longrightarrow \quad \langle L(x), y \rangle \leq 0.$$

There are many examples of **Z**-transformations including

$$L(X) = X - \sum_{i=1}^{k} B_i X {B_i}^T$$

that ha applications in control theory [10]. For many other examples, we refer the readers to Section 3 of [8].

Gowda and Sznajder in 2007 [6] characterized **GUS**-property of L where both L and $-L$ are **Z**-transformations (so-called *Lyapunov-like* transformation (Section 7 of [6])). They showed (Theorem 7.1 of [6]) that such L has the **GUS**-property if and only if L is positive stable (that is, every eigenvalue of L has positive real parts) and positive semidefinite. Recently, Kong, Tao, et al. characterized the **GUS**-property of a **Z**-transformation on a Lorentz cone (Theorem 3.2 of [9]).

We show in this paper that for a linear transformation $L : V \to V$ where V is a Euclidean Jordan algebra with a symmetric cone K such that $L(K) \subseteq K$:

$$\mathbf{GUS} \cap \mathbf{Z} = [\text{strictly copositive on } K] \cap \mathbf{Z} = [\text{monotone}] \cap \mathbf{P} \cap \mathbf{Z}.$$

The composition of the paper is as follows: In section 2, we provide notations, definitions and preliminaries. In section 3, the characterization of the **GUS**-property of a cone-preserving (i.e., $L(K) \subseteq K$) **Z**-transformation is established. In section 4, we specialize the result to the Stein transformation S_A with $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$ and get some matrix theoretic result relating the **GUS**-property. A conclusion is given in section 5.

2. Definitions and Preliminaries

First, we describe all the notations which are used in this paper. The '$*$' denotes the Hadamard product (entry-wise product) of matrices whereas '$\circ$' denotes the Jordan product. For a vector $d \in R^n$, $d \geq (>)0$ means every component of d is nonnegative (positive). The notation $d \leq 0$ means $-d \geq 0$. We write $Diag(d)$ to mean a diagonal matrix whose diagonal is the vector d. The notation $\|d\|$ denotes the Euclidean norm, that is, $\|d\| = \sqrt{\sum_{i=1}^{n} {d_i}^2}$.

For a diagonal matrix D, $diag(D)$ means a vector whose entries are the diagonal of D. We write $\rho(A)$ to denote the spectral radius of A, i.e., the maximum distance from the origin to an eigenvalue of A in the complex plane. Finally, $tr(AB)$ means the sum of the diagonal elements of the matrix product AB.

We now list definitions and state preliminary results that go along with the corresponding definition.

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$ where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in V$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$, and
(iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in V$.

In a Euclidean Jordan algebra V, the set of squares

$$K := \{x \circ x : x \in V\}$$

is a *symmetric cone* (Faraut and Korányi [3], p.46). This K is a self-dual (i.e., $K = K^*$) closed convex cone, and proper. For standard examples of Euclidean Jordan algebras, we refer the readers to p. 465 of [6].

This symmetric cone K induces a (partial) order on V (section 2.1 of [7]):

$$x \leq y \quad (\text{or } y \geq x) \iff \quad y - x \in K.$$

Since K is self-dual, closed and convex cone, let Π_K denote the metric projection onto K, that is, for an $x \in V$, $x^+ := \Pi_K(x)$ if and only if $x^+ \in K$ and $\|x - x^+\| \leq \|x - y\|$ for all $y \in K$. It is well known that x^+ is unique (by Moreau decomposition [11]) and any $x \in V$ can be written as

$$x = x^+ - x^-, \quad \text{where} \quad x^- = x^+ - x,$$

with $x^+, x^- \geq 0$ and $\langle x^+, x^- \rangle = 0$.

In a Euclidean Jordan algebra V, for a given $x \in V$, Gowda and Sznajder (p.464 of [6]) defined the *Lyapunov transformation* $L_x : V \to V$ as

$$L_x(z) = x \circ z,$$

and called the elements x and y *operator commute* if $L_xL_y = L_yL_x$. Moreover, for $x, y \in K$, if $\langle x, y\rangle = 0$, then x and y operator commute and $x \circ y = 0$ (Proposition 2.2 of [6]).

A linear transformation L has the

- **GUS** (Globally-Uniquely-Solvable)-property if for all $q \in V$, LCP(L, K, q) has a unique solution.
- **P**-property if
  $[x \circ L(x) \leq 0$ and x and $L(x)$ operator commute$] \implies x = 0$.
- **Q**-property if for all $q \in V$, LCP(L, K, q) has a solution.
  It is known (see Theorem 12, 14, and 17 of [7]) that

$$\mathbf{GUS} \quad \Longrightarrow \quad \mathbf{P} \quad \Longrightarrow \quad \mathbf{Q}.$$

A linear transformation $L : V \to V$ is called

- *copositve on* K if $\langle L(x), x\rangle \geq 0 \quad \forall x \in K$; *strictly copositive on* K if $\langle L(x), x\rangle > 0$ for all nonzero $x \in K$.
- a **Z**-transformation with respect to K (written $L \in \mathbf{Z}(K)$ for short) if

$$x \in K, \quad y \in K^*, \quad \text{and} \quad \langle x, y\rangle = 0 \quad \Rightarrow \quad \langle L(x), y\rangle \leq 0.$$

  From the definition, it is clear that if $L \in \mathbf{Z}(K)$, then $L^T \in \mathbf{Z}(K^*)$.
  Moreover, when $L \in \mathbf{Z}(K)$ where K is a symmetric cone in a Euclidean Jordan algebra, the following are equivalent (extracted from Theorem 6 of [8]):
  (i) L has the **Q**-property with respect to K.
  (ii) L^{-1} exists and $L^{-1}(K) \subseteq K$.
- *Monotone*:
  A linear transformation $L : V \to V$ is called *monotone* if $\langle L(x), x\rangle \geq 0 \quad \forall x \in V$.

Specializing Corollary 6 of [4] to our case, we get that if $L \in \mathbf{Z}(K)$ is strictly copositive on K (where K is a symmetric cone in a Euclidean Jordan algebra), then L is monotone.

In addition, Theorem 22 [7] states that for a monotone transformation,

$$\mathbf{P} \quad \Longleftrightarrow \quad \mathbf{GUS}.$$

3. GUS-property of a cone-preserving Z-transformation on a Euclidean Jordan Algebra

We will call a **Z**-transformation with respect to K as simply **Z**-transformation if not otherwise specified. We first show that for a **Z**-transformation $L : V \to V$ with $L(K) \subseteq K$, **GUS**-property is equivalent to L being strictly copositive on the symmetric cone K.

Theorem 3.1. *Consider a Euclidean Jordan algebra $(V, \circ, \langle\cdot,\cdot\rangle)$ and a symmetric cone K in V. For a **Z**-transformation $L : V \to V$ with $L(K) \subseteq K$, the following are equivalent:*

(a) *L is strictly copositive on K.*
(b) *L has the* **GUS**-*property, i.e., $LCP(L, K, q)$ has a unique solution for all $Q \in V$.*

Proof. For $(a) \Rightarrow (b)$: Note that our assumption leads to L being monotone. Suppose there exists $q \in V$ with two distinct solutions x_1 and x_2 in K. Let $y_i = L(x_i) + q \in K$ for $i = 1, 2$, and let $z = x_1 - x_2$. Linearity and monotonicity of L together imply

$$0 \leq \langle z, L(z)\rangle = \langle x_1 - x_2, y_1 - y_2\rangle = -\langle x_2, y_1\rangle - \langle x_1, y_2\rangle \leq 0,$$

whence $\langle z, L(z)\rangle = 0$. Then, by writing $z = z^+ - z^-$, we get

$$\begin{aligned} 0 &= \langle z^+ - z^-, L(z^+ - z^-)\rangle \\ &= \langle z^+, L(z^+)\rangle + \langle z^-, L(z^-)\rangle - \langle z^+, L(z^-)\rangle - \langle z^-, L(z^+)\rangle. \end{aligned}$$

Since both z^+ and z^- are in K with at least one of them nonzero, the sum of the first two terms is *positive* by strict copositivity of L on K. Moreover, since the symmetric cone K is self-dual and $L \in \mathbf{Z}(K)$, $L^T \in \mathbf{Z}(K)$. Then $\langle z^+, L(z^-)\rangle = \langle L^T(z^+), z^-\rangle$ with z^+ and z^- orthogonal, therefore, $\langle z^+, L(z^-)\rangle$ and $\langle z^-, L(z^+)\rangle$ are both nonpositive by **Z**-property of L^T. However, these observations lead to an absurd conclusion, namely,

$$0 = \langle z, L(z)\rangle > 0.$$

Hence L has the **GUS**-property.

For $(b) \Rightarrow (a)$: Suppose there exists $0 \neq x \in K$ such that

$$\langle x, L(x)\rangle \leq 0. \tag{2}$$

Since L has the **GUS**-property, it has the **Q**-property. So L^{-1} exists and $L^{-1}(K) \subseteq K$. Since $L(K) \subseteq K$ and $0 \neq x \in K$, one can find $0 \neq y \in K$ such that $x = L^{-1}(y)$. Then (2) becomes $\langle L^{-1}(y), y\rangle = \langle x, y\rangle \leq 0$. Since both x, y are elements of K, this means $x \circ y = 0 = x \circ L(x)$ for some nonzero x, and x and $L(x)$ operator commute. But note that since **GUS** $\Rightarrow$ **P** for L, this is a contradiction. This completes the proof. □

Now we establish the main result of the paper.

Corollary 3.2. *For a* **Z**-*transformation L on a Euclidean Jordan algebra with a symmetric cone K with $L(K) \subseteq K$, the following are equivalent:*

(a) *L is strictly copositive on K.*
(b) *L has the* **GUS**-*property.*
(c) *L is monotone and has the* **P**-*property.*

Proof. The equivalence of (a) and (b) follows from Theorem 3.1. For $(b) \Rightarrow (c)$: if L has the **GUS**-property, then by Theorem 3.1, L is strictly copositive on K, and hence L is monotone, and **GUS** $\Rightarrow$ **P**. Hence (c) holds. For the converse, **P** $=$ **GUS** for a monotone transformation. This completes the proof. □

4. GUS-property of the Stein Transformation $S_A : \mathbf{S}^n \to \mathbf{S}^n$ when $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$

In this section, we restrict our attention to the space of real symmetric matrices $\mathbf{S}^n$ and the Stein transformation $L = S_A : \mathbf{S}^n \to \mathbf{S}^n$, defined by

$$S_A(X) := X - AXA^T, \quad A \in R^{n\times n}.$$

We recall below some known results for S_A:

(a) $S_A \in \mathbf{Z}(\mathbf{S}^n_+)$ (Example 3, Section 3 [8]).
(b) $\rho(A) < 1 \iff S_A \in \mathbf{P}$ (Theorem 11 [5]).

Using Corollary 3.2, we try to characterize the **GUS**-property of S_A in terms of the matrix A when $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$.

First we recall useful results for nonnegative matrices:

Lemma 4.1 (Part of Theorem (1.1) of [2]). *If A is a nonnegative square matrix, then*

(a) *$\rho(A)$, the spectral radius of A, is an eigenvalue,*
(b) *A has a nonnegative eigenvector corresponding to $\rho(A)$.*

We characterize the strict copositivity of S_A in terms of the matrix A, and relate it to the **GUS**-property.

Theorem 4.2. *For $A \in R^{n\times n}$, consider the Stein Transformation $S_A(X) = X - AXA^T$ with $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$. Then the following are equivalent.*

(a) *S_A is strictly copositive on $\mathbf{S}^n_+$.*
(b) *$\rho(U^TAU * U^TAU) < 1$ for any U orthogonal.*
(c) *S_A has the **GUS**-property, i.e., SDLCP(S_A, Q) has a unique solution for all $Q \in \mathbf{S}^n$.*

Proof. For $(a) \Rightarrow (b)$: suppose S_A is strictly copositive on $\mathbf{S}^n_+$, i.e., $\langle X, S_A(X)\rangle > 0$ for all $0 \neq X \in \mathbf{S}^n_+$. Fix an arbitrary orthogonal matrix U and an arbitrary vector $d \geq 0$. Let $B = U^TAU$, $D = Diag(d)$, and $X = UDU^T$. Then,

$$\begin{aligned} \langle X, S_A(X)\rangle &= \langle UDU^T, UDU^T - AUDU^TA^T\rangle \\ &= \langle D, D - BDB^T\rangle = \langle d, (I - B*B)d\rangle \\ &= \langle d, d\rangle - \langle d, (B*B)d\rangle \\ &> 0 \quad \forall\, 0 \neq d \geq 0. \end{aligned}$$

So,

$$\langle d, (B*B)d\rangle < \langle d, d\rangle \quad \forall\, 0 \neq d \geq 0. \tag{3}$$

From Lemma 4.1, the matrix $B*B$ has a nonnegative (hence *real*) eigenvector corresponding to the *real* eigenvalue $\rho(B*B)$. If x is such an eigenvector, then $\langle x, (B*B)x\rangle = \rho(B*B)\langle x, x\rangle < \langle x, x\rangle$ by (3). Hence $\rho(B*B) < 1$. Since U is arbitrary, we get the desired result.

For $(b) \Rightarrow (c)$: suppose S_A is not GUS. Then there exists $Q \in \mathbf{S}^n$ such that two distinct symmetric positive semidefinite matrices X_1 and X_2 solve SDLCP(S_A, Q). Let $Y_i = S_A(X_i)+Q$ for $i = 1, 2$, and let $Z = X_1 - X_2$. Then by linearity of S_A,

$$\langle Z, S_A(Z)\rangle = \langle X_1 - X_2, Y_1 - Y_2\rangle \leq 0.$$

Let us write $Z = UDU^T$ in a way that diagonal of D is the eigenvalues of Z in decreasing order. That is, $D = D^+ - D^-$ where D^+ has only positive eigenvalues of Z (if any) or zero on its diagonal and D^- has *absolute values* of negative eigenvalues of Z (if any) or zero on its diagonal. Note that both $D^+, D^- \succeq 0$, $D^+D^- = 0$, and at least one of D^+ or D^- is nonzero since $Z \neq 0$. Now let $B = U^TAU$, and $d = diag(D)$, to get

$$\begin{aligned} 0 &\geq \langle Z, S_A(Z)\rangle = \langle d, (I - B * B)d\rangle \\ &= \langle d^+ - d^-, (I - B * B)(d^+ - d^-)\rangle \\ &= \langle d^+, (I - B * B)d^+\rangle + \langle d^-, (I - B * B)d^-\rangle \\ &\quad -\langle d^+, (I - B * B)d^-\rangle - \langle d^-, (I - B * B)d^+\rangle. \end{aligned}$$

Note that
$\langle d^+, (I-B*B)d^+\rangle = \langle UD^+U^T, UD^+U^T - AUD^+U^TA^T\rangle$, and a similar equation holds for $\langle d^-, (I - B * B)d^-\rangle$ as well. Since both d^+ and d^- are nonnegative with at least one of d^+ or d^- is nonzero, the sum of these two terms is *positive* by strictly copositivity of S_A. Moreover,
$\langle d^+, (I - B * B)d^-\rangle = \langle d^+, d^-\rangle - \langle d^+, (B * B)d^-\rangle \leq 0$ because d^+ and d^- are orthogonal and $(B * B)$ is a nonnegative matrix. Same conclusion holds for $\langle d^-, (I - B * B)d^+\rangle$. However, these observations lead to an absurd conclusion, namely, $0 \geq \langle Z, S_A(Z)\rangle > 0$. Hence S_A is GUS and $(b) \Rightarrow (c)$ is established.

Finally, the implication $(c) \Rightarrow (a)$ is obtained by applying the result of Theorem 3.1 for $V = \mathbf{S}^n, K = \mathbf{S}^n_+$, and $L = S_A$. □

Now, We give a matrix-theoretic characterization of the **GUS**-property of the Stein Transformation $S_A(X) = X - AXA^T$ with $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$ below and show that, interestingly, strict copositivity and strict monotonicity are equivalent in such case, i.e., if we let $K = \mathbf{S}^n_+$ and $V = \mathbf{S}^n$, then

$$\langle X, S_A(X)\rangle > 0 \quad \forall X \in K \quad \Longleftrightarrow \quad \langle X, S_A(X)\rangle > 0 \quad \forall X \in V.$$

That is, S_A positive definite on the symmetric cone K implies S_A positive definite for the whole space V.

Corollary 4.3. *For $A \in R^{n\times n}$, consider the Stein Transformation $S_A(X) = X - AXA^T$ with $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$. Then the following are equivalent.*

(a) *$I \pm A$ is positive definite.*
(b) $\rho(A) < 1$
(c) $1 \notin \sigma(A)\sigma(A)$
(d) *S_A has the* **GUS***-property, i.e., SDLCP(S_A, Q) has a unique solution for all $Q \in \mathbf{S}^n$.*
(e) *S_A is strictly copositive on $\mathbf{S}^n_+$.*
(f) *S_A is strictly monotone.*

(g) $\rho(U^T AU * U^T AU) < 1$ *for any* U *orthogonal.*

Proof. The equivalence of the statements $(a), (b)$, and (c) is from Theorem 2.5 (p233) of [13]. The equivalence of the statements (c) and (f) is from Theorem 2.4 (p232) of [13]. On the other hand, the equivalence of the statements (d) and (e) is from Theorem 4.2 of this paper. Moreover, $(d) \Rightarrow (b)$ ($\mathbf{GUS} \Rightarrow \mathbf{P}$), and $(f) \Rightarrow (d)$ (strict monotonicity $\Rightarrow$ **GUS**). Hence, the statements (a) through (f) are all equivalent. Finally, the equivalence of (g) and (d) follows from Theorem 4.2 of this paper. $\square$

5. Conclusion

In this paper, we showed that for a **Z**-transformation L with respect to a symmetric cone K in a Euclidean Jordan algebra such that $L(K) \subseteq K$, the following implications hold:

$$\text{strictly copositive on } K \quad \Longleftrightarrow \quad \mathbf{GUS} \quad \Longleftrightarrow \quad \text{monotone} + \mathbf{P}.$$

By specializing the result to the Stein transformation S_A on the space of real symmetric matrices such that $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$, we have

$$S_A \ \mathbf{GUS} \quad \Longleftrightarrow \quad I \pm A \text{ positive definite.}$$

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Yoon J. Song received her M.Sc. and Ph.D. at University of Maryland Baltimore County, USA. She has been at Soongsil University since 2010. Her research interests are semidefinite complementarity problems in optimization.

Department of Mathematics, Soongsil University, Seoul, 156-743, Korea.
e-mail: yoonjsong@ssu.ac.kr