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SUFFICIENT CONDITIONS FOR SOME HAMILTONIAN PROPERTIES AND K-CONNECTIVITY OF GRAPHS

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Abstract. For a connected graph $G = (V, E)$, its inverse degree is defined as $\sum_{v \in V} \frac{1}{d(v)}$. Using an upper bound for the inverse degree of a graph obtained by Cioabă in [4], we in this note present sufficient conditions for some Hamiltonian properties and k-connectivity of a graph.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [3]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. We use $\delta = d_1 \leq d_2 \leq \cdots \leq d_n = \Delta$ to denote the degree sequence of G. If G is connected, we define its inverse degree as $\sum_{v \in V} \frac{1}{d(v)}$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. A graph G is called Hamilton-connected if for each pair of vertices in G there is a Hamiltonian path between them. In this note, we will use an upper bound for the inverse degree of a graph obtained by Cioabă in [4] to present sufficient conditions for Hamiltonian, traceable, Hamilton-connected, and k-connected graphs.

2. Main results

The main results of this paper are as follows.

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Theorem 2.1. *Let G be a connected graph of order $n \geq 3$ and size e. If*
$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < 1 + \frac{1}{n-2} + \frac{1}{\Delta},$$
then G is Hamiltonian.

Theorem 2.2. *Let G be a connected graph of order $n \geq 4$ and size e. If*
$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < \frac{n}{n-2} + \frac{1}{n-3} + \frac{1}{\Delta},$$
then G is traceable.

Theorem 2.3. *Let G be a connected graph of order $n \geq 3$ and size e. If*
$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < \frac{1}{2} + \frac{1}{n-2} + \frac{2}{\Delta},$$
then G is Hamilton-connected.

Theorem 2.4. *Let G be a connected graph of order $n \geq k + 1 \geq 3$ and size e. If*
$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < \frac{2}{n+k-3} + \frac{n-k+1}{2(n-2)} + \frac{k-1}{\Delta},$$
then G is k-connected.

3. Lemmas

In order to prove the theorems above, we need the following results as our lemmas.

Lemma 3.1 ([1])**.** *Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If*
$$d_k \leq k < \frac{n}{2} \Longrightarrow d_{n-k} \geq n - k,$$
then G is Hamiltonian.

Lemma 3.2 ([1])**.** *Let G be a graph of order $n \geq 2$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If*
$$d_k \leq k - 1 \leq \frac{n}{2} - 1 \Longrightarrow d_{n+1-k} \geq n - k,$$
then G is traceable.

Lemma 3.3 ([1])**.** *Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If*
$$2 \leq k \leq \frac{n}{2},\ d_{k-1} \leq k \Longrightarrow d_{n-k} \geq n - k + 1,$$
then G is Hamilton-connected.

Lemma 3.4 ([2]). *Let G be a graph of order $n \geq 2$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ and let $1 \leq k \leq n-1$. If*

$$1 \leq i \leq \lfloor \frac{n-k+1}{2} \rfloor,\ d_i \leq i+k-2 \Longrightarrow d_{n-k+1} \geq n-i,$$

then G is k-connected.

Lemma 3.5 ([4]). *Let G be a connected graph of order n and size e. Then*

$$\sum_{v \in V} \frac{1}{d(v)} \leq \frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n-1-\frac{2e}{n}\right).$$

Notice that Lemma 3.1 is Corollary 3 on Page 209 in [1] or Theorem 4.5 on Page 57 in [3], Lemma 3.2 is Corollary 6 on Page 210 in [1], Lemma 3.3 is Theorem 12 on Page 218 in [1], Lemma 3.4 is the Corollary on Page 163 in [2], and Lemma 3.5 is from Theorem 9 on Page 1963 in [4].

4. Proofs

Proof of Theorem 2.1. Let G be a graph satisfying the conditions in Theorem 2.1. Suppose that G is not Hamiltonian. Then, from Lemma 3.1, there exists an integer k such that $d_k \leq k < \frac{n}{2}$ and $d_{n-k} \leq n-k-1$. Obviously, $k \geq 1$. Therefore, from Lemma 3.5, we have that

$$\begin{aligned}
&\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n-1-\frac{2e}{n}\right)\\
&\geq \sum_{v \in V} \frac{1}{d(v)}\\
&= \frac{1}{d_1} + \cdots + \frac{1}{d_k} + \frac{1}{d_{k+1}} + \cdots + \frac{1}{d_{n-k}} + \frac{1}{d_{n+1-k}} + \cdots + \frac{1}{d_n}\\
&\geq \frac{k}{d_k} + \frac{n-2k}{d_{n-k}} + \frac{k}{d_n}\\
&\geq \frac{k}{k} + \frac{n-2k}{n-k-1} + \frac{k}{\Delta}\\
&\geq 1 + \frac{1}{n-2} + \frac{1}{\Delta},
\end{aligned}$$

a contradiction. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let G be a graph satisfying the conditions in Theorem 2.2. Suppose that G is not traceable. Then, from Lemma 3.2, there exists an integer k such that $d_k \leq k-1 \leq \frac{n}{2}-1$ and $d_{n+1-k} \leq n-k-1$. Obviously, $k \geq 2$. Therefore, from Lemma 3.5, we have that

$$\begin{aligned}
&\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n-1-\frac{2e}{n}\right)\\
&\geq \sum_{v \in V} \frac{1}{d(v)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d_1} + \cdots + \frac{1}{d_k} + \frac{1}{d_{k+1}} + \cdots + \frac{1}{d_{n+1-k}} + \frac{1}{d_{n+2-k}} + \cdots + \frac{1}{d_n} \\
&\geq \frac{k}{d_k} + \frac{n+1-2k}{d_{n+1-k}} + \frac{k-1}{d_n} \\
&\geq \frac{k}{k-1} + \frac{n+1-2k}{n-k-1} + \frac{k-1}{\Delta} \\
&\geq 1 + \frac{1}{k-1} + \frac{1}{n-3} + \frac{1}{\Delta} \\
&\geq 1 + \frac{1}{\frac{n}{2}-1} + \frac{1}{n-3} + \frac{1}{\Delta} \\
&= \frac{n}{n-2} + \frac{1}{n-3} + \frac{1}{\Delta},
\end{aligned}
$$

a contradiction. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Let G be a graph satisfying the conditions in Theorem 2.3. Suppose that G is not Hamilton-connected. Then, from Lemma 3.3, there exists an integer k such that $2 \leq k \leq \frac{n}{2}$, $d_{k-1} \leq k$, and $d_{n-k} \leq n-k$. Therefore, from Lemma 3.5, we have that

$$
\begin{aligned}
&\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) \\
&\geq \sum_{v \in V} \frac{1}{d(v)} \\
&= \frac{1}{d_1} + \cdots + \frac{1}{d_{k-1}} + \frac{1}{d_k} + \cdots + \frac{1}{d_{n-k}} + \frac{1}{d_{n-k+1}} + \cdots + \frac{1}{d_n} \\
&\geq \frac{k-1}{d_{k-1}} + \frac{n-2k+1}{d_{n-k}} + \frac{k}{d_n} \\
&\geq \frac{k-1}{k} + \frac{n-2k+1}{n-k} + \frac{k}{\Delta} \\
&\geq 1 - \frac{1}{k} + \frac{1}{n-2} + \frac{2}{\Delta} \\
&\geq \frac{1}{2} + \frac{1}{n-2} + \frac{2}{\Delta},
\end{aligned}
$$

a contradiction. This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. Let G be a graph satisfying the conditions in Theorem 2.4. Suppose that G is not k-connected. Then, from Lemma 3.4, there exists an integer j such that $1 \leq j \leq \lfloor \frac{n-k+1}{2} \rfloor \leq \frac{n-k+1}{2}$, $d_j \leq j + k - 2$, and $d_{n-k+1} \leq n - j - 1$. Therefore, from Lemma 3.5, we have that

$$
\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right)
$$

$$
\begin{aligned}
&\geq \sum_{v \in V} \frac{1}{d(v)}\\
&= \frac{1}{d_1} + \cdots + \frac{1}{d_j} + \frac{1}{d_{j+1}} + \cdots + \frac{1}{d_{n+1-k}} + \frac{1}{d_{n+2-k}} + \cdots + \frac{1}{d_n}\\
&\geq \frac{j}{d_j} + \frac{n+1-k-j}{d_{n+1-k}} + \frac{k-1}{d_n}\\
&\geq \frac{j}{j+k-2} + \frac{n+1-k-j}{n-j-1} + \frac{k-1}{\Delta}\\
&\geq \frac{1}{\frac{n-k+1}{2} + k - 2} + \frac{n+1-k-\frac{n+1-k}{2}}{n-2} + \frac{k-1}{\Delta}\\
&= \frac{2}{n+k-3} + \frac{n-k+1}{2(n-2)} + \frac{k-1}{\Delta},
\end{aligned}
$$

a contradiction. This completes the proof of Theorem 2.4. □

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