COMMON FIXED POINT THEOREMS OF MEIR-KEELER TYPE ON MULTIPLICATIVE METRIC SPACES

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Abstract. In this paper, we present some common fixed point theorems for two pairs of weakly compatible self-mappings on multiplicative metric spaces satisfying a generalized Meir-Keeler type contractive condition. The results obtained in this paper extend, improve and generalize some well known comparable results in literature.

1. Introduction

The classical results of Banach contraction principle [3], which is one of the most celebrated fixed point theorem, have been the inspiration for many researchers working in the area of metric fixed theory. In 1969, Meir and Keeler [14] obtained a remarkable generalization of the Banach contraction principle and since then the theorem has been extended in many directions. In 1976, Jungck [10] generalized the Banach contraction principle by introducing the idea of commuting maps. Introducing weakly commuting maps, Sessa [21] generalized the concept of commuting maps. Then Jungck generalized this idea, first to compatible mappings [11] and then to weakly compatible mappings [9].

In 2008, Bashirov et al. [4] studied the usefulness of a new calculus, called multiplicative calculus and defined a new distance, the so called multiplicative distance between two nonnegative real numbers as well as between two positive square matrices, by using the concept of a multiplicative absolute value. This provides the basis for multiplicative metric space. In 2012, Ozavsar and Cevikel [15] introduced the concept of multiplicative metric spaces by using the idea of multiplicative distance,

and gave some topological properties in such space. They also introduced the concept of multiplicative Banach's contraction mapping and proved fixed point results of such mapping on multiplicative metric spaces.

In 2002, Aamri and El-Moutawakil [1] introduced the notion of $(E.A)$ property for self mappings which contained the class of non-compatible mappings in metric spaces. It was pointed out that $(E.A)$ property allows replacing the completeness requirement of the space with a more natural condition of closedness of the range as well as relaxes the complexness of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. In 2009, Abbas et al. [2] introduced the concept of common property $(E.A)$. For more on $(E.A)$ and common $(E.A)$ properties, we refer to [1] and [12]. Recently in 2012, Chauhan et al. [6] introduce the notion of the joint common limit in the range of mappings property called $(JCLR)$ property and proved a common fixed point theorem for a pair of weakly compatible mappings using $(JCLR)$ property in fuzzy metric space.

The aim of this paper is to present some common fixed point theorems for two pairs of weakly compatible self-mappings in multiplicative metric spaces satisfying a generalized Meir-Keeler type contractive condition. The results obtained in this paper extend improve and generalize some well known comparable results in literature, in particular the results obtained in [5], [8], [13], [17-20].

2. Preliminaries

Definition 2.1 ([4]). Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

(i) $d(x,y) \geq 1$ for all $x, y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) $d(x,y) \leq d(x,z) \cdot d(z,y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

The pair (X,d) is called a *multiplicative metric space.*

Example 2.2 ([15]). Let R_+^n be the collection of n-tuples of positive real numbers. Let $d : R_+^n \times R_+^n \to R$ be defined as follows:

$d(x,y) = ||x_1 y_1 \cdot ||x_2 y_2 \cdots ||x_n y_n,$

where $x = (x_1, x_2, ..., x_n),\ y = (y_1, y_2, ..., y_n) \in R_+^n$ and $|\cdot| : R_+ \to R_+$ is defined as follows: $|a| = \begin{cases} a, & \text{if } a \geq 1, \\ ..1a, & \text{if } a < 1. \end{cases}$

Then it is obvious that all conditions of a multiplicative metric are satisfied.

Example 2.3. Let $d : R \times R \to [1, \infty)$ be defined as $d(x, y) = e^{|x-y|}$, where $x, y \in R$. Then d is a multiplicative metric.

Definition 2.4 ([15]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists a natural number N such that $n \geq N$, then $x_n \in B_\epsilon(x)$. The sequence $\{x_n\}$ is said to be a *multiplicative converging to x*, denoted by $x_n \to x$ $(n \to \infty)$.

Proposition 2.5 ([15]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x$ $(n \to \infty)$ if and only if $d(x_n, x) \to 1$ $(n \to \infty)$.

Definition 2.6 ([15]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. The sequence is called a *multiplicative Cauchy sequence* if it holds for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$.

Proposition 2.7 ([15]). Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1$ $(n, m \to \infty)$.

Definition 2.8 ([15]). A multiplicative metric space (X, d) is said to be *multiplicative complete* if every multiplicative Cauchy sequence in (X, d) is multiplicative convergent in X.

Proposition 2.9 ([15]). Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces, $S : X \to Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then S is multiplicative continuous at $x \in X$ if and only if $S(x_n) \to S(x)$ for every sequence $\{x_n\}$ with $x_n \to x$ $(n \to \infty)$.

Proposition 2.10 ([15]). Let (X, d_X) be a multiplicative metric space, $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x$, $y_n \to y$ $(n \to \infty)$, $x, y \in X$. Then $d(x_n, y_n) \to d(x, y)$ $(n \to \infty)$.

Definition 2.11 ([7]). The self-maps S and T of a multiplicative metric space (X, d) are said to be *compatible* if $\lim_{n \to \infty} d(STx_n, TSx_n) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$, for some $t \in X$.

Definition 2.12 ([7]). Suppose that S and T are two self mappings on a multiplicative metric space (X, d). The pair (S, T) is called *weakly compatible* if $Sx = Tx$, $x \in X$ implies $STx = TSx$. That is, $d(Sx, Tx) = 1 \Rightarrow d(STx, TSx) = 1$.

Remark 2.13. Compatible mappings must be weakly compatible, but the converse is not true.

3. MAIN RESULTS

We start our work by introducing the following three concepts on multiplicative metric spaces.

Definition 3.1. Let S and T be two self mappings on a multiplicative metric space (X, d). We say that S and T satisfy the *property* $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$$

for some $z \in X$.

Example 3.2. Let $X = [0, \infty)$ and let (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$. Define self-mappings T and S on X by $T(x) = 2x - 1$ and $S(x) = x^2$ for all x in X. For $x_n = 1 - ..1n$, we have,

$$\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = 1 \in X.$$

Thus the pair (S, T) satisfies the property $(E.A)$.

Definition 3.3. Let A, B, S and T be four self mappings on a multiplicative metric space (X, d). The pairs (A, S) and (B, T) are said to satisfy the *common property* $(E.A)$ if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z$$

for some $z \in X$.

Example 3.4. Let $X = [1, \infty)$ and (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Define self maps A, B, S and T on X by

$Ax = 3x - 1$ for all x, $Bx = x$ for all x, $Sx = x + 1$ for all x, and $Tx = 4 - x$ if $1 \le x \le 3$ and $Tx = x - 2$ if $x > 3$.

Take $x_n = 1 + ..1n$ and $y_n = 2 - ..1n$, one can easily verify that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = 2$$

This shows that the pairs (A, S) and (B, T) satisfy common property $(E.A)$.

Definition 3.5. Let A, B, S and T be four self mappings on a multiplicative metric space (X, d). The pairs (A, S) and (B, T) are said to satisfy the $(JCLR)$ *property*, if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = Sz = Tz$$

for some $z \in X$.

Example 3.6. Let $X = [0, \infty)$ and (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Define self maps A, B, S and T on X by

$$Ax = \begin{cases} 2, & \text{if } 0 \le x \le 2, \\ x + 3, & \text{if } 2 < x \le \infty, \end{cases} \qquad Sx = \begin{cases} 4 - x, & \text{if } 0 \le x \le 2, \\ x + 5, & \text{if } x > 2, \end{cases}$$

$$Bx = \begin{cases} 2, & \text{if } 0 \le x \le 2, \\ 3x - 1, & \text{if } 2 < x \le \infty, \end{cases} \qquad Tx = \begin{cases} x, & \text{if } 0 \le x \le 2, \\ x + 1, & \text{if } x > 2. \end{cases}$$

Take $\{x_n = 2 - ..1n\}$ and $\{y_n = 2 - ..1n\}$. Then

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n \lim_{n \to \infty} By_n = 2 = T(2) = S(2).$$

Thus the pairs (A, S) and (B, T) satisfy the $(JCLR)$ property.

Theorem 3.7. *Let A, B, S and T be four self mappings on a multiplicative metric space (X, d) satisfying the following conditions*

(3.7.1) *$AX \subseteq TX$ and $BX \subseteq SX$;*

(3.7.2) *given an $\varepsilon > 1$ and for all $x, y \in X$, there exists a $\delta \in (1, \varepsilon)$, such that*

(1) $$\varepsilon \le m(x, y) < \varepsilon + \delta \implies d(Ax, By) < \varepsilon$$

$$\text{where } m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\}$$

(3.7.3) *one of AX, BX, SX or TX is a complete subspace of X.*

Then

(I) *A and S have a coincidence point,*

(II) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) as well as (B, T) are weakly compatible, then the maps A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X by using (3.7.1), we have

$$Ax_{2n-2} = Tx_{2n-1} = y_{2n-1}$$

and

(2) $$Bx_{2n-1} = Sx_{2n} = y_{2n}.$$

We claim that $\{y_n\}$ is a Cauchy sequence. Let $d_n = d(y_n, y_{n+1})$.

Two cases arises. Suppose that $d_n = 1$ for some $n = 2k - 1$. Then $d(y_{2k-1}, y_{2k}) = 1$. This gives $y_{2k-1} = y_{2k}$, which implies that $Tx_{2k-1} = Ax_{2k-2} = Sx_{2k} = Bx_{2k-1}$, so T and B have a coincidence point. Further, if $d_n = 1$ for some $n = 2k$, then $d(y_{2k}, y_{2k+1}) = 1$. This gives $y_{2k} = y_{2k+1}$, which implies that $Tx_{2k+1} = Ax_{2k} = Sx_{2k} = Bx_{2k-1}$, so A and S have a coincidence point.

Now suppose that $d_n \neq 1$ for all n.

If for some $x, y \in X$, $d(x, y) = 1$ then we get $Ax = Sx$ and $By = Ty$. Hence the result

If $m(x, y) > 1$ for all $x, y \in X$, then by (1), we have

(3) $$d(Ax, By) < m(x, y)$$

Hence, we have

$$
\begin{aligned}
d_{2n-1} &= d(y_{2n-1}, y_{2n}) = d(Ax_{2n-2}, Bx_{2n-1}) \\
&< m(x_{2n-2}, x_{2n-1}) \\
&= \max\{d(Sx_{2n-2}, Tx_{2n-1}), d(Ax_{2n-2}, Sx_{2n-2}), d(Bx_{2n-1}, Tx_{2n-1})\} \\
&= \max\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n-2}), d(y_{2n}, y_{2n-1})\} \\
&= \max\{d_{2n-2}, d_{2n-1}\} = d_{2n-2}
\end{aligned}
$$

Therefore,

(4) $$d_{2n-1} < d_{2n-2}$$

Similarly,

$$d_{2n} < d_{2n-1}.$$

Hence we deduce that $d_n < d_{n-1}$ for all n.

Thus $\{d_n\}$ is a strictly decreasing sequence of positive real numbers. Hence converges to some limit, say p i.e.,

(5) $$\lim_{n \to \infty} d_n = p$$

Next we claim that $p = 1$. If $p \neq 1$, then by (5), there exists a $\delta > 1$ and a natural number r such that for each $n \geq r$,

(6) $$p < d(y_n, y_{n+1}) = d_n \leq p + \delta$$

In particular

$$
\begin{aligned}
m(x_{2n-1}, x_{2n}) &= \max\{d(Sx_{2n-1}, Tx_{2n}), d(Ax_{2n-1}, Sx_{2n-1}), d(Bx_{2n}, Tx_{2n})\} \\
&= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n})\} \\
&= \max\{d_{2n-1}, d_{2n}\} = d_{2n-1},
\end{aligned}
$$

and we get

$$
p < d_{2n-1} \le p + \delta
$$

Therefore, by using (1)

$$
d(Ax_{2n}, Bx_{2n-1}) = d(y_{2n+1}, y_{2n}) = d_{2n} < p,
$$

a contradiction. Hence $p = 1$; i.e.,

$$
\lim_{n \to \infty} d_n = \lim_{n \to \infty} d(y_n, y_{n+1}) = 1
$$

Now for any positive integer $m > n$, we have

$$
d(y_n, y_m) \le d(y_n, y_{n+1}) \cdot d(y_{n+1}, y_{n+2})...d(y_{m-1}, y_m)
$$

Since, $\lim_{n \to \infty} d(y_n, y_{n+1}) = 1$, it follows that

$$
\lim_{n \to \infty} d(y_n, y_m) \le 1 \cdot 1...1 = 1
$$

which shows that $\{y_n\}$ is a multiplicative Cauchy sequence in X.

Now suppose that SX is a complete subspace of X. Then the subsequence $y_{2n} = Sx_{2n}$ must have a limit in SX, call it z, and $v \in S^{-1}(z)$, so that $Sv = z$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n}\}$, the sequence $\{y_n\}$ also converges to z.

First we claim that $Av = z$. Suppose not, then on setting $x = v$ and $y = x_{2n-1}$ in (3), we get

$$
\begin{aligned}
d(Av, Bx_{2n-1}) &< m(v, x_{2n-1}) \\
&= \max\{d(Sv, Tx_{2n-1}), d(Av, Sv), d(Bx_{2n-1}, Tx_{2n-1})\}
\end{aligned}
$$

Taking the limit as $n \to \infty$, we have

$$
\begin{aligned}
d(Av, z) &< \max\{d(z, z), d(Av, z), d(z, z)\} \\
&= d(z, Av)
\end{aligned}
$$

a contradiction. Therefore $Av = z = Sv$. Hence the pair (A, S) has a point of coincidence. As $AX \subseteq TX$, $Av = z$ implies that $z \in TX$. Let $w \in T^{-1}(z)$, then $Tw = z$.

Next we claim that $Bw = z$. Suppose not, again by using (3), we get

$$
\begin{aligned}
d(y_{2n+1}, Bw) &= d(Ay_{2n}, Bw) < m(y_{2n}, w) \\
&= \max\{d(Sy_{2n}, Tw), d(Sy_{2n}, Ay_{2n}), d(Bw, Tw)\}
\end{aligned}
$$

Taking the limit as $n \to \infty$, we have

$$
\begin{aligned}
d(z, Bw) &< \max\{d(z, z), d(z, z), d(Bw, z)\} \\
&= d(Bw, z)
\end{aligned}
$$

a contradiction. Therefore $Bw = z = Tw$. Thus the pair (B, T) has a point of coincidence. Hence we have shown that $z = Sv = Av = Bw = Tw$.

The same result is obtained if we assume TX to be complete. Indeed, if AX is complete, then $z \in AX \subseteq TX$ and if BX is complete, then $z \in BX \subseteq SX$.

As the pairs (A, S) and (B, T) are weakly compatible, then $Az = ASv = SAv = Sz$ and $Bz = BTw = TBw = Tz$.

Next we claim that $Az = z$. If not, then by (3), we have

$$
\begin{aligned}
d(Az, z) &= d(Az, Bw) < m(z, w) \\
&= \max\{d(Sz, Tw), d(Sz, Az), d(Bw, Tw)\} \\
&= \max\{d(Az, z), d(Az, Az), d(z, z)\} = d(Az, z)
\end{aligned}
$$

a contradiction. Therefore, $Az = z$. Similarly, one can easily show that $Bz = z$. Thus z is a common fixed point of A, B, S and T. Uniqueness of the fixed point is an easy consequence of inequality (3.7.2). Hence the result. $\square$

Example 3.8. Let $X = [3, \infty)$ and (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Define self maps A, B, S and T on X by

$Ax = 3$ for all x, $\qquad Bx = \begin{cases} 3, & \text{if } x = 3 \text{ or } x > 5, \\ x + 1, & \text{if } 3 < x \leq 5, \end{cases}$

$Sx = \begin{cases} 3, & \text{if } x = 3, \\ 5, & \text{if } 3 < x \leq 5, \\ x - 2, & \text{if } x > 5, \end{cases} \qquad Tx = \begin{cases} 3, & \text{if } x = 3, \\ x + 5, & \text{if } x > 3. \end{cases}$

Then the self maps A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 3$. Moreover the maps satisfy neither the φ−contractive condition nor the Banach type contractive condition. Also, one may verify that the self maps A, B, S and T are discontinuous at the common fixed point $x = 3$ and SX is a complete subspace of X.

Now we shall improve the above theorem by using common property $(E.A)$, Since it relaxes containment of the range of one map into the range of other, which is utilized to construct the sequence of joint iterates in common fixed point considerations.

Theorem 3.9. *Let A, B, S and T be four self mappings on a multiplicative metric space (X, d) satisfying (3.7.2) and the following conditions*

(3.9.1) *the pairs (A, S) and (B, T) satisfy common property $(E.A)$,*

(3.9.2) *SX and TX are closed subsets of X.*

Then

(I) *A and S have a coincidence point,*

(II) *B and T have a coincidence point.*

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then the maps A, B, S and T have a unique common fixed point in X.

Proof. In view of (3.9.1), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z$$

for some $z \in X$. Since SX is closed subset of X, there exists a point $u \in X$ such that $z = Su$.

We claim that $Au = z$. If not, then by (3.7.2) or equation (3), take $x = u$, $y = y_n$. Then

$$d(Au, By_n) < m(u, y_n) = \max\{d(Su, Ty_n), d(Su, Au), d(By_n, Ty_n)\}$$

Taking the limit as $n \to \infty$, we have

$$d(Au, z) < \max\{d(z, z), d(z, Au), d(z, z)\} = d(z, Au)$$

a contradiction. Therefore, $Au = z = Su$, which shows that u is a coincidence point of the pair (A, S).

Since TX is also a closed subset of X, $\lim_{n \to \infty} Ty_n = z \in TX$, and hence there exists a $v \in X$ such that $Tv = z = Au = Su$.

Now we show that $Bv = z$. If $Bv \neq z$, then by using (3.7.2), take $x = u$, $y = v$, we get

$$d(Au, Bv) < m(u, v) = \max\{d(Su, Tv), d(Su, Au), d(Bv, Tv)\}$$

Which implies that

$$d(z, Bv) < \max\{d(z, z), d(z, z), d(Bv, z)\} = d(Bv, z)$$

which is a contradiction. Therefore, $Bv = z = Tv$, which shows that v is a coincidence point of the pair (B, T).

Since the pairs (A, S) and (B, T) are weakly compatible and

$$Au = Su, Bv = Tv, Az = ASu = SAu = Sz, Bz = BTv = TBv = Tz.$$

If $Az \neq z$, then by using (3.7.2), take $x = z$, $y = v$, we get

$$d(Az, Bv) < m(z, v) = \max\{d(Sz, Tv), d(Sz, Az), d(Bv, Tv)\},$$

$$d(Az, z) < \max\{d(Az, z), d(Az, Az), d(Bv, \ Bv)\} = d(Az, z)$$

a contradiction. Therefore, $Az = z = Sz$.

Similarly, one can prove that $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz$, and z is a common fixed point of A, B, S and T. Uniqueness of the fixed point is an easy consequence of the inequality (3.7.2). Hence the result. □

Example 3.10. Let $X = [0, \infty)$ and (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Define self maps A, B, S and T on X by

$$Ax = \begin{cases} 2, & \text{if } 0 \leq x \leq 2, \\ 3, & \text{if } x > 2, \end{cases} \qquad Bx = \begin{cases} 2, & \text{if } 0 \leq x \leq 2, \\ 1, & \text{if } x > 2, \end{cases}$$

$$Sx = \begin{cases} 4 - x, & \text{if } 0 \leq x \leq 2, \\ 6, & \text{if } x > 2, \end{cases} \qquad Tx = \begin{cases} x, & \text{if } 0 \leq x \leq 2, \\ 9, & \text{if } x > 2. \end{cases}$$

Take $\{x_n = 2 - ..1n\}$ and $\{y_n = 2 - ..1n\}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n \lim_{n\to\infty} By_n = 2 \in X.$$

Thus the pairs (A, S) and (B, T) satisfy common property $(E.A)$. One can easily verify that the self maps A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 2$. Here SX and TX are closed subspaces of X. Moreover the maps neither satisfy the φ−contractive condition nor the Banach type contractive condition. Also, one may notice that that neither $BX \nsubseteq SX$ nor $AX \nsubseteq TX$ and the self maps A, B, S and T are discontinuous at the common fixed point $x = 2$.

Finally, it is observed that common property $(E.A)$ requires the completeness or closedness of the subspaces for the existence of the common fixed point. So an attempt has been made to drop the closedness of the subspaces from Theorem 3.9 by using the $(JCLR)$ property.

Theorem 3.11. *Let A, B, S and T be four self mappings on a multiplicative metric space (X, d) satisfying (3.7.2) and the following conditions*

(3.11.1) *the pairs (A, S) and (B, T) satisfy the $(JCLR)$ property.*
Then
(*I*) *A and S have a coincidence point,*
(*II*) *B and T have a coincidence point.*
Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then the maps A, B, S and T have a unique common fixed point in X.

Proof. As the pairs (A, S) and (B, T) satisfy the $(JCLR)$ property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = Su = Tu$$

for some $u \in X$.

First we assert that $Au = Su$. By (3.7.2) or equation (3), take $x = u$, $y = y_n$, we get

$$d(Au, By_n) < m(u, y_n) = \max\{d(Su, Ty_n), d(Su, Au), d(By_n, Ty_n)\}$$

Taking the limit as $n \to \infty$, we have

$$d(Au, Su) < \max\{d(Sz, Su), d(Su, Au), d(Su, Su)\} = d(Su, Au)$$

a contradiction. Therefore, $Au = Su$, which shows that u is a coincidence point of the pair (A, S).

Secondly, we assert that $Bu = Tu$. By using (3.7.2), take $x = u$, $y = u$ to get

$$d(Au, Bu) < m(u, u) = \max\{d(Su, Tu), d(Su, Au), d(Bu, Tu)\},$$

$$d(Tu, Bu) < \max\{d(Su, Su), d(Tu, Tu), d(Bu, Tu)\} = d(Tu, Bu),$$

a contradiction. Hence $Bu = Tu$, which shows that u is a coincidence point of the pair (B, T). Thus we have $Tu = Bu = Au = Su$.

Now, we assume that $z = Tu = Bu = Au = Su$. Since the pairs (A, S) and (B, T) are weakly compatible and

$$Au = Su, Bu = Tu, Az = ASu = SAu = Sz \text{ and } Bz = BTu = TBu = Tz.$$

If $Az \neq z$, then, by using inequality (3.7.2), take $x = z$, $y = u$, to obtain

$$d(Az, Bu) < m(z, u) = \max\{d(Sz, Tu), d(Sz, Az), d(Bu, Tu)\},$$

$$d(Az, z) < \max\{d(Az, z), d(Az, Az), d(z, z)\} = d(Az, z),$$

a contradiction. Therefore, $Az = z = Sz$.

Similarly, one can prove that $Bz = Tz = z$. Hence $Az = Bz = Sz = Tz$ and z is a common fixed point of A, B, S and T. Uniqueness of the fixed point is an easy consequence of inequality (3.7.2). Hence the result. $\qquad\qquad\qquad\square$

Example 3.12. Let $X = [0, \infty)$ and (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Define self maps A, B, S and T on X by

$$Ax = \begin{cases} 2, & \text{if } 0 \leq x \leq 2 \text{ or } x > 5, \\ x + 1, & \text{if } 2 < x \leq 5, \end{cases} \qquad Sx = \begin{cases} 4 - x, & \text{if } 0 \leq x \leq 2, \\ 6 + x, & \text{if } x > 2, \end{cases}$$

$$Bx = \begin{cases} 2, & \text{if } 0 \leq x \leq 2 \text{ or } x > 5, \\ x + 2, & \text{if } 2 < x \leq 5, \end{cases} \qquad Tx = \begin{cases} x, & \text{if } 0 \leq x \leq 2, \\ 9 + x, & \text{if } x > 2. \end{cases}$$

Take $\{x_n = 2 - ..1n\}$ and $\{y_n = 2 - ..1n\}$. Then

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n \lim_{n \to \infty} By_n = 2 = T(2) = S(2).$$

Thus the pairs (A, S) and (B, T) satisfy the $(JCLR)$ property. One can easily verify that the self maps A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 2$. Moreover the maps neither satisfy the φ−contractive condition nor the Banach type contractive condition. Also, one may notice that that neither $BX \not\subseteq SX$ nor $AX \not\subseteq TX$ and the self maps A, B, S and T are discontinuous at the common fixed point $x = 2$.

References

1. M. Aamri & D. El Moutawakil: Some new common fixed point theorems under strict contractive conditions. *J. Math. Anal. Appl.* **270** (2002), no. 1, 181-188 .

2. M. Abbas, I. Altun & D. Gopal: Common fixed point theorems for non compatible mappings in fuzzy metric spaces. *Bull. Math. Anal. Appl.* **1** (2009), no. 2, 47-56.

3. S. Banach: Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. *Fundam. Math.* **3** (1922), 133-181.

4. A.E. Bashirov, E.M. Kurplnara & A. Ozyapici: Multiplicative calculus and its applicatiopns. *J. Math. Anal. Appl.* **337** (2008), 36-48.

5. H. Bouhadjera & A. Djoudi: On fixed point theorems of Meir and Killer type. *An. St. Univ. Ovidius Constanta.* **16** (2008), no. 2, 39-46.

6. S. Chauhan, W. Sintunavarat & P. Kumam: Common Fixed point theorems for weakly compatible mappings in fuzzy metric spaces using $(JCLR)$ property. *Applied Mathematics* **3** (2012), no. 9, 976-982.

7. F. Gu & Y.J. Cho: Common fixed point results for four maps satisfying ϕ-contractive condition in multiplicative metric spaces. *Fixed Point Theory Appl.* DOI:10.1186/s13663-015-0412-4.

8. K. Jha, R. Pant & S.L. Singh: Common fixed point for compatible mappings in metric spaces. *Radovi Mat.* **12** (2003), 107-114.

9. G. Jungck: Common fixed points for noncontinuous nonself maps on non-metric spaces. *Far East J. Math. Sci.* **4** (1996), 199-215.

10. _____: Commuting mappings and fixed points. *Am. Math. Mon.* **73** (1976), 261-263.

11. _____: Compatible mappings and common fixed points. *Int. J. Math. Math. Sci.* **9** (1986), no. 4, 771-779.

12. W. Lui, J. Wu & Z. Li: Common fixed points of single-valued and multi-valued maps. *Int. J. Math. Sci.* **19** (2005), 3045-3055.

13. M. Maiti & T.K. Pal: Generalizations of two fixed point theorems. *Bull. Calcutta Math. Soc.* **70** (1978), 57-61.

14. A. Meir & E. Keeler: A theorem on contraction mappings. *J. Math. Anal. Appl.* **28** (1969), 326-329.

15. M. Ozavsar & A.C. Cevikel: Fixed points of multiplicative contraction mappings on multiplicative metric spaces. arXiv:1205.5131v1 [math.GM], 2012.

16. R.P. Pant & K. Jha: A generalization of Meir-Keeler type common fixed point for four mappings. *J. Natural and Physical Sciences* **16** (2002), no. 1-2, 77-84.

17. _____: A generalization of Meir-Keeler type fixed point theorems for four mappings. *Ultra Science* **15** (2003), no. 1, 97-102.

18. S. Park & B.E. Rhoades: Meir-Keeler contractive conditions. *Math. Japonica* **26** (1981), no. 1, 13-20.

19. V. Popa: A general common fixed point theorem of Meir and Killer type for non continuous weak compatible mappings. *Filomat (Nis).* **18** (2004), 33-40.

20. J.H.N. Rao & K.P.R. Rao: Generalizations of fixed point theorem of Meir and Keeler type. *Indian J. Pure Appl. Math.* **16** (1985), no. 1, 1249-1262.

21. S. Sessa: On a weak commutativity condition of mappings in fixed point consideration. *Publ. Inst. Math. Soc.* **32** (1982), 149-153.

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