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QUADRATIC ρ-FUNCTIONAL INEQUALITIES

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Abstract. In this paper, we solve the quadratic ρ-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where ρ is a fixed complex number with $|\rho|<1$, and

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right\| \\ &\leq \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|, \end{aligned} \tag{0.2}$$

where ρ is a fixed complex number with $|\rho|<\frac{1}{2}$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [11] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

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is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [10] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

The functional equation

$$4f\left(\frac{x+y}{2}\right) + (x-y) = 2f(x) + 2f(y)$$

is called a *Jensen type quadratic equation*. See [2, 4, 7, 9, 12] for more information on the stability problems of functional equations.

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in complex Banach spaces.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, let G be a 2-divisible abelian group. Assume that X is a real or complex normed space with norm $\|\cdot\|$ and that Y is a complex Banach space with norm $\|\cdot\|$.

2. QUADRATIC ρ-FUNCTIONAL INEQUALITY (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. *If a mapping $f : G \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \le \left\|\rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \end{aligned} \tag{2.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \le |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\\ &\quad\le \left\|\rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\|\\ &\quad= |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. $\square$

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (2.1) in complex Banach spaces.

Theorem 2.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\\ &\quad\le \left\|\rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{2\theta}{2^r - 4}\|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.3), we get $\|2f(0)\| \le |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$\|f(2x) - 4f(x)\| \le 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \frac{2}{2^r}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{2}{2^r}\sum_{j=l}^{m-1} \frac{4^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} &\|h(x+y) + h(x-y) - 2h(x) - 2h(y)\| \\ &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 4^n|\rho| \left\|4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| \\ &\quad + \lim_{n\to\infty} \frac{4^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) \\ &= |\rho| \left\|4h\left(\frac{x+y}{2}\right) + h(x-y) - 2h(x) - 2h(y)\right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x+y)+h(x-y)-2h(x)-2h(y)\| \leq \left\|\rho\left(4h\left(\frac{x+y}{2}\right) + h(x-y) - 2h(x) - 2h(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= 4^n \left\|h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\ &\leq 4^n \left(\left\|h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\leq \frac{4 \cdot 4^n}{(2^r - 4)2^{nr}}\theta\|x\|^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (2.4). □

Theorem 2.3. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.3). *Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{4 - 2^r}\|x\|^r \tag{2.7}$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{\theta}{2}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x)\right\| \\ &\leq \sum_{j=l}^{m-1}\frac{2^{rj}}{4^j}\frac{\theta}{2}\|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

Remark 2.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Quadratic ρ-functional Inequality (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.2) in complex Banach spaces.

Lemma 3.1. *If a mapping $f : G \to Y$ satisfies*

$$\left\|4f\left(\frac{x+y}{2}\right) + f\left(x-y\right) - 2f(x) - 2f(y)\right\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \tag{3.1}$$

for all $x, y \in G$, then $f : G \to Y$ is quadratic.

Proof. Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0$ and so

$$4f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &= \left\|4f\left(\frac{x+y}{2}\right) + f\left(x-y\right) - 2f(x) - 2f(y)\right\| \\ &\leq |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (3.1) in complex Banach spaces.

Theorem 3.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|4f\left(\frac{x+y}{2}\right) + f\left(x-y\right) - 2f(x) - 2f(y)\| \\ &\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2^r\theta}{2^r - 4}\|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.3), we get $\|f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{4^j}{2^{rj}}\theta\|x\|^r \qquad (3.6)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

It follows from (3.3) that

$$\begin{aligned}&\left\|4h\left(\frac{x+y}{2}\right) + h\left(x-y\right) - 2h(x) - 2h(y)\right\| \\ &= \lim_{n\to\infty} 4^n \left\|4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 4^n \left\|\rho\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\quad + \lim_{n\to\infty}\frac{4^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r) \\ &= \|\rho(h(x+y) + h(x-y) - 2h(x) - 2h(y))\|\end{aligned}$$

for all $x, y \in X$. So

$$\left\|4h\left(\frac{x+y}{2}\right) + h\left(x-y\right) - 2h(x) - 2h(y)\right\| \le \|\rho(h(x+y)+h(x-y)-2h(x)-2h(y))\|$$

for all $x, y \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.4). Then we have

$$\begin{aligned}\|h(x) - T(x)\| &= 4^n\left\|h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\ &\le 4^n\left(\left\|h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\le \frac{2\cdot 4^n\cdot 2^r}{(2^r-4)2^{nr}}\theta\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (3.4). □

Theorem 3.3. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.3)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2^r\theta}{4 - 2^r}\|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{2^r\theta}{4}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x)\right\| \\ &\le \frac{2^r\theta}{4}\sum_{j=l}^{m-1}\frac{2^{rj}}{4^j}\|x\|^r \end{aligned} \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 3.2. □

Remark 3.4. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

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