J. Appl. Math. & Informatics Vol. **34**(2016), *No. 3 - 4, pp. 179 - 192*
http://dx.doi.org/10.14317/jami.2016.179

NEW CONCEPTS OF PRODUCT INTERVAL-VALUED FUZZY GRAPH

A.A. TALEBI, HOSSEIN RASHMANLOU* AND REZA AMERI

ABSTRACT. In this paper, we introduce product interval-valued fuzzy graphs and prove several results which are analogous to interval-valued fuzzy graphs. We conclude by giving properties for a product interval-valued fuzzy graph.

AMS Mathematics Subject Classification : 05C99.
Key words and phrases : Interval-valued fuzzy graph, product interval-valued fuzzy graph.

1. Introduction

In 1965, Zadeh [22] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. The theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, engineering, statics, graph theory, computer networks, decision making and automata theory. In 1975, Rosenfeld [8] introduced the concept of fuzzy graphs, and proposed another elaborated definition, including fuzzy vertex and fuzzy edges, and several fuzzy analogs of graph theoretic concepts such as paths, cycles, connectedness and etc. Zadeh [22] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy set [23] in which the values of the membership degrees are intervals of numbers instead of the number. Interval-valued fuzzy set provid a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. The first definition of interval-valued fuzzy graph was proposed by Akram and Dudek [1]. Rashmanlou et al. [9, 10, 11, 12, 13, 14] studied bipolar fuzzy graphs, balanced interval-valued fuzzy graph, complete interval-valued fuzzy graphs and some properties of highly irregular interval-valued fuzzy graphs.

Received August 10, 2015. Revised September 2, 2015. Accepted September 17, 2015.
*Corresponding author.
© 2016 Korean SIGCAM and KSCAM.

Samanta and Pal [17, 18, 19, 20, 21] defined fuzzy tolerance graphs, fuzzy threshold graphs, fuzzy planar graphs, fuzzy k-competition graphs and p-competition fuzzy graphs and irregular bipolar fuzzy graphs. In this paper we develop the concept of product interval-valued fuzzy graphs of interval-valued fuzzy graphs, further investigate properties of product interval-valued fuzzy graphs. The definitions that we used in this paper are standard. For other notations, the readers are referred to [2, 3, 4, 5, 6, 7, 15, 16].

2. Preliminaries

Definition 2.1. The interval-valued fuzzy set A in V is defined by $A = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) : \ x \in V\}$, where $\mu_{A^-}(x)$ and $\mu_{A^+}(x)$ are fuzzy subsets of V such that $\mu_{A^-}(x) \leq \mu_{A^+}(x)$ $\forall x \in V$.

For any two interval-valued sets $A = \{(x, [\mu_A^-(x), \mu_A^+(x)] \mid x \in V\}$ and $B = \{(x, [\mu_B^-(x), \mu_B^+(x)] \mid x \in V\}$ in V we define:

$$A \cup B = \{(x, [\max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))] \mid x \in V\}.$$

Definition 2.2. By an interval-valued fuzzy graph of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = [\mu_{A^-}, \mu_{A^+}]$ is an interval-valued fuzzy set on V and $B = [\mu_{B^-}, \mu_{B^+}]$ is an interval-valued fuzzy set on E, such that $\mu_B^-(xy) \leq \min(\mu_A^-(x), \mu_A^-(y))$, $\mu_B^+(xy) \leq \min(\mu_A^+(x), \mu_A^+(y))$.

Definition 2.3. Let $G = (A, B)$ be an interval-valued fuzzy graph of a graph $G^* = (V, E)$. If $\mu_{B^-}(xy) \leq \mu_{A^-}(x) \times \mu_{A^-}(y)$ and $\mu_{B^+}(xy) \leq \mu_{A^+}(x) \times \mu_{A^+}(y)$, for all $x, y \in V$, then the interval-valued fuzzy graph G is called product interval-valued fuzzy graph of G^*.

Remark 2.1. If $G = (A, B)$ is a product interval-valued fuzzy graph, then since $\mu_{A^-}(x)$ and $\mu_{A^-}(y)$ are less than or equal to 1, it follows that $\mu_{B^-}(xy) \leq \mu_{A^-}(x) \times \mu_{A^-}(y) \leq \mu_{A^-}(x) \wedge \mu_{A^-}(y)$ and $\mu_{B^+}(xy) \leq \mu_{A^+}(x) \times \mu_{A^+}(y) \leq \mu_{A^+}(x) \wedge \mu_{A^+}(y)$, for all $x, y \in V$.

Thus every product interval-valued fuzzy graph is an interval-valued fuzzy graph.

Definition 2.4. A product interval-valued fuzzy graph $G = (A, B)$ is said to be complete if $\mu_{B^-}(xy) = \mu_{A^-}(x) \times \mu_{A^-}(y)$ and $\mu_{B^+}(xy) = \mu_{A^+}(x) \times \mu_{A^+}(y)$, for all $x, y \in V$.

Proposition 2.5. *Let $G = (A, B)$ be a complete product interval-valued fuzzy graph where μ_{A^-} and μ_{A^+} are normal. Then $\mu^n_{B^-}(xy) = \mu_{B^-}(xy)$ and $\mu^n_{B^+}(xy) = \mu_{B^+}(xy)$, for all $x, y \in V$ in which for all positive integer $n \geq 2$, $\mu^n_{B^-}(xy) = \bigvee_{z \in V} \{\mu^{n-1}_{B^-}(xz) \times \mu_{B^-}(zy)\}$ and $\mu^n_{B^+}(xy) = \bigvee_{z \in V} \{\mu^{n-1}_{B^+}(xz) \times \mu_{B^+}(zy)\}$.*

Proof. We prove by method of induction on n. Let $n \leq 2$, $x, y \in V$. We have

$$\mu^2_{B^-}(xy) = \bigvee_{z \in V} \{\mu_{B^-}(xz) \times \mu_{B^-}(zy)\}$$

$$= \bigvee_{z\in V} \{(\mu_{A^-}(x) \times \mu_{A^-}(z)) \times (\mu_{A^-}(z) \times \mu_{A^-}(y))\}$$
$$= \bigvee_{z\in V} \{\mu_{A^-}(x) \times \mu_{A^-}(y) \times \mu_{A^-}(z)^2\}$$

Since $\mu_{A^-}(z)^2 \leq 1$, for all z, $(\mu_{A^-}(z) \leq 1)$, $\mu^2_{B^-}(xy) \leq \bigvee_{z\in V}\{\mu_{A^-}(x) \times \mu_{A^-}(y)\} = \mu_{A^-}(x) \times \mu_{A^-}(y)$. Hence

$$\mu^2_{B^-}(xy) \leq \mu_{B^-}(xy) \tag{1}$$

Since μ_{A^-} is normal, $\mu_{A^-}(t) = 1$, for some $t \in V$. Then

$$\begin{aligned}\mu^2_{B^-}(xy) &= \bigvee_{z\in V} \{\mu_{A^-}(x) \times \mu_{A^-}(y) \times \mu_{A^-}(z)^2\}\\ &\geq \mu_{A^-}(x) \times \mu_{A^-}(y) \times \mu_{A^-}(t)^2\\ &= \mu_{A^-}(x) \times \mu_{A^-}(y)\ (\mu_{A^-}(t)^2 = 1).\end{aligned}$$

Therefore

$$\mu^2_{B^-}(xy) \geq \mu_{B^-}(xy). \tag{2}$$

Now from (1) and (2), we get $\mu^2_{B^-}(xy) = \mu_{B^-}(xy)$. Also

$$\begin{aligned}\mu^2_{B^+}(xy) &= \bigvee_{z\in V} \{\mu_{B^+}(xz) \times \mu_{B^+}(zy)\}\\ &= \bigvee_{z\in V} \{[\mu_{A^+}(x) \times \mu_{A^+}(z)] \times [\mu_{A^+}(z) \times \mu_{A^+}(y)]\}\\ &= \bigvee_{z\in V} \{\mu_{A^+}(x) \times \mu_{A^+}(y) \times \mu_{A^+}(z)^2\}.\end{aligned}$$

Since $\mu_{A^+}(z)^2 \leq 1$, for all z, $[\mu_{A^+}(z) \leq 1]$, $\mu^2_{B^+}(xy) \leq \bigvee_{z\in V}\{\mu_{A^+}(x) \times \mu_{A^+}(y)\} = \mu_{A^+}(x) \times \mu_{A^+}(y)$. Hence,

$$\mu^2_{B^+}(xy) \leq \mu_{B^+}(xy). \tag{3}$$

Since μ_{A^+} is normal, $\mu_{A^+}(t) = 1$, for some $t \in V$. Then

$$\begin{aligned}\mu^2_{B^+}(xy) &= \bigvee_{z\in V} \{\mu_{A^+}(x) \times \mu_{A^+}(y) \times \mu_{A^+}(z)^2\}\\ &\geq \mu_{A^+}(x) \times \mu_{A^+}(y) \times \mu_{A^+}(t)^2\\ &= \mu_{A^+}(x) \times \mu_{A^+}(y).\end{aligned}$$

Hence,

$$\mu^2_{B^+}(xy) \geq \mu_{B^+}(xy)\ [\text{Since } \mu^2_{B^+}(xy) = \mu_{A^+}(x) \times \mu_{A^+}(y)]. \tag{4}$$

From (3) and (4) we get $\mu^2_{B^+}(xy) = \mu_{B^+}(xy)$. Let $\mu^k_{B^-}(xy) = \mu_{B^-}(xy)$ and $\mu^k_{B^+}(xy) = \mu_{B^+}(xy)$. We will prove that $\mu^{k+1}_{B^-}(xy) = \mu_{B^-}(xy)$ and $\mu^{k+1}_{B^+}(xy) = \mu_{B^+}(xy)$. We have

$$\mu^{k+1}_{B^-}(xy) = \bigvee_{z\in V} \{\mu^k_{B^-}(xz) \times \mu_{B^-}(zy)\}$$

$$= \bigvee_{z \in V} \{\mu_{B^-}(xz) \times \mu_{B^-}(zy)\} = \mu^2_{B^-}(xy) = \mu_{B^-}(xy).$$

Similarly, we get $\mu^{k+1}_{B^+}(xy) = \mu_{B^+}(xy)$. □

Definition 2.6. The complement of a product interval-valued fuzzy graph $G = (A, B)$ is an interval-valued fuzzy graph $G^c = (A^c, B^c)$ where $A^c = A = [\mu_{A^-}, \mu_{A^+}]$ and $B^c = [\mu^c_{B^-}, \mu^c_{B^+}]$ is defined by

$$\begin{cases} \mu^c_{B^-}(xy) = \mu_{A^-}(x) \times \mu_{A^-}(y) - \mu_{B^-}(xy), \\ \mu^c_{B^+}(xy) = \mu_{A^+}(x) \times \mu_{A^+}(y) - \mu_{B^+}(xy). \end{cases} x, y \in V$$

Remark 2.2. The complement of a product interval-valued fuzzy graph is denoted by G^c. It follows that G is a product interval-valued fuzzy graph. Throughout this paper suppose that $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are product interval-valued fuzzy graph of $G^*_1 = (A_1, B_1)$ and $G^*_2 = (A_2, B_2)$, respectively.

Definition 2.7. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued fuzzy graph. The union $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is defined as follows:

$$(A) \begin{cases} (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \mu^-_{A_1}(x), & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \mu^-_{A_2}(x), & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \max(\mu^-_{A_1}(x), \mu^-_{A_2}(x)), & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$(B) \begin{cases} (\mu^+_{A_1} \cup \mu^+_{A_2})(x) = \mu^+_{A_1}(x), & \text{if } x \in V_1 \ and\ x \notin V_2, \\ (\mu^+_{A_1} \cup \mu^+_{A_2})(x) = \mu^+_{A_2}(x), & \text{if } x \in V_2 \ and\ x \notin V_1, \\ (\mu^+_{A_1} \cup \mu^+_{A_2})(x) = \max(\mu^+_{A_1}(x), \mu^+_{A_2}(x)), & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$(C) \begin{cases} (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) = \mu^-_{B_1}(xy), & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) = \mu^-_{B_2}(xy), & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) = \max(\mu^-_{B_1}(xy), \mu^-_{B_2}(xy)), & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

$$(D) \begin{cases} (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) = \mu^+_{B_1}(xy), & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) = \mu^+_{B_2}(xy), & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) = \max(\mu^+_{B_1}(xy), \mu^+_{B_2}(xy)), & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

Proposition 2.8. *The union of two product interval-valued fuzzy graphs is a product interval-valued fuzzy graph.*

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be product interval-valued fuzzy graph. We prove that $G_1 \cup G_2$ is a product interval-valued fuzzy graph of the graph $G^*_1 \cup G^*_2 = (V_1 \cup V_2, E_1 \cup E_2)$. Let $xy \in E_1 \cap E_2$. Then

$$\begin{aligned} (\mu^-_{B_1} \cup \mu^-_{B_2}) &= \max(\mu^-_{B_1}(xy), \mu^-_{B_2}(xy)) \\ &\le \max(\mu^-_{A_1}(x) \times \mu^-_{A_1}(y), \mu^-_{A_2}(x) \times \mu^-_{A_2}(y)) \end{aligned}$$

$$\begin{aligned}&\leq \max\{\mu^-_{A_1}(x), \mu^-_{A_2}(x)\} \times \max\{\mu^-_{A_1}(y), \mu^-_{A_2}(y)\}\\&= (\mu^-_{A_1} \cup \mu^-_{A_2})(x) \times (\mu^-_{A_1} \cup \mu^-_{A_2})(y),\end{aligned}$$

$$\begin{aligned}(\mu^+_{B_1} \cup \mu^+_{B_2}) &= \max(\mu^+_{B_1}(xy), \mu^+_{B_2}(xy))\\&\leq \max(\mu^+_{A_1}(x) \times \mu^+_{A_1}(y), \mu^+_{A_2}(x) \times \mu^+_{A_2}(y))\\&\leq \max\{\mu^+_{A_1}(x), \mu^+_{A_2}(x)\} \times \max\{\mu^+_{A_1}(y), \mu^+_{A_2}(y)\}\\&= (\mu^+_{A_1} \cup \mu^+_{A_2})(x) \times (\mu^+_{A_1} \cup \mu^+_{A_2})(y),\end{aligned}$$

If $xy \in E_1$ and $xy \notin E_2$ then

$$\begin{aligned}(\mu^-_{B_1} \cup \mu^-_{B_2})(xy) &= \mu^-_{B_1}(xy) \leq \mu^-_{A_1}(x) \times \mu^-_{A_1}(y)\\&= (\mu^-_{A_1} \cup \mu^-_{A_2})(x) \times (\mu^-_{A_1} \cup \mu^-_{A_2})(y),\\(\mu^+_{B_1} \cup \mu^+_{B_2})(xy) &= \mu^+_{B_1}(xy) \leq \mu^+_{A_1}(x) \times \mu^+_{A_1}(y)\\&= (\mu^+_{A_1} \cup \mu^+_{A_2})(x) \times (\mu^+_{A_1} \cup \mu^+_{A_2})(y).\end{aligned}$$

Similarly if $xy \in E_2$ and $xy \in E_1$, then we get $(\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \leq (\mu^-_{A_1} \cup \mu^-_{A_2})(x) \times (\mu^-_{A_1} \cup \mu^-_{A_2})(y)$, $(\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \leq (\mu^+_{A_1} \cup \mu^+_{A_2})(x) \times (\mu^+_{A_1} \cup \mu^+_{A_2})(y)$. □

Proposition 2.9. *Let $G^*_1 = (V_1, E_1)$ and $G^*_2 = (V_2, E_2)$ be crisp graphs with $V_1 \cap V_2 = \emptyset$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subset of V_1, V_2, E_1 and E_2 respectively. Then, $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is a product interval-valued fuzzy graph of $G^*_1 \cup G^*_2$ if and only if $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are product interval-valued fuzzy graph of G^*_1 and G^*_2, respectively.*

Proof. Let $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ be an product interval-valued fuzzy graph of $G^*_1 \cup G^*_2$. Let $xy \in E_1$. Then $xy \notin E_2$ and $x, y \in V_1$. Hence

$$\begin{aligned}\mu^-_{B_1}(xy) &= (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \leq (\mu^-_{A_1} \cup \mu^-_{A_2})(x) \times (\mu^-_{A_1} \cup \mu^-_{A_2})(y)\\&= \mu^-_{A_1}(x) \times \mu^-_{A_1}(y),\\\mu^+_{B_1}(xy) &= (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \leq (\mu^+_{A_1} \cup \mu^+_{A_2})(x) \times (\mu^+_{A_1} \cup \mu^+_{A_2})(y)\\&= \mu^+_{A_1}(x) \times \mu^+_{A_1}(y).\end{aligned}$$

Therefore $G_1 = (A_1, B_1)$ is a product interval-valued fuzzy graphs. Similarly, we can prove that $G_2 = (A_2, B_2)$ is a product interval-valued fuzzy graph. By proposition (2.8) we get the converse. □

Proposition 2.10. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be product interval-valued fuzzy graph of G^*_1 and G^*_2, respectively, and let $V_1 \cap V_2 = \emptyset$. Then $G_1 \cup G_2$ is complete if and only if G_1 and G_2 are complete.*

Proof. It is obvious. □

Example 2.11. Let $G^*_1 = (V_1, E_1)$ and $G^*_2 = (V_2, E_2)$ be graphs such that $V_1 = \{a, b, c\}, E_1 = \{ab, bc, ac\}, V_2 = \{a, b, d\}$ and $E_2 = \{ab, bd\}$.

Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

$$A_1 = \left\langle \left(\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.2}\right), \left(\frac{a}{0.1}, \frac{b}{0.4}, \frac{c}{0.5}\right) \right\rangle \qquad A_2 = \left\langle \left(\frac{a}{0.2}, \frac{b}{0.4}, \frac{d}{0.6}\right), \left(\frac{a}{0.3}, \frac{b}{0.5}, \frac{d}{0.7}\right) \right\rangle$$

$$B_1 = \left\langle \left(\frac{ab}{0.06}, \frac{ac}{0.04}, \frac{bc}{0.06}\right), \left(\frac{ab}{0.04}, \frac{ac}{0.05}, \frac{bc}{0.2}\right) \right\rangle \quad B_2 = \left\langle \left(\frac{ab}{0.08}, \frac{ad}{0.12}\right), \left(\frac{ab}{0.15}, \frac{ad}{0.21}\right) \right\rangle.$$

We have

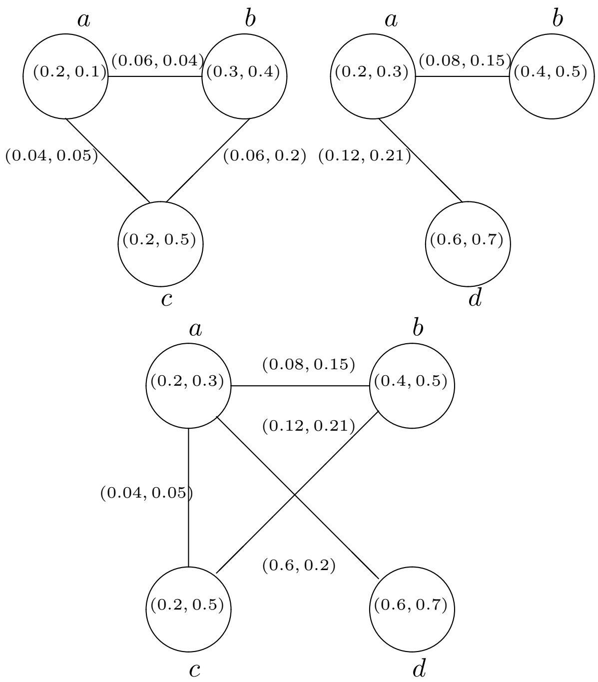

FIGURE 1. Union of G_1 and G_2 $(G_1 \cup G_2)$

It is clear that G_1 and G_2 are complete, but $G_1 \cup G_2$ is not.

Definition 2.12. The joint $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ of two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is defined as follows:

$$(A) \begin{cases} (\mu^-_{A_1} + \mu^-_{A_2})(x) = (\mu^-_{A_1} \cup \mu^-_{A_2})(x) \\ (\mu^+_{A_1} + \mu^+_{A_2})(x) = (\mu^+_{A_1} \cup \mu^+_{A_2})(x) \end{cases} \quad \text{if } x \in V_1 \cup V_2,$$

$$(B) \begin{cases} (\mu^-_{B_1} + \mu^-_{B_2})(xy) = (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \\ (\mu^+_{B_1} + \mu^+_{B_2})(xy) = (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \end{cases} \quad \text{if } xy \in E_1 \cup E_2,$$

$$(C) \begin{cases} (\mu^-_{B_1} + \mu^-_{B_2})(xy) = \mu^-_{A_1}(x) \times \mu^-_{A_2}(y) \\ (\mu^+_{B_1} + \mu^+_{B_2})(xy) = \mu^+_{A_1}(x) \times \mu^+_{A_2}(y) \end{cases} \quad \text{if } xy \in E',$$

where E' denote the set of all arcs joining the vertices V_1 and V_2.

Proposition 2.13. *If G_1 and G_2 are product interval-valued fuzzy graph, then $G_1 + G_2$ is a product interval-valued fuzzy graph.*

Proof. In view of Proposition (2.8) it is sufficient to verify when $xy \in E'$. In this case we have:

$$(\mu^-_{B_1} + \mu^-_{B_2})(xy) = \mu^-_{A_1}(x) \times \mu^-_{A_2}(y) \leq (\mu^-_{A_1} + \mu^-_{A_2})(x) \times (\mu^-_{A_1} + \mu^-_{A_2})(y),$$
$$(\mu^+_{B_1} + \mu^+_{B_2})(xy) = \mu^+_{A_1}(x) \times \mu^+_{A_2}(y) \leq (\mu^+_{A_1} + \mu^+_{A_2})(x) \times (\mu^+_{A_1} + \mu^+_{A_2})(y).$$

□

Example 2.14. In previous example we have

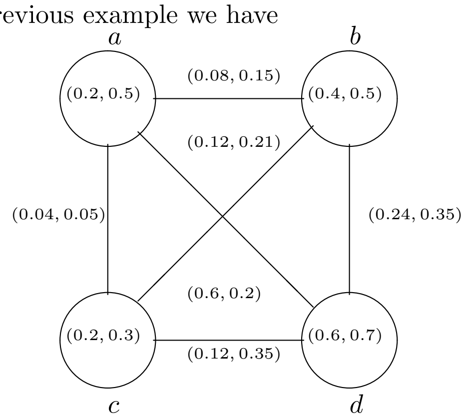

FIGURE 2. Join of G_1 and G_2 $(G_1 + G_2)$

It is clear that G_1 and G_2 are complete, but $G_1 + G_2$ is not complete.

Proposition 2.15. *Let G_1 and G_2 be product interval-valued fuzzy graph such that $V_1 \cap V_2 = \emptyset$. Then, $G_1 + G_2$ is complete if and only if G_1 and G_2 are both complete.*

Proof. Let G_1 and G_2 be complete and $u, v \in V_1$. Then

$$\begin{aligned}(\mu_{B_1^-} + \mu_{B_2^-})(uv) &= \mu_{B_1^-}(uv) = \mu_{A_1^-}(u) \times \mu_{A_1^-}(v)\\ &= (\mu_{A_1^-} + \mu_{A_2^-})(u) \times (\mu_{A_1^-} + \mu_{A_2^-})(v),\\ (\mu_{B_1^+} + \mu_{B_2^+})(uv) &= \mu_{B_1^+}(uv) = \mu_{A_1^+}(u) \times \mu_{A_1^+}(v)\\ &= (\mu_{A_1^+} + \mu_{A_2^+})(u) \times (\mu_{A_1^+} + \mu_{A_2^+})(v).\end{aligned}$$

If $u, v \in V_2$ then we have the same argument as above. Now Suppose that $u \in V_1$ and $v \in V_2$. We get $(\mu_{B_1^-} + \mu_{B_2^-})(uv) = \mu_{A_1^-}(u) \times \mu_{A_2^-}(v)$, whereas $(\mu_{A_1^-} + \mu_{A_2^-})(u) \times (\mu_{A_1^-} + \mu_{A_2^-})(v) = \mu_{A_1^-}(u) \times \mu_{A_2^-}(v)$. Thus, $(\mu_{B_1^-} + \mu_{B_2^-})(uv) = (\mu_{A_1^-} + \mu_{A_2^-})(u) \times (\mu_{A_1^-} \times \mu_{A_2^-})(v)$. Also $(\mu_{B_1^+} + \mu_{B_2^+})(uv) = \mu_{A_1^+}(u) \times \mu_{A_2^+}(v)$ whereas $(\mu_{A_1^+} + \mu_{A_2^+})(u) \times (\mu_{A_1^+} + \mu_{A_2^+})(v) = \mu_{A_1^+}(u) \times \mu_{A_2^+}(v)$. Therefore,

$(\mu_{B_1^+} + \mu_{B_2^+})(uv) = (\mu_{A_1^+} + \mu_{A_2^+})(u) \times (\mu_{A_1^+} + \mu_{A_2^+})(v)$. Hence, $G_1 + G_2$ is complete.

Conversely, assume that $G_1 + G_2$ is complete. First we show that G_1 is complete. Let $u, v \in V_1$. Then

$$(\mu_{B_1^-} + \mu_{B_2^-})(uv) = \mu_{B_1^-}(uv), \tag{5}$$

$$(\mu_{B_1^+} + \mu_{B_2^+})(uv) = \mu_{B_1^+}(uv). \tag{6}$$

Since $G_1 + G_2$ is complete,

$$(\mu_{B_1^-} + \mu_{B_2^-})(uv) = (\mu_{A_1^-} + \mu_{A_2^-})(u) \times (\mu_{A_1^-} + \mu_{A_2^-})(v) = \mu_{A_1^-}(u) \times \mu_{A_2^-}(v), \tag{7}$$

$$(\mu_{B_1^+} + \mu_{B_2^+})(uv) = (\mu_{A_1^+} + \mu_{A_2^+})(u) \times (\mu_{A_1^+} + \mu_{A_2^+})(v) = \mu_{A_1^+}(u) \times \mu_{A_2^+}(v). \tag{8}$$

Now using (5),(6),(7) and (8) we get $\mu_{B_1^-}(uv) = \mu_{A_1^-}(u) \times \mu_{A_1^-}(v)$ and $\mu_{B_1^+}(uv) = \mu_{A_1^+}(u) \times \mu_{A_1^+}(v)$. Therefore G_1 is complete. Similarly, we may prove that G_2 is complete. □

Proposition 2.16. *Let G_1 and G_2 be product interval-valued fuzzy graph of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, such that $V_1 \cap V_2 = \emptyset$. Then, $(G_1 + G_2)^c = G_1^c \cup G_2^c$.*

Proof. Let $u \in V_1$. Then, $(\mu_{A_1^-} + \mu_{A_2^-})^c(u) = (\mu_{A_1^-} + \mu_{A_2^-})(u) = \mu_{A_1^-}(u)$ and $\max(\mu^c_{A_1^-}(u), \mu^c_{A_2^-}(u)) = \max(\mu_{A_1^-}(u), \mu_{A_2^-}(u)) = \mu_{A_1^-}(u)$.

Hence $(\mu_{A_1^-} + \mu_{A_2^-})^c(u) = (\mu^c_{A_1^-} \cup \mu^c_{A_2^-})(u)$. Similarly we can prove that $(\mu_{A_1^-} + \mu_{A_2^-})^c(u) = (\mu^c_{A_1^-} \cup \mu^c_{A_2^-})(u)$, for all $u \in V_2$. Now suppose that $uv \in X_1$. Then $u, v \in V_1$, and we have

$$\begin{aligned}(\mu_{B_1^-} + \mu_{B_2^-})^c(uv) &= (\mu_{A_1^-} + \mu_{A_2^-})(u) \times (\mu_{A_1^-} + \mu_{A_2^-})(v) - (\mu_{B_1^-} + \mu_{B_2^-})(uv) \\ &= \mu_{A_1^-}(u) \times \mu_{A_1^-}(v) - \mu_{B_1^-}(uv) = \mu^c_{B_1^-}(uv).\end{aligned}$$

Also we have $\max(\mu^c_{B_1^-}(uv), \mu^c_{B_2^-}(uv)) = \mu^c_{B_1^-}(uv)$. Hence $(\mu_{B_1^-} + \mu_{B_2^-})^c(uv) = (\mu^c_{B_1^-} \cup \mu^c_{B_2^-})(uv)$. Similarly we can prove that $(\mu_{B_1^-} + \mu_{B_2^-})^c(uv) = (\mu^c_{B_1^-} \cup \mu^c_{B_2^-})(uv)$, for all $uv \in X_2$.

Now assume that $(u, v) \in X'$. Then $u \in V_1$ and $v \in V_2$. Thus

$$\begin{aligned}(\mu_{B_1^-} + \mu_{B_2^-})^c(uv) &= (\mu_{A_1^-} + \mu_{A_2^-})(u) \times (\mu_{A_1^-} + \mu_{A_2^-})(v) - (\mu_{B_1^-} + \mu_{B_2^-})(uv) \\ &= \mu_{A_1^-}(u) \times \mu_{A_1^-}(v) - (\mu_{A_1^-}(u) \times \mu_{A_2^-}(v)) = 0.\end{aligned}$$

Also $\max(\mu^c_{B_1^-}(uv), \mu^c_{B_2^-}(uv)) = 0$, since $u \in V_1$ and $v \in V_2$. Therefore $(G_1 + G_2)^c = G_1^c \cup G_2^c$. □

Proposition 2.17. *Let G_1 and G_2 be product interval-valued fuzzy graph. Then $(G_1 \cup G_2)^c = G_1^c + G_2^c$.*

Proof. Let $u \in V_1$. Then, $(\mu_{A_1^-} \cup \mu_{A_2^-})^c(u) = (\mu_{A_1^-} \cup \mu_{A_2^-})(u) = \mu_{A_1^-}(u)$ and $(\mu^c_{A_1^-} + \mu^c_{A_2^-})(u) = \max(\mu^c_{A_1^-}(u), \mu^c_{A_2^-}(u)) = \max(\mu_{A_1^-}(u), \mu_{A_2^-}(v)) = \mu_{A_1^-}(u)$. Hence, $(\mu_{A_1^-} \cup \mu_{A_2^-})^c(u) = (\mu^c_{A_1^-} + \mu^c_{A_2^-})(u)$, for all $u \in V_1$. Similarly we can prove when $u \in V_2$. Now suppose that $uv \in X_1$, then

$$(\mu_{B_1^-} \cup \mu_{B_2^-})^c(uv) = (\mu_{A_1^-} \cup \mu_{A_2^-})(u) \times (\mu_{A_1^-} \cup \mu_{A_2^-})(v) - (\mu_{B_1^-} \cup \mu_{B_2^-})(uv)$$
$$\mu_{A_1^-}(u) \times \mu_{A_1^-}(v) - \mu_{B_1^-}(uv) = \mu^c_{B_1^-}(uv) = (\mu^c_{B_1^-} + \mu^c_{B_2^-})(uv).$$

If $uv \in X_2$, then $u, v \in V_2$, and hence

$$(\mu_{B_1^-} \cup \mu_{B_2^-})^c(uv) = (\mu_{A_1^-} \cup \mu_{A_2^-})(u) \times (\mu_{A_1^-} \cup \mu_{A_2^-})(v) - (\mu_{B_1^-} \cup \mu_{B_2^-})(uv)$$
$$\mu_{A_2^-}(u) \times \mu_{A_2^-}(v) - \mu_{B_2^-}(uv) = \mu^c_{B_2^-}(uv) = (\mu^c_{B_1^-} + \mu^c_{B_2^-})(uv).$$

If $uv \in X'$, then $u \in V_1$, and $v \in V_2$. Hence,

$$\begin{aligned}(\mu_{B_1^-} \cup \mu_{B_2^-})^c(uv) &= (\mu_{A_1^-} \cup \mu_{A_2^-})(u) \times (\mu_{A_1^-} \cup \mu_{A_2^-})(v) - (\mu_{B_1^-} \cup \mu_{B_2^-})(uv)\\ &= \mu_{A_1^-}(u) \times \mu_{A_2^-}(v) \text{ (Since } \mu_{B_1^-}(uv) = \mu_{B_2^-}(uv) = 0)\\ &= \mu^c_{A_1^-}(u) \times \mu^c_{A_2^-}(v) = (\mu^c_{B_1^-} + \mu^c_{B_2^-})(uv).\end{aligned}$$

Therefore $(\mu_{B_1^-} \cup \mu_{B_2^-})^c = \mu^c_{B_1^-} + \mu^c_{B_2^-}$. Similarly, we get $(\mu_{A_1^+} \cup \mu_{A_2^+})^c = \mu^c_{A_1^+} + \mu^c_{A_2^+}$ and $(\mu_{B_1^+} \cup \mu_{B_2^+})^c = \mu^c_{B_1^+} + \mu^c_{B_2^+}$. Let G_1^* and G_2^* be two graphs whose vertex sets are V_1 and V_2, respectively. Consider a new graph $G^* = G_1^* \times G_2^*$ whose vertex set is $V_1 \times V_2$ and edge set is a subset of $(V_1 \times V_2) \times (V_1 \times V_2)$.

Let G_1 and G_2 be product interval-valued fuzzy graph of G_1^* and G_2^*, respectively. If $v_1 \in V_1$ and $v_2 \in V_2$, we define:
$\mu^-_{A_1} \times \mu^-_{A_2}(v_1, v_2) = \mu^-_{A_1}(v_1) \times \mu^-_{A_2}(v_2)$ and $\mu^+_{A_1} \times \mu^+_{A_2}(v_1, v_2) = \mu^+_{A_1}(v_1) \times \mu^+_{A_2}(v_2)$.
Also, if $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$, then we define:

$$\mu^-_{B_1} \times \mu^-_{B_2}((u_1, u_2)(v_1, v_2)) = \mu^-_{B_1}(u_1 v_1) \times \mu^-_{B_2}(u_2 v_2),$$
$$\mu^+_{B_1} \times \mu^+_{B_2}((u_1, u_2)(v_1, v_2)) = \mu^+_{B_1}(u_1 v_1) \times \mu^+_{B_2}(u_2 v_2).$$

So, $A = [\mu^-_{A_1} \times \mu^-_{A_2}, \mu^+_{A_1} \times \mu^+_{A_2}$ is an interval-valued fuzzy sub set on $V = V_1 \times V_2$ and $B = [\mu^-_{B_1} \times \mu^-_{B_2}, \mu^+_{B_1} \times \mu^+_{B_2}$ is an interval-valued fuzzy subset of $(V_1 \times V_2) \times (V_1 \times V_2)$. In fact $G = (A, B)$ is an interval-valued fuzzy graph of $G_1^* \times G_2^*$ that is denoted by $G_1 \times G_2$. □

Proposition 2.18. *Let G_1 and G_2 be product interval-valued fuzzy graph. Then $G_1 \times G_2$ is a product interval-valued fuzzy graph.*

Proof. Let $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. Then we have

$$\begin{aligned}(\mu_{B_1^-} \times \mu_{B_2^-})((u_1, u_2)(v_1, v_2)) &= \mu_{B_1^-}(u_1 v_1) \times \mu_{B_2^-}(u_2 v_2)\\ &\le (\mu_{A_1^-}(u_1) \times \mu_{A_1^-}(v_1)) \times (\mu_{A_2^-}(u_2) \times \mu_{A_2^-}(v_2))\\ &= (\mu_{A_1^-}(u_1) \times \mu_{A_2^-}(u_2)) \times (\mu_{A_1^-}(v_1) \times \mu_{A_2^-}(v_2))\\ &= (\mu_{A_1^-} \times \mu_{A_2^-})(u_1, u_2) \times (\mu_{A_1^-} \times \mu_{A_2^-})(v_1, v_2).\end{aligned}$$

Similarly, we can prove that $(\mu_{B_1^+}\times\mu_{B_2^+})((u_1,u_2)(v_1,v_2)) \leq (\mu_{A_1^+}\times\mu_{A_2^+})(u_1,u_2)\times(\mu_{A_1^+}\times\mu_{A_2^+})(v_1,v_2)$, for all $u_1,v_1\in V_1$ and $u_2,v_2\in V_2$. This complete the proof. □

Definition 2.19. The product interval-valued fuzzy graph $G_1\times G_2$ is referred to as the multiplication of the product interval-valued fuzzy graphs G_1 and G_2.

Proposition 2.20. *Let G_1 and G_2 be product interval-valued fuzzy graph. Then, $G_1\times G_2$ is complete if and only if both G_1 and G_2 are complete.*

Proof. Let G_1 and G_2 be complete, $u_1,v_1\in V_1$ and $u_2,v_2\in V_2$. Then

$$\begin{aligned}(\mu^-_{B_1}\times\mu^-_{B_2})((u_1,u_2)(v_1,v_2)) &= \mu^-_{B_1}(u_1v_1)\times\mu^-_{B_2}(u_2v_2)\\ &= [\mu^-_{A_1}(u_1)\times\mu^-_{A_1}(v_1)]\times[\mu^-_{A_2}(u_2)\times\mu^-_{A_2}(v_2)]\\ &= [\mu^-_{A_1}(u_1)\times\mu^-_{A_1}(u_2)]\times[\mu^-_{A_1}(v_1)\times\mu^-_{A_2}(v_2)]\\ &= (\mu^-_{A_1}\times\mu^-_{A_2})(u_1,u_2)\times(\mu^-_{A_1}\times\mu^-_{A_2})(v_1,v_2).\end{aligned}$$

Similarly, we can prove that if $u_1,v_1\in V_1$ and $u_2,v_2\in V_2$, then

$$(\mu^+_{B_1}\times\mu^+_{B_2})((u_1,u_2)(v_1,v_2)) = (\mu^+_{A_1}\times\mu^+_{A_2})(u_1,u_2)\times(\mu^+_{A_1}\times\mu^+_{A_2})(v_1,v_2)$$

Therefore $G_1\times G_2$ is complete. Conversely, let $G_1\times G_2$ be complete. We will prove that G_1 and G_2 both are complete. Suppose that G_1 is not complete. Then, there exist $u_1,v_1\in V_1$ for which one of the following inequalities hold.

$$\mu^-_{B_1}(u_1v_1)<\mu^-_{A_1}(u_1)\times\mu^-_{A_1}(v_1),\ \mu^+_{B_2}(u_1v_1)<\mu^+_{A_2}(u_1)\times\mu^+_{A_2}(v_1).$$

Assume that $\mu^-_{B_1}(u_1v_1)<\mu^-_{A_1}(u_1)\times\mu^-_{A_1}(v_1)$.

Now by considering $((u_1,u_2),(v_1,v_2))\in(V_1\times V_2)\times(V_1\times V_2)$, we have

$$\begin{aligned}(\mu^-_{B_1}\times\mu^-_{B_2})((u_1,u_2),(v_1,v_2)) &= \mu^-_{B_1}(u_1v_1)\times\mu^-_{B_2}(u_2v_2)\\ &< [\mu^-_{A_1}(u_1)\times\mu^-_{A_1}(v_1)]\times[\mu^-_{A_2}(u_2)\times\mu^-_{A_2}(v_2)]\\ &= [\mu^-_{A_1}(u_1)\times\mu^-_{A_2}(u_2)]\times[\mu^-_{A_1}(v_1)\times\mu^-_{A_2}(v_2)]\\ &= (\mu^-_{A_1}\times\mu^-_{A_2})(u_1,u_2)\times(\mu^-_{A_1}\times\mu^-_{A_2})(v_1,v_2).\end{aligned}$$

This is a contradiction, since $G_1\times G_2$ is complete. Similarly, if $\mu^+_{B_2}(u_1,v_1)<\mu_{A_2^+}(u_1)\times\mu_{A_2^+}(v_1)$, a contradiction can be obtained. Hence G_1 is complete. By the same argument as above we can prove that G_2 is complete. □

Proposition 2.21. *Let $V_1=\{v_{11},v_{12},\cdots,v_{1n}\}$ and $V_2=\{v_{21},v_{22},\cdots,v_{2n}\}$ be the vertex sets of graphs G_1 and G_2, respectively . Further, let $G=(A,B)$ be the multiplication of G_1 and G_2. Then the following equations have solutions in $[0,1]$.*

(i) $x_i\times y_i=\mu^-_A(v_{1i},v_{2j})$ $(i=1,2,\cdots,n,\ j=1,2,\cdots,m)$,

(ii) $z_{ik}\times w_{jl}=\mu^-_B((v_{1i},v_{2j})(v_{1k},v_{2l}))$ $(i,k=1,2,\cdots,n,\ j,l=1,2,\cdots,m)$,

(iii) $x_i\times y_j=\mu^+_A(v_{1i},v_{2j})$ $(i=1,2,\cdots,n,\ j=1,2,\cdots,m)$

(iv) $z_{ik}\times w_{jl}=\mu^+_B((v_{1i},v_{2j})(v_{1k},v_{2l}))$ $(i,k=1,2,\cdots,n,\ j,l=1,2,\cdots,m)$.

Proof. Let G be the multiplication of product interval-valued fuzzy graphs G_1 and G_2. Then $(\mu_A^-, \mu_B^-) = (\mu_{A_1}^- \times \mu_{A_2}^-, \mu_{B_1}^- \times \mu_{B_2}^-)$ and $(\mu_A^+, \mu_B^+) = (\mu_{A_1}^+ \times \mu_{A_2}^+, \mu_{B_1}^+ \times \mu_{B_2}^+)$. Now we have

$$\mu_A^-(v_{1i}, v_{2j}) \quad = \quad (\mu_{A_1}^- \times \mu_{A_2}^-)(v_{1i}, v_{2j}) = \mu_{A_1}(v_{1i}) \times \mu_{A_2}^-(v_{2j}) = x_i \times y_j,$$

where $x_i = \mu_{A_1}^-(v_{1i}) \in [0,1]$ and $y_j = \mu_{A_2}^-(v_{2j}) \in [0,1]$. If $v_{1i}, v_{1k} \in V_1$ and $v_{2j}, v_{2k} \in V_2$, then

$$\begin{aligned}\mu_B^-((v_{1i}, v_{2j})(v_{1k}, v_{2l})) &= (\mu_{B_1}^- \times \mu_{B_2}^-)((v_{1i}, v_{2j})(v_{1k}, v_{2l})) \\ &= \mu_{B_1}^-(v_{1i}, v_{1k}) \times \mu_{B_2}^-(v_{2j}, v_{2l}) = z_{ik} \times w_{jl}\end{aligned}$$

where $z_{ik} = \mu_{B_1}^-(v_{1i}, v_{1k}) \in [0,1]$ and $w_{jl} = \mu_{B_2}^-(v_{2j}, v_{2l}) \in [0,1]$. Therefore, the equations (i) and (ii) have solutions in $[0,1]$. Similarly, by the same argument as above we can prove that the equations (iii) and (iv) have solutions. □

Theorem 2.22. *Let G^* be a product of two graphs G_1^* and G_2^*. Let $G = (A, B)$ be a product interval-valued fuzzy graph of G^* where μ_B^- and μ_B^+ are normal. Moreover, suppose that the following equations have solutions in* $[0,1]$,
(i) $x_i \times y_i = \mu_A^-(v_{1i}, v_{2j})$ $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$,
(ii) $z_{ik} \times w_{jl} = \mu_B^-((v_{1i}, v_{2j})(v_{1k}, v_{2l}))$ $(i, k = 1, 2, \cdots, n,\ j, l = 1, 2, \cdots, m,)$
(iii) $s_i \times t_j = \mu_A^+(v_{1i}, v_{2j})$ $(i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$,
(iv) $p_{ik} \times w_{jl} = \mu_B^+((v_{1i}, v_{2j})(v_{1k}, v_{2l}))$ $(i, k = 1, 2, \cdots, n,\ j, l = 1, 2, \cdots, m)$.
Then G is the multiplication of a product interval-valued fuzzy graph of G_1^* and a product interval-valued fuzzy graph of G_2^*.

Proof. Define

$\mu_{A_1}^- : V_1 \longrightarrow [0,1],\ \mu_{A_1}^-(v_{1i}) = x_i$
$\mu_{A_2}^- : V_2 \longrightarrow [0,1],\ \mu_{A_2}^-(v_{2i}) = y_j$
$\mu_{B_1}^- : V_1 \times V_1 \longrightarrow [0,1],\ \mu_{B_1}^-(v_{1i}, v_{1k}) = z_{ik}$
$\mu_{B_2}^- : V_2 \times V_2 \longrightarrow [0,1],\ \mu_{B_2}^-(v_{2i}, v_{2k}) = w_{jl}$
$\mu_{A_1}^+ : V_1 \longrightarrow [0,1],\ \mu_{A_1}^+(v_{1i}) = s_i$
$\mu_{A_2}^+ : V_2 \longrightarrow [0,1],\ \mu_{A_2}^+(v_{2j}) = t_j$
$\mu_{B_1}^+ : V_1 \times V_1 \longrightarrow [0,1],\ \mu_{B_1}^+(v_{1i}, v_{1k}) = r_{ik}$
$\mu_{B_2}^+ : V_2 \times V_2 \longrightarrow [0,1],\ \mu_{B_2}^+(v_{2i}, v_{2l}) = u_{jl}.$

We will prove the following.
(i) $G_1 = (A_1, B_1)$ is a product interval-valued fuzzy graph of G_1^*.
(ii) $G_2 = (A_2, B_2)$ is a product interval-valued fuzzy graph of G_2^*.
(iii) $\mu_A^- = \mu_{A_1}^- \times \mu_{A_2}^-, \mu_B^- = \mu_{B_1}^- \times \mu_{B_2}^-$.
(iv) $\mu_A^+ = \mu_{A_1}^+ \times \mu_{A_2}^+, \mu_B^+ = \mu_{B_1}^+ \times \mu_{B_2}^+$.
If $v_{1i}, v_{1k} \in V_1$ then for all $v_{2j}, v_{2l} \in V_2$ we have

$$\begin{aligned}\mu_B^-((v_{1i}, v_{2j})(v_{1k}, v_{2l})) &\le \mu_A^-(v_{1i}, v_{2j}) \times \mu_A^-(v_{1k}, v_{2l}) \\ (x_i \times y_i) \times (x_k \times y_l) &= (x_i \times x_k) \times (y_j \times y_l) \\ &\le x_i \times x_k \text{ (Since } y_j,\ y_l \le 1).\end{aligned}$$

Hence, $z_{ik}\times w_{jl} \leq \mu^-_{A_1}(v_{1i})\times\mu^-_{A_1}(v_{1k})$, for all j, l. Since μ^-_B is normal, $\mu^-_B((v_{1p}, v_{2a})(v_{1q}, v_{2b})) = 1$, for some p, q, a and b. Thus $z_{pq} \times w_{ab} = 1$, and so $z_{pq} = w_{ab} = 1$ (Since $z_{pq}.w_{ab} \in [0, 1]$). Replacing j by a and l by b, we get

$$\mu^-_{B_1}(v_{1i}) = z_{1k} = z_{1k} \times w_{ab} \leq \mu^-_{A_1}(v_{1i}) \times \mu_{A_1}(v_{1k}).$$

Similarly, we can get $\mu^+_{B_1}(v_{1i}) \leq \mu^+_{A_1}(v_{1i}) \times \mu^+_{A_1}(v_{1k})$. This prove that $G_1 = (A_1, B_1)$ is a product interval-valued fuzzy graph of G^*_1. Similarly, we can prove that $G_2 = (A_2, B_2)$ is a product fuzzy graph of G^*_2. Now if $v_{1i} \in V_1$ and $v_{2i} \in V_2$, then

$$\mu^-_{A_1} \times \mu^-_{A_2}(v_{1i}, v_{2i}) = \mu^-_{A_1}(v_{1i}) \times \mu^-_{A_2}(v_{2i}) = x_i \times y_i = \mu^-_A(v_{1i}, v_{2i})$$
$$\mu^+_{A_1} \times \mu^+_{A_2}(v_{1i}, v_{2i}) = \mu^+_{A_1}(v_{1i}) \times \mu^+_{A_2}(v_{2i}) = s_i \times t_j = \mu^+_A(v_{1i}, v_{2i}).$$

This prove that $\mu^-_A = \mu^-_{A_1} \times \mu^-_{A_2}$ and $\mu^+_A = \mu^+_{A_1} \times \mu^+_{A_2}$. If $v_{1i}, v_{1k} \in V_1$ and $v_{2i}, v_{2k} \in V_2$, then

$$\begin{aligned}\mu^-_{B_1} \times \mu^-_{B_2}((v_{1i}, v_{2j})(v_{1k}, v_{2l})) &= \mu^-_{B_1}(v_{1i}, v_{1i}) \times \mu^-_{B_2}(v_{2j}, v_{2l})\\ &= z_{1k} \times w_{jl} = \mu^-_B(v_{1i}, v_{2j})(v_{1k}, v_{2l})).\end{aligned}$$

Thus, $\mu^-_B = \mu^-_{B_1} \times \mu^-_{B_2}$. Similarly, we can prove that $\mu_B+ = \mu^+_{B_1} \times \mu^+_{B_2}$. □

3. Application of related theorems

An interval-valued fuzzy set is an extension of Zadeh's fuzzy set theory whose range of membership degree is $[0, 1]$. The interval-valued fuzzy graph is a generalized structure of a fuzzy graph which gives more precision, flexibility, and compatibility with a system when compared with the fuzzy graphs. The natural extension of the research work on interval-valued fuzzy graph is product interval-valued fuzzy graphs. Note that one of the most widely studied classes of interval-valued fuzzy graphs is product interval-valued fuzzy graph. They show up in many contexts. These results can be applied in database theory, geographical information system roughness in graphs, roughness in hyper- graphs, soft graphs, and soft hypergraphs. Fuzzy cognitive maps (FCMs) are used in science, engineering, and the social sciences to represent the causal structure of a body of knowledge (be it empirical knowledge, traditional knowledge, or a personal view); for some examples. An FCM of the type that we shall consider in this paper is described by a set of factors and causal relationships between pairs of factors. A factor can have a direct positive or direct negative impact (or both) on another factor or on itself. In addition, a numerical weight is assigned to each direct impact; these weights are usually taken to be in the interval $[0, 1]$. Graph-theoretic tools are used to analyze FCMs. In particular, algorithms for computing a transitive closure of the FCM, from which all, not just direct, impacts together with their weights can be read. Two models can be constructed in the probabilistic model, the absolute value of the weight of an impact is interpreted as the probability that the impact occurs, while in the fuzzy model, it is interpreted as the degree of truth. In both cases, the FCM is represented as an interval-valued fuzzy directed graph; the definition of the transitive closure, however, depends on the model. Here, product interval-valued fuzzy graphs are introduced to improve the solution of the problems. The problem of the probabilistic transitive closure of an interval-valued fuzzy directed graph is an interval-valued version of the network reliability problem called s, t-connectedness (for all pairs of vertices s and t). Some of these results mentioned in the paper will help the reduction-recovery algorithm, complete state enumeration, the basic inclusion-exclusion algorithm, and the boolean algebra approach. This adaptation is far from trivial, as care must be taken to generate not only directed paths, but rather all minimal directed walks, and to distinguish between positive and negative minimal directed walks.

4. Conclusions

Graph theory has several interesting applications in system analysis, operations research, computer applications, and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy systems. It is known that fuzzy graph theory has numerous applications in modern science and engineering, neural networks, expert systems, medical diagnosis, town planning and control theory. In this paper, we have introduced product interval-valued fuzzy graphs and proved several interesting results which are analogous to interval-valued fuzzy graphs. In our future work, we will focus on applications of product interval-valued fuzzy graphs in other sciences.

Acknowledgement

The authors are extremely grateful to the Editor in Chief Prof. Cheon Seoung Ryoo and anonymous referees for giving them many valuable comments and helpful suggestions which helps to improve the presentation of this paper.

References

1. M. Akram and Wieslaw A. Dudek, *Interval-valued fuzzy graphs*, Computer and Mathematics with Applications, **61** (2011), 289-299.
2. S. Arumugam and S. Velammal, *Edge domination in graph*, Taiwanese Journal of Mathematics, **2** (1998), 173-179.
3. S. Ayyaswamy and C. Natarajan, *Strong (Weak) domination intuitionistic fuzzy graphs*, International Journal of Computational and Mathematical Sciences, 2010.
4. F. Harary, *Graph Theory*, Addition Wesley, Third Printing, October 1972.
5. J.N. Mordeson and P.S. Nair, *Fuzzy Graphs and Fuzzy Hyper Graphs*, Physica-Verlag, Heidelberg, 2000.
6. A. Nagoor gani and M. Basher Ahamed , *Order and size in fuzzy graphs*, Bulletin of Pure and Applies Science, **22** (2003), 145-148.
7. A. Nagoor gani and P. Vadivel, *Relatio between the parameter of Independent Domination and Irredundance in fuzzy Graph*, International Journal of Algorithms Computing and Mathenmatics, **2** (2009), 15-19.

8. A. Rosenfeld , *Fuzzy graphs*, in:L. A. Zadeh, K. S. Fu, Moshimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, New York, (1975), 77-95.
9. H. Rashmanlou, S. Samanta, M. Pal and R.A. Borzooei, *A study on Bipolar fuzzy graphs*, Journal of Intelligent and Fuzzy Systems, **28** (2015), 571-580.
10. H. Rashmanlou, S. Samanta, M. Pal and R.A. Borzooei, *Bipolar fuzzy graphs with categorical properties*, International Journal of Computational Intelligent Systems, **8** (2015), 808-818.
11. H. Rashmanlou, S. Samanta, M. Pal and R.A. Borzooei, *Product of Bipolar fuzzy graphs and their degree*, International Journal of General Systems, to appear.
12. H. Rashmanlou and M. Pal, *Balanced interval-valued fuzzy graph*, Journal of Physical Sciences, **17** (2013), 43-57.
13. H. Rashmanlou and Y.B. Jun, *Complete interval-valued fuzzy graphs*, Annals of Fuzzy Mathematics and Informatics, **6** (2013), 677-687.
14. H. Rashmanlou and M. Pal, *Some properties of highly irregular interval-valued fuzzy graphs*, World Applied Sciences Journal, **27** (2013), 1756-1773.
15. A. Somasundaram and S. Somasundaram, *Domination in Fuzzy Graph-I*, Pattern Recognition Letters, **19** (1998), 787-791.
16. A. Somasundaram, *Domination in Fuzzy Graph-II*, Journal of Fuzzy Mathematics, 2004.
17. S. Samanta and M. Pal, *Fuzzy tolerance graphs*, International Journal Latest Trend Mathematics, **1** (2011), 57-67.
18. S. Samanta and M. Pal, *Fuzzy threshold graphs*, CiiT International Journal of Fuzzy Systems, **3** (2011), 360-364.
19. S. Samanta, M. Pal and A. Pal, *New concepts of fuzzy planar graph*, International Journal of Advanced Research in Artificial Intelligence, **3** (2014), 52-59.
20. S. Samanta and M. Pal, *Irregular bipolar fuzzy graphs*, Inernational Journal of Applications of Fuzzy Sets, **2** (2012), 91-102.
21. S. Samanta and M. Pal, *Fuzzy k-competition graphs and p-competition fuzzy graphs*, Fuzzy Information and Engineering, **5** (2013), 191-204.
22. L.A. Zadeh, *The concept of a linguistic and application to approximate reasoning I*, Information Sciences, **8** (1975), 199-249.
23. L.A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

A.A. Talebi

Department of Mathematics, University of Mazandaran, Babolsar, Iran.
e-mail: a.talebi@umz.ac.ir

Hossein Rashmanlou

Department of Computer Science, University College of Rouzbahan, Sari, Iran.
e-mail: Rashmanlou.1987@gmail.com

Reza Ameri

Department of Mathematics, Tehran University, Tehran, Iran.
e-mail: rameri@ut.ac.ir