J. Appl. Math. & Informatics Vol. **34***(2016), No. 3 - 4, pp. 249 - 255*
http://dx.doi.org/10.14317/jami.2016.249

SOME INTEGRALS ASSOCIATED WITH MULTIINDEX MITTAG-LEFFLER FUNCTIONS

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Abstract. The object of the present paper is to establish two interesting unified integral formulas involving Multiple (multiindex) Mittag-Leffler functions, which is expressed in terms of Wright hypergeometric function. Some deduction from these results are also considered.

AMS Mathematics Subject Classification : 33C60, 33E12.
Key words and phrases : Multiple (multiindex) Mittag-Leffler Function, Wright Hypergeometric Function and Integrals.

1. Introduction

A number of integral formulas involving a variety of special functions have been developed by many authors (see [2, 3, 4, 5], also see [7] and [10]) motivated by their work. We presents two integral formulae involving the Multiple (multiindex) Mittag-Leffler function, which are expressed in terms of Wright Hypergeometric function. Some interesting cases of our main results are also considered.

The generalization of the generalized hypergeometric series ${}_pF_q$ (1.9) is due to Fox [1] and Wright ([12, 13, 14]) who studied the asymptotic expansion of the generalized Wright Hypergeometric function defined by (see [7, p.21]).

$$ {}_p\psi_q\left[\begin{matrix}(\alpha_1,\ A_1),\ (\alpha_2, A_2),\ \cdots, (\alpha_p, A_p);\\ (\beta_1,\ B_1),\ (\beta_2, B_2),\ \cdots,\ (\beta_q, B_q);\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q}\Gamma(\beta_j + b_j k)}\frac{z^k}{k!} \qquad (1.1) $$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0 $$

Received July 28, 2015. Revised October 6, 2015. Accepted October 21, 2015. *Corresponding author.

A special case of (1) is

$$ {}_p\psi_q\left[\begin{matrix}(\alpha_1,1),\ (\alpha_2,1),\ \cdots,\ (\alpha_p,1);\\ (\beta_1,1),\ (\beta_2,1),\ \cdots,\ (\beta_q,1);\end{matrix} z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\ {}_pF_q\left[\begin{matrix}(\alpha_1),\ \cdots,\ \cdots,\ (\alpha_p);\\ (\beta_1,1),\ \cdots,\ \cdots,\ (\beta_q);\end{matrix} z\right], $$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [8, section 1.5])

$$ {}_pF_q\left[\begin{matrix}(\alpha_1),\ (\alpha_2),\ \cdots,\ (\alpha_p);\\ (\beta_1),\ (\beta_2),\ \cdots,\ (\beta_q);\end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} = {}_pF_q(\alpha_1,\cdots\alpha_p;\beta_1,\cdots\beta_q;z) $$

where $(\lambda)_n$ is called the pochhammer's symbol [8].

Kiryakova [11] defined the multiple (multiindex) Mittag-Leffler function as follows. Let $m > 1$ be an integer, $\rho_1,\cdots\rho_m > 0$ and $\mu_1,\cdots,\mu_m$ be arbitrary real numbers. By means of "multiindices" $(\rho_i)(\mu_i)$, we introduce the so-called multiindex(m-tuple,multiple) Mittag-Leffler functions.

$$ E_{(\frac{1}{\rho_i}),(\mu_i)}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})} \tag{1.2} $$

In what follows the relations of (1.2) with some known special functions:

(i) For m=2, if we put $\frac{1}{\rho_1}=\alpha$, $\frac{1}{\rho_2}=0$ and $\mu_1=1$, $\mu_2=1$, in (1.2), we have

$$ E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)} \tag{1.3} $$

(ii) For m=2, if we put $\frac{1}{\rho_1}=\alpha$, $\frac{1}{\rho_2}=0$ and $\mu_1=\beta$, $\mu_2=1$, in (1.2), we have

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)} \tag{1.4} $$

(iii) For m=2, if we put $\frac{1}{\rho_1}=1$, $\frac{1}{\rho_2}=1$ and $\mu_1=\nu+1$, $\mu_2=1$ and replacing z by $\frac{-z^2}{4}$, in (1.2), we have (see [11])

$$ E_{(1,1),(1+\nu,1)}\left(\frac{-z^2}{4}\right) = \left(\frac{2}{z}\right)^{\nu}\ J_\nu(z) \tag{1.5} $$

where $J_\nu(z)$ is a Bessel function of first kind (see [8, 9]).

(iv) For m=2, if we put $\frac{1}{\rho_1}=1$, $\frac{1}{\rho_2}=1$ and $\mu_1=\frac{3-\nu+\mu}{2}$, $\mu_2=\frac{3+\nu+\mu}{2}$ and replacing z by $\frac{-z^2}{4}$, in (1.2), we have (see [11])

$$ E_{(1,1),(\frac{3-\nu+\mu}{2},\frac{3+\nu+\mu}{2})}\left(\frac{-z^2}{4}\right) = \frac{1}{z^{\mu+1}}\ 4S_{\mu,\nu}(z) \tag{1.6} $$

where $S_{\mu,\nu}(z)$ is a Struve function (see [8, 9]).

(v) For m=2, if we put $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 1$ and $\mu_1 = \frac{3}{2}$, $\mu_2 = \frac{3+2\nu}{2}$ and replacing z by $\frac{-z^2}{4}$, in (1.2), we have (see [11])

$$E_{(1,1),(\frac{3}{2},\frac{3+2\nu}{2})}\left(\frac{-z^2}{4}\right) = \frac{1}{z^{\mu+1}}\ 4H_\nu(z) \tag{1.7}$$

where $H_\nu(z)$ is a Lommel function (see [8, 9]).

In the present investigations, we shall be invoking following relations, see Obhettinger [6].

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^\mu \frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)} \tag{1.8}$$

provided $0 < R(\mu) < R(\lambda)$.

2. Main results

Two generalized integral formulae, which have been established in this section, are expressed in terms of generalized (Wrigt) hypergeometric function, Multiple Mittag-Leffler, with suitable arguments in the integrands, is invoked in the analysis of the results under investigation.

First Integral

The following integral formula holds true:

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}E_{(\frac{1}{\rho_i}),(\mu_i)}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\mu}a^{\mu-\lambda}\Gamma(2\mu)\ {}_3\psi_{m+2}\left[\begin{matrix}(1+\lambda,1), & (\lambda-\mu,1),\ (1,1); \\ \\ (\mu_1,\dfrac{1}{\rho_1}),\ \ldots,\ (\mu_m,\dfrac{1}{\rho_m}), & (\lambda,1),\ (1+\lambda+\mu,1);\end{matrix}\ \frac{y}{a}\right]. \tag{2.1}$$

Second Integral

The following integral formula holds true:

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}E_{(\frac{1}{\rho_i}),(\mu_i)}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\mu}a^{\mu-\lambda}\Gamma(\lambda-\mu)\ {}_3\psi_{m+2}\left[\begin{matrix}(1+\lambda,1), & (2\mu,2), \quad (1,1); \\ \\ (\mu_1,\dfrac{1}{\rho_1}),\ \ldots,\ (\mu_m,\dfrac{1}{\rho_m}), & (\lambda,1),\ (1+\lambda,2);\end{matrix}\ \frac{y}{2}\right]. \tag{2.2}$$

Proof of (2.1).

In order to derive (2.1), we denote the left- hand side of (2.1) by I, expressing $E_{(\frac{1}{\rho_i}),\mu_i}(z)$ as a series with the help of (1.2) and then interchanging the order of integration and summation, which is justified by uniform convergence of the involved series under the given conditions, we get

$$\begin{aligned}
\mathrm{I} &= \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{(\frac{1}{\rho_i}),(\mu_i)}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx \\
&= \sum_{k=0}^{\infty} \frac{(y)^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots(\mu_m+\frac{k}{\rho_m})} \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda-k}\, dx
\end{aligned}$$

Evaluating the above integral with the help of (1.8), we get

$$\mathrm{I} = 2^{1-\mu}\mathrm{a}^{\mu-\lambda}\Gamma(2\mu)\sum_{\mathrm{k}=0}^{\infty} \frac{\Gamma(1+\mathrm{k}+\lambda)\Gamma(\lambda+\mathrm{k}-\mu)\Gamma(1+\mathrm{k})}{\Gamma(\mu_1+\frac{\mathrm{k}}{\rho_1})\ldots\Gamma(\mu_\mathrm{m}+\frac{\mathrm{k}}{\rho_\mathrm{m}})\Gamma(1+\mathrm{k}+\lambda+\mu)\Gamma(\lambda+\mathrm{k})}\left(\frac{\mathrm{y}}{\mathrm{a}}\right)^{\mathrm{k}}\frac{1}{\mathrm{k}!}.$$

Finally, summing the above series with the help of (1.1), we arrive at the right hand side of (2.1). This completes the proof of first result.

Proof of (2.2).

Similarly, to derive (2.2), we denote the left- hand side of (2.2) by I', expressing $E_{(\frac{1}{\rho_i}),\mu_i}(z)$ as a series with the help of (1.2) and then interchanging the order of integration and summation, which is justified by uniform convergence of the involved series under the given conditions, we get

$$\begin{aligned}
\mathrm{I}' &= \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{(\frac{1}{\rho_i}),(\mu_i)}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx \\
&= \sum_{k=0}^{\infty} \frac{(y)^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots(\mu_m+\frac{k}{\rho_m})} \int_0^\infty x^{\mu+k-1}(x+a+\sqrt{x^2+2ax})^{-\lambda-k}\, dx
\end{aligned}$$

Evaluating the above integral with the help of (1.8), we get

$$\mathrm{I}' = 2^{1-\mu}\mathrm{a}^{\mu-\lambda}\Gamma(2\mu)\sum_{\mathrm{k}=0}^{\infty} \frac{\Gamma(1+\mathrm{k}+\lambda)\Gamma(2\mathrm{k}+2\mu)\Gamma(\mathrm{k}+1)}{\Gamma(\mu_1+\frac{\mathrm{k}}{\rho_1})\ldots\Gamma(\mu_\mathrm{m}+\frac{\mathrm{k}}{\rho_\mathrm{m}})\Gamma(1+2\mathrm{k}+\lambda)\Gamma(\lambda+\mathrm{k})}\left(\frac{\mathrm{y}}{2}\right)^{\mathrm{k}}\frac{1}{\mathrm{k}!}.$$

Finally, summing the above series with the help of (1.1), we arrive at the right hand side of (2.2). This completes the proof of second result.

3. Special Cases

In this section, we define some special cases of our main results:

$$\begin{aligned}
1.\quad & \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\; E_\alpha\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx \\
&= 2^{1-\mu}\; a^{\mu-\lambda}\;\Gamma(2\mu)\;{}_3\psi_3\left[\begin{matrix}(1+\lambda,1), & (\lambda-\mu,1), & (1,1); \\ & & & \frac{y}{a} \\ (1,\alpha), & (1+\mu+\lambda,1), & (\lambda,1); \end{matrix}\right]
\end{aligned} \tag{3.1}$$

$$2.\ \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\ E_\alpha\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\mu}\ a^{\mu-\lambda}\ \Gamma(\lambda-\mu)\ {}_3\psi_3\left[\begin{matrix}(1+\lambda,1), & (2\mu,2), & (1,1); & \\ & & & \dfrac{y}{2}\\ (1,\alpha), & (1+\lambda,2), & (\lambda,1); & \end{matrix}\right] \tag{3.2}$$

The above results (3.1) and (3.2) can be established with the help of integrals (2.1) and (2.2) by taking m=2, $\frac{1}{\rho_1}=\alpha$, $\frac{1}{\rho_2}=0$, $\mu_1=1$, $\mu_2=1$ and using equation (1.3).

$$3.\ \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\ E_{\alpha,\beta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\mu}\ a^{\mu-\lambda}\ \Gamma(2\mu)\ {}_3\psi_3\left[\begin{matrix}(1+\lambda,1), & (\lambda-\mu,1), & (1,1); & \\ & & & \dfrac{y}{a}\\ (\beta,\alpha), & (1+\mu+\lambda,1), & (\lambda,1); & \end{matrix}\right] \tag{3.3}$$

$$4.\ \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\ E_{\alpha,\beta}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\mu}\ a^{\mu-\lambda}\ \Gamma(\lambda-\mu)\ {}_3\psi_3\left[\begin{matrix}(1+\lambda,1), & (2\mu,2), & (1,1); & \\ & & & \dfrac{y}{2}\\ (\beta,\alpha), & (1+\lambda,2), & (\lambda,1); & \end{matrix}\right] \tag{3.4}$$

The above results (3.3) and (3.4) can be established with the help of integrals (2.1) and (2.2) by taking m=2, $\frac{1}{\rho_1}=\alpha$, $\frac{1}{\rho_2}=0$, $\mu_1=\beta$, $\mu_2=1$ and using equation (1.4).

$$5.\ \int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda+\frac{\nu}{2}}\ J_\nu\left[2i\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)^{\frac{1}{2}}\right]dx$$
$$= i^\nu\ (y)^{\frac{\nu}{2}}\ 2^{1-\mu}\ a^{\mu-\lambda}\ \Gamma(2\mu)\ {}_2\psi_3\left[\begin{matrix}(1+\lambda,1), & (\lambda-\mu,1); & & \\ & & & \dfrac{y}{a}\\ (1+\mu+\lambda,1), & (1+\nu,1), & (\lambda,1); & \end{matrix}\right] \tag{3.5}$$

$$6.\ \int_0^\infty x^{\frac{2\mu-\nu-2}{2}}(x+a+\sqrt{x^2+2ax})^{-\lambda+\frac{\nu}{2}}\ J_\nu\left[2i\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)^{\frac{1}{2}}\right]dx$$
$$= i^\nu\ (y)^{\frac{\nu}{2}}\ 2^{1-\mu}\ a^{\mu-\lambda}\ \Gamma(\lambda-\mu)\ {}_2\psi_3\left[\begin{matrix}(1+\lambda,1), & (2\mu,2); & & \\ & & & \dfrac{y}{2}\\ (1+\lambda,2), & (1+\nu,1), & (\lambda,1); & \end{matrix}\right] \tag{3.6}$$

The above results (3.5) and (3.6) can be established with the help of integrals (2.1) and (2.2) by taking m=2, $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 1$, $\mu_1 = \nu + 1$, $\mu_2 = 1$, replacing z by $\frac{-z^2}{4}$ and using equation (1.5) (see [11]).

$$7.\quad \int_0^\infty x^{\mu-1}(x + a + \sqrt{x^2 + 2ax})^{\frac{\mu-2\lambda+1}{2}} S_{\mu,\nu}\left[2i\left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right)^{\frac{1}{2}}\right] dx$$
$$= i^{\mu+1}\ (y)^{\frac{\mu+1}{2}}\ a^{\mu-\lambda}\ \Gamma(2\mu)\ {}_3\left[\begin{array}{l} (1+\lambda, 1), \quad (\lambda-\mu, 1), \quad (1, 1); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{y}{a} \\ (1+\mu+\lambda, 1),\ (\frac{3-\nu+\mu}{2}, 1),\ (\frac{3+\nu+\mu}{2}, 1),\ (\lambda, 1); \end{array}\right] \tag{3.7}$$

$$8.\quad \int_0^\infty x^{\frac{\mu-3}{2}}(x + a + \sqrt{x^2 + 2ax})^{\frac{\mu-2\lambda+1}{2}}\ S_{\mu,\nu}\left[2i\left(\frac{xy}{x + a + \sqrt{x^2 + 2ax}}\right)^{\frac{1}{2}}\right] dx$$
$$= i^{\mu+1}\ (y)^{\frac{\mu+1}{2}}\ a^{\mu-\lambda}\ \Gamma(\lambda-\mu)\ {}_3\psi_4\left[\begin{array}{l} (1+\lambda, 1), \quad (2\mu, 2), \quad (1, 1); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{y}{2} \\ (1+\lambda, 2),\ (\frac{3-\nu+\mu}{2}, 1),\ (\frac{3+\nu+\mu}{2}, 1),\ (\lambda, 1); \end{array}\right] \tag{3.8}$$

The above results (3.7) and (3.8) can be established with the help of integrals (2.1) and (2.2) by taking m=2, $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 1$, $\mu_1 = \frac{3-\nu+\mu}{2}$, $\mu_2 = \frac{3+\nu+\mu}{2}$, replacing z by $\frac{-z^2}{4}$ and using equation (1.6) (see [11]).

$$9.\quad \int_0^\infty x^{\mu-1}(x + a + \sqrt{x^2 + 2ax})^{\frac{\nu-2\lambda+1}{2}} H_\nu\left[2i\left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right)^{\frac{1}{2}}\right] dx$$
$$= i^{\mu+1}\ (y)^{\frac{\nu+1}{2}}\ 2^{\nu-\mu}\ a^{\mu-\lambda}\Gamma(2\mu)\ {}_3\psi_4\left[\begin{array}{l} (1+\lambda, 1), \quad (\lambda-\mu, 1), \quad (1, 1); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{y}{a} \\ (1+\mu+\lambda, 1),\ (\frac{3}{2}, 1),\ (\frac{3+2\nu}{2}, 1),\ (\lambda, 1); \end{array}\right] \tag{3.9}$$

$$10.\quad \int_0^\infty x^{\frac{2\mu-\nu-3}{2}}(x + a + \sqrt{x^2 + 2ax})^{\frac{\mu-2\lambda+1}{2}} H_\nu\left[2i\left(\frac{xy}{x + a + \sqrt{x^2 + 2ax}}\right)^{\frac{1}{2}}\right] dx$$
$$= i^{\mu+1}\ (y)^{\frac{\mu+1}{2}}\ 2^{\nu-\mu}\ a^{\mu-\lambda}\Gamma(\lambda-\mu)\ {}_3\psi_4\left[\begin{array}{l} (1+\lambda, 1),\ (2\mu, 2),\ (1, 1); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{y}{2} \\ (1+\lambda, 2),\ (\frac{3}{2}, 1),\ (\frac{3+2\nu}{2}, 1),\ (\lambda, 1); \end{array}\right] \tag{3.10}$$

The above results (3.9) and (3.10) can be established with the help of integrals (2.1) and (2.2) by taking m=2, $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 1$, $\mu_1 = \frac{3}{2}$, $\mu_2 = \frac{3+2\nu}{2}$, replacing z by $\frac{-z^2}{4}$ and using equation (1.7).

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