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INITIAL SOFT L-FUZZY PREPROXIMITIES

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Abstract. In this paper, we introduce the notions of soft L-fuzzy preproximities in complete residuated lattices. We prove the existence of initial soft L-fuzzy preproximities. From this fact, we define subspaces and product spaces for soft L-fuzzy preproximity spaces. Moreover, we give their examples.

1. Introduction

Hájek [5] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structures [6, 7-9]. Recently, Molodtsov [11] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1, 4]. Pawlak's rough set [12, 13] can be viewed as a special case of soft rough sets [4]. The topological structures of soft sets have been developed by many researchers [2, 7-9, 15-17].

Čimoka et.al [3] introduced L-fuzzy syntopogenous structures as fundamentals and application to L-fuzzy topologies, L-fuzzy proximities and L-fuzzy uniformities in a complete residuated lattice. Kim [7] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft $F : A \to P(U)$ where L is a complete residuated lattice. Kim [7-9] introduced the soft topological structures, soft L-fuzzy quasi-uniformities and soft L-fuzzy topogenous orders in complete residuated lattices.

In this paper, we prove the existence of initial soft L-fuzzy preproximities. From this fact, we define subspaces and product spaces for soft L-fuzzy preproximity spaces. Moreover, we give their examples.

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2. Preliminaries

Definition 2.1 ([5, 6]). An algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ is called a *complete residuated lattice* if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \to, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution * which is defined by $x \oplus y = (x^* \odot y^*)^*$ and $x^* = x \to 0$.

Lemma 2.2 ([5, 6]). *For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.*

(1) $1 \to x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$,

(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(9) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(10) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(12) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(13) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(14) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,

(15) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(16) $x \odot y \odot (z \odot w) \leq (x \odot z) \oplus (y \odot w)$.

(17) $x \to y = y^* \to x^*$.

Definition 2.3 ([7-9]). Let X be an initial universe of objects and E the set of parameters (attributes) in X. A pair (F, A) is called a *fuzzy soft set* over X, where $A \subset E$ and $F : A \to L^X$ is a mapping. We denote $S(X, A)$ as the family of all fuzzy soft sets under the parameter A.

Definition 2.4 ([7-9])**.** Let (F,A) and (G,A) be two fuzzy soft sets over a common universe X.

(1) (F,A) is a fuzzy soft subset of (G,A), denoted by $(F,A) \leq (G,A)$ if $F(\epsilon) \leq G(\epsilon)$, for each $\epsilon \in A$.

(2) $(F,A) \wedge (G,A) = (F \wedge G, A)$ if $(F \wedge G)(\epsilon) = F(\epsilon) \wedge G(\epsilon)$ for each $\epsilon \in A$.

(3) $(F,A) \vee (G,A) = (F \vee G, A)$ if $(F \vee G)(\epsilon) = F(\epsilon) \vee G(\epsilon)$ for each $\epsilon \in A$.

(4) $(F,A) \odot (G,A) = (F \odot G, A)$ if $(F \odot G)(\epsilon) = F(\epsilon) \odot G(\epsilon)$ for each $\epsilon \in A$.

(5) $(F,A)^* = (F^*, A)$ if $F^*(\epsilon) = (F(\epsilon))^*$ for each $\epsilon \in A$.

(6) $(F,A) \oplus (G,A) = (F \oplus G, A)$ if $(F \oplus G)(\epsilon) = (F^*(\epsilon) \odot G^*(\epsilon))^*$ for each $\epsilon \in A$.

Definition 2.5 ([8,9])**.** Let $S(X,A)$ and $S(Y,B)$ be the families of all fuzzy soft sets over X and Y, respectively. The mapping $f_\phi : S(X,A) \to S(Y,B)$ is a soft mapping where $f : X \to Y$ and $\phi : A \to B$ are mappings.

(1) The image of $(F,A) \in S(X,A)$ under the mapping f_ϕ is denoted by $f_\phi((F,A)) = (f_\phi(F), B)$ where

$$f_\phi(F)(b)(y) = \begin{cases} \bigvee_{a \in \phi^{-1}(\{b\})} (f_\phi(F(a))(y), & \text{if } \phi^{-1}(\{b\}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The inverse image of $(G,B) \in S(Y,B)$ under the mapping f_ϕ is denoted by $f_\phi^{-1}((G,B)) = (f_\phi^{-1}(G), A)$ where

$$f_\phi^{-1}(G)(a)(x) = f_\phi^{-1}(G(\phi(a)))(x), \ \forall a \in A, x \in X.$$

(3) The soft mapping $f_\phi : S(X,A) \to S(Y,B)$ is called injective (resp. surjective, bijective) if f and ϕ are both injective (resp. surjective, bijective).

Lemma 2.6 ([8,9])**.** *Let $f_\phi : S(X,A) \to S(Y,B)$ be a soft mapping. Then we have the following properties. For $(F,A), (F_i,A) \in S(X,A)$ and $(G,B), (G_i,B) \in S(Y,B)$,*

(1) $(G,B) \geq f_\phi(f_\phi^{-1}((G,B)))$ *with equality if f is surjective,*

(2) $(F,A) \leq f_\phi^{-1}(f_\phi((F,A)))$ *with equality if f is injective,*

(3) $f_\phi^{-1}(\bigvee_{i \in I}(G_i,B)) = \bigvee_{i \in I} f_\phi^{-1}((G_i,B))$,

(4) $f_\phi^{-1}(\bigwedge_{i \in I}(G_i,B)) = \bigwedge_{i \in I} f_\phi^{-1}((G_i,B))$,

(5) $f_\phi(\bigvee_{i \in I}(F_i,A)) = \bigvee_{i \in I} f_\phi((F_i,A))$,

(6) $f_\phi(\bigwedge_{i \in I}(F_i,A)) \leq \bigwedge_{i \in I} f_\phi((F_i,A))$ *with equality if f is injective,*

(7) $f_\phi^{-1}((G_1,B) \odot (G_2,B)) = f_\phi^{-1}((G_1,B)) \odot f_\phi^{-1}((G_2,B))$,

(8) $f_\phi^{-1}((G_1,B) \oplus (G_2,B)) = f_\phi^{-1}((G_1,B)) \oplus f_\phi^{-1}((G_2,B))$,

(9) $f_\phi((F_1,A)\odot(F_2,A)) \le f_\phi((F_1,A))\odot f_\phi((F_2,A))$ *with equality if f is injective.*

(10) $f_\phi((F_1,A)\oplus(F_2,A)) \le f_\phi((F_1,A))\oplus f_\phi((F_2,A))$.

Definition 2.7. A function $\delta : L^X \times L^X \to L$ is called a *soft L-fuzzy pre-proximity* on X if it satisfies the following conditions:

(SP1) $\delta((1_X,A),(0_X,A)) = 0$ and $\delta((0_X,A),(1_X,A)) = 0$.

(SP2) If $(F,A) \le (F_1,A)$ and $(G,A) \le (G_1,A)$, then
$\delta((F,A),(G,A)) \le \delta((F_1,A),(G_1,A))$.

(SP3) If $\delta((F,A),(G,A)) \ne 1$, then $(F,A) \le (G,A)^*$.

(SP4)

$$\delta((F_1,A)\odot(F_2,A),(H_1,A)\oplus(H_2,A)) \le \delta((F_1,A),(H_1,A)) \oplus \delta((F_2,A),(H_2,A)).$$

The triple (X,A,δ) is said to be a *soft L-fuzzy pre-proximity space.*

A soft L-fuzzy pre-proximity space is called a *soft L-fuzzy quasi-proximity* if (SQ)

$$\delta((F,A),(G,A)) \ge \bigwedge_{(H,A)\in S(X,A)}\{\delta((F,A),(H,A)) \oplus \delta((H,A)^*,(G,A))\}.$$

A soft L-fuzzy pre-proximity space is called *perfect* if

(P) $\delta(\bigvee_{i\in I}(F_i,A),(G,A)) \le \bigvee_{i\in I}\delta((F_i,A),(G,A))$.

Let (X,A,δ_1) and (X,A,δ_2) be soft L-fuzzy pre-proximity spaces. We say that δ_1 is *finer* than δ_2 (δ_2 is *coarser* than δ_1) if $\delta_1((F,A),(G,A)) \le \delta_2((F,A),(G,A))$ for all $(F,A),(G,A) \in S(X,A)$.

Let (X,A,δ_X) and (Y,B,δ_Y) be soft L-fuzzy pre-proximity spaces and $f_\phi : X \to Y$ be a soft map. Then f is called a fuzzy proximity soft map if $\forall (F,A),(G,A) \in S(X,A)$, $\delta_X((F,A),(G,A)) \le \delta_X((f_\phi((F,A)),(f_\phi((G,A)))$.

Remark 2.8. (1) If a complete residuated lattice $(L,\le,\odot,\oplus,^*)$ is a completely distributive lattice $(L,\le,\wedge,\vee,^*)$ with a strong negation * with $\odot=\wedge$ and $\oplus=\vee$, the above definition coincide with that in the sense [3].

(2) Let (X,A,δ) be a soft L-fuzzy pre-proximity space. By (SP4), we have

$$\delta(\odot_{i=1}^p(F_i,A),\oplus_{k=1}^p(G_k,A)) \le \bigwedge_{\sigma\in K}(\oplus_{i=1}^p\delta((F_i,A),(G_{\sigma(i)},A)))$$

where $K=\{\sigma \mid \sigma:\{1,2,...,p\}\to\{1,2,...,p\}$ is a bijective function$\}$.

(3) Let L be an idempotent complete residuated lattice, that is, $x\odot x = x$, for each $x\in L$. Since $(F,A)\odot(F,A) = (F,A)$ and $(G,A)\oplus(G,A) = (G,A)$, then $\delta((F,A),(G_1,A)\oplus(G_2,A)) \le \delta((F,A),(G_1,A)) \oplus \delta((F,A),(G_2,A))$ and $\delta((F_1,A)\odot(F_2,A),(G,A)) \le \delta((F_1,A),(G,A)) \oplus \delta((F_2,A),(G,A))$.

3. Initial soft L-fuzzy Preproximities

Theorem 3.1. *Let $\{(X_i, A_i, \delta_i) \mid i \in \Gamma\}$ be a family of soft L-fuzzy pre-proximity spaces. Let X be a set and, for each $i \in \Gamma$, $f_i : X \to X_i$ and $\phi_i : A \to A_i$ mappings. Define the function $\delta : S(X, A) \times S(X, A) \to L$ on X by*

$$\begin{aligned}&\delta((F, A), (G, A))\\&= \bigwedge\Big\{ \bigwedge_{\sigma\in K} \Big\{ \oplus_{j=1}^{p} \Big(\bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_j, A)), (f_i)_{\phi_i}((G_{\sigma(j)}, A)))\Big) \Big\}\Big\},\end{aligned}$$

where the first $\bigwedge$ is taken over all two finite families $\{(F_j, A) \mid (F, A) = \odot_{j=1}^{p}(F_j, A)\}$, $\{(G_k, A) \mid (G, A) = \oplus_{j=1}^{p}(G_{\sigma(j)}, A)\}$ and

$$K = \{\sigma \mid \sigma : \{1, ..., p\} \to \{1, ..., p\} \quad \textit{is a bijective function}\}.$$

Then:

(1) δ is the coarsest soft L-fuzzy pre-proximity on X which all $(f_i)_{\phi_i}$,$i \in \Gamma$, are fuzzy proximity soft maps.

(2) If $\{(X_i, A_i, \delta_i) \mid i \in \Gamma\}$ is a family of soft L-fuzzy quasi-proximity spaces, δ is a soft L-fuzzy quasi-proximity on X.

(3) A map $f_\phi : (Y, B, \delta_0) \to (X, A, \delta)$ is a fuzzy proximity soft map iff each $(f_i)_{\phi_i} \circ f_\phi : (Y, B, \delta_0) \to (X_i, A_i, \delta_i)$ is a fuzzy proximity soft map.

Proof. (1) First, we will show that δ is a soft L-fuzzy pre-proximity on X.

(SP1) Since $\delta((F, A), (0_X, A)) \le \delta_i((f_i)_{\phi_i}((F, A)), (0_{X_i}, A_i)) = 0$ for all $(F, A) \in S(X, A)$, it is clear.

(SP2) It follows from the definition of δ.

(SP3) We will show that if $(F, A) \not\le (G, A)^*$, then $\delta((F, A), (G, A)) = 1$.

Let $(F, A) \not\le (G, A)^*$. Then, for every two finite families $\{(F_j, A) \mid (F, A) = \odot_{j=1}^{p}(F_j, A)\}$ and $\{(G_k, A) \mid (G, A) = \oplus_{k=1}^{p}(G_k, A)\}$ and $\sigma \in K$, there exist $j_0, \sigma(j_0)$, x_0 such that $(F_{j_0}, A)(x_0) \not\le (G_{\sigma(j_0)}, A)(x_0)^*$. It follows that, for all $i \in \Gamma$,

$$(f_i)_{\phi_i}((F_{j_0}, A))((f_i)_{\phi_i}(x_0)) \not\le (f_i)_{\phi_i}((G_{\sigma(j_0)}, A))((f_i)_{\phi_i}(x_0))^*.$$

Since δ_i is a soft L-fuzzy pre-proximity on X_i,for each $i \in \Gamma$, by (SP3),

$$\delta_i((f_i)_{\phi_i}((F_{j_0}, A)), (f_i)_{\phi_i}((G_{\sigma(j_0)}, A))) = 1.$$

So, $\bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_{j_0}, A)), (f_i)_{\phi_i}((G_{\sigma(j_0)}, A))) = 1$. By Lemma 2.2(3), it follows

$$\oplus_{j=1}^{p}\Big(\bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_j, A)), (f_i)_{\phi_i}((G_{\sigma(j)}, A)))\Big) = 1,$$

for every two finite families $\{(F_j,A) \mid (F,A) = \odot_{j=1}^{p}(F_j,A)\}$ and $\{(G_k,A) \mid (G,A) = \oplus_{k=1}^{p}(G_k,A)\}$ and $\sigma \in K$. Hence $\delta((F,A),(G,A)) = 1$.

(SP4) Suppose there exist $(F_i,A),(G_i,A) \in S(X,A)$ such that

$$\begin{array}{l}\delta((F_1,A)\odot(F_2,A),(G_1,A)\oplus(G_2,A)) \\ \not\leq \delta((F_1,A),(G_1,A)) \oplus \delta((F_2,A),(G_2,A)).\end{array}$$

By the definition of $\delta((F_1,A),(G_1,A))$ and Lemma 2.2(6), there exist two finite families $\{(F_{1_j},A) \mid (F_1,A) = \odot_{j=1}^{p}(F_{1_j},A)\}$ and $\{(G_{1_{\sigma(j)}},A) \mid (G_1,A) = \oplus_{j=1}^{p}(G_{1_{\sigma(j)}},A)\}$ with a bijective function σ, we have

$$\begin{array}{l}\delta((F_1,A)\odot(F_2,A),(G_1,A)\oplus(G_2,A)) \\ \not\leq \Big\{ \oplus_{j=1}^{p} \Big(\bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_{1_j},A)),(f_i)_{\phi_i}((G_{1_{\sigma(j)}},A)))\Big)\Big\} \oplus \delta((F_2,A),(G_2,A))\end{array}$$

Again, by the definition of $\delta((F_2,A),(G_2,A))$ and Lemma 2.2(6), there exist two finite families $\{(F_{2_k},A) \mid (F_2,A) = \odot_{k=1}^{q}(F_{2_k},A)\}$ and $\{(G_{2_{\epsilon(k)}},A) \mid (F_2,A) = \oplus_{k=1}^{q}(G_{2_{\epsilon(k)}},A)\}$ with a bijective function ϵ, we have

$$\begin{array}{l}\delta((F_1,A)\odot(F_2,A),(G_1,A)\oplus(G_2,A)) \\ \not\leq \Big\{ \oplus_{j=1}^{p} \Big(\bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_{1_j},A)),(f_i)_{\phi_i}((G_{1_{\sigma(j)}},A)))\Big)\Big\} \\ \quad \oplus \Big\{ \oplus_{k=1}^{q} \Big(\bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_{2_k},A),(f_i)_{\phi_i}((G_{2_{\epsilon(k)}},A))\Big)\Big\}\end{array}$$

By Lemma 2.2(6), for each $j,\sigma(j)$ and $k,\epsilon(k)$, there exist $i_j, i_k \in \Gamma$ such that

$$\begin{array}{l}\delta((F_1,A)\odot(F_2,A),(G_1,A)\oplus(G_2,A)) \\ \not\leq \Big\{ \oplus_{j=1}^{p} \Big(\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_{1_j},A)),(f_{i_j})_{\phi_{i_j}}((G_{1_{\sigma(j)}},A)))\Big)\Big\} \\ \quad \oplus \Big\{ \oplus_{k=1}^{q} \Big(\delta_{i_k}((f_{i_j})_{\phi_{i_k}}((F_{2_k},A)),(f_{i_j})_{\phi_{i_k}}((G_{2_{\epsilon(k)}},A)))\Big)\Big\}\end{array}$$

On the other hand, since

$$(F_1,A)\odot(F_2,A) = \Big(\odot_{j=1}^{p} (F_{1_j},A)\Big) \odot \Big(\odot_{k=1}^{q} (F_{2_k},A)\Big),$$

$$(G_1,A)\oplus(G_2,A) = \Big(\oplus_{j=1}^{p} (G_{1_{\sigma(j)}},A)\Big) \oplus \Big(\oplus_{k=1}^{q} (G_{2_{\epsilon(k)}},A)\Big),$$

for a bijective function $\sigma \cup \epsilon$, we have

$$\begin{array}{l}\delta((F_1,A)\odot(F_2,A),(G_1,A)\oplus(G_2,A)) \\ \leq \Big\{ \oplus_{j=1}^{p} \Big(\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_{1_j},A)),(f_{i_j})_{\phi_{i_j}}((G_{1_{\sigma(j)}},A)))\Big)\Big\} \\ \quad \oplus \Big\{ \oplus_{k=1}^{q} \Big(\delta_{i_k}((f_{i_j})_{\phi_{i_k}}((F_{2_k},A)),(f_{i_j})_{\phi_{i_k}}((G_{2_{\epsilon(k)}},A)))\Big)\Big\}.\end{array}$$

It is a contradiction. Hence the condition (SP4) holds.

Second, from the definition of δ, for two families $\{(F,A) \mid (F,A) = (F,A)\}$ and $\{(G,A) \mid (G,A) = (G,A)\}$, since

$$\begin{array}{rl}\delta((F,A),(G,A)) & \leq \bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F,A)),(f_i)_{\phi_i}((G,A))) \\ & \leq \delta_i((f_i)_{\phi_i}((F,A)),(f_i)_{\phi_i}((G,A))),\end{array}$$

for each $i \in \Gamma$, $(f_i)_{\phi_i} : (X, A, \delta) \to (X_i, A_i, \delta_i)$ is a fuzzy proximity soft map.

If all $(f_i)_{\phi_i} : (X, A, \delta_0) \to (X_i, A_i, \delta_i)$ are fuzzy proximity soft maps, then, for all two finite families $\{(F_j, A) \mid (F, A) = \odot_{j=1}^{p}(F_j, A)\}$ and $\{(G_k, A) \mid (G, A) = \oplus_{k=1}^{p}(G_k, A)\}$ and $\sigma \in K$,

$$\begin{aligned}\delta((F, A), (G, A)) &= \bigwedge\{\oplus_{j=1}^{p} \bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_j, A)), (f_i)_{\phi_i}((G_{\sigma(j)}, A)))\} \\ &\geq \bigwedge\{\oplus_{j=1}^{p}\delta_0((F_j, A), (G_{\sigma(j)}, A))\} \\ &\geq \delta_0((F, A), (G, A)). \quad \text{(by Remark 2.8(2))}\end{aligned}$$

Thus, $\delta_0((F, A), (G, A)) \leq \delta((F, A), (G, A))$ for each $(F, A), (G, A) \in S(X, A)$.

(2) Let $\{(X_i, A_i, \delta_i) \mid i \in \Gamma\}$ be a family of soft L-fuzzy quasi-proximity spaces. We will show that δ is an soft L-fuzzy quasi-proximity on X.

Suppose there exist $(F, A), (G, A) \in S(X, A)$ such that

$$\delta((F, A), (G, A)) \not\geq \bigwedge_{(H,A)\in S(X,A)} \{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.$$

By the definition of δ, there are finite families $\{(F_j, A) \mid (F, A) = \odot_{j=1}^{p}(F_j, A)\}$ and $\{(G_k, A) \mid (G, A) = \oplus_{k=1}^{p}(G_k, A)\}$ and a bijective function σ such that

$$\begin{aligned}&\oplus_{j=1}^{p}\Big\{ \bigwedge_{i\in\Gamma} \delta_i((f_i)_{\phi_i}((F_j, A)), (f_i)_{\phi_i}((G_{\sigma(j)}, A)))\Big\} \\ &\not\geq \bigwedge_{(H,A)\in S(X,A)}\{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.\end{aligned}$$

It follows that for any $j, \sigma(j)$, there exists an $i_j \in \Gamma$ such that

$$\begin{aligned}&\oplus_{j=1}^{p}\Big\{\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_j, A)), (f_{i_j})_{\phi_{i_j}}((G_{\sigma(j)}, A)))\Big\} \\ &\not\geq \bigwedge_{(H,A)\in S(X,A)}\{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.\end{aligned}$$

Since δ_{i_j} is a soft L-fuzzy quasi-proximity on X_{i_j}, by (SQ), there exists $(H_{i_j}, A_{i_j}) \in S(X_{i_j}, A_{i_j})$ such that

(B)

$$\begin{aligned}&\oplus_{j=1}^{p}\Big\{\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_j, A)), (H_{i_j}, A_{i_j}))) \oplus \delta_{i_j}((H^*_{i_j}, A_{i_j}), (f_{i_j})_{\phi_{i_j}}((G_{\sigma(j)}, A)))\Big\} \\ &\not\geq \bigwedge_{(H,A)\in S(X,A)}\{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.\end{aligned}$$

On the other hand, put $(H, A) = \oplus_{j=1}^{p}(f_{i_j})^{-1}_{\phi_{i_j}}((H_{i_j}, A_{i_j}))$. Since

$$(f_{i_j})_{\phi_{i_j}}((f_{i_j})^{-1}_{\phi_{i_j}}((H_{i_j}, A_{i_j}))) \leq (H_{i_j}, A_{i_j}),$$

for the identity $\sigma(j) = j$, then

$$\begin{aligned}&\delta((F, A), (H, A)) \leq \oplus_{k=1}^{p}\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_j, A)), (f_{i_j})_{\phi_{i_j}}((f_{i_j})^{-1}_{\phi_{i_j}}((H_{i_j}, A_{i_j})))) \\ &\leq \oplus_{k=1}^{p}\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_j, A)), (H_{i_j}, A_{i_j})).\end{aligned}$$

Since $(H,A)^* = \odot_{j=1}^{p}(f_{i_j})^{-1}_{\phi_{i_j}}((H_{i_j},A_{i_j})^*)$, for $\sigma \in K$, we have

$$\begin{aligned}\delta((H^*,A),(G,A)) &\leq \oplus_{j=1}^{p}\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((f_{i_j})^{-1}_{\phi_{i_j}}((H_{i_j},A_{i_j}))^*),(f_{i_j})_{\phi_{i_j}}((G_{\sigma(j)},A))) \\ &\leq \oplus_{j=1}^{p}\delta_{i_j}((H_{i_j},A_{i_j})^*,(f_{i_j})_{\phi_{i_j}}((G_{\sigma(j)},A))).\end{aligned}$$

It implies

$$\begin{aligned}&\delta((F,A),(H,A)) \oplus \delta((H^*,A),(G,A)) \\ &\leq \oplus_{j=1}^{p}\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_j,A)),(H_{i_j},A_{i_j})) \oplus \Big\{\oplus_{j=1}^{p}\delta_{i_j}((H^*_{i_j},A_{i_j}),(f_{i_j})_{\phi_{i_j}}((G_{\sigma(j)},A)))\Big\} \\ &= \oplus_{j=1}^{p}\Big\{\delta_{i_j}((f_{i_j})_{\phi_{i_j}}((F_j,A)),(H_{i_j},A_{i_j})) \oplus \delta_{i_j}((H^*_{i_j},A_{i_j}),(f_{i_j})_{\phi_{i_j}}((G_{\sigma(j)},A)))\Big\}.\end{aligned}$$

It is a contradiction for (B). Thus, the result follows.

(3) Necessity of the composition condition is clear since the composition of fuzzy proximity soft maps is a fuzzy proximity soft map.

Each two finite families $\{(F_j,A) \mid f_\phi((F,A)) = \odot_{j=1}^{p}(F_j,A)\}$ and $\{(G_k,A) \mid f_\phi((G,A)) = \oplus_{k=1}^{p}(G_k,A)\}$ and each $\sigma \in K$, we have

$$\begin{aligned}&\delta(f_\phi((F,A)),f_\phi((G,A))) \\ &= \bigwedge\Big\{\oplus_{j=1}^{p}\Big(\bigwedge_{i\in\Gamma}\delta_i((f_i)_{\phi_i}((F_j,A)),(f_i)_{\phi_i}((G_{\sigma(j)},A)))\Big)\Big\}.\end{aligned}$$

It follows

$$(F,A) \leq f_\phi^{-1}(f_\phi((F,A))) = \odot_{j=1}^{p}f_\phi^{-1}((F_j,A)) \text{ and } (G,A) \leq \oplus_{k=1}^{p}f_\phi^{-1}((G_{\sigma(j)},A)).$$

Since $(f_i)_{\phi_i} \circ f_\phi$ is a fuzzy proximity soft map and for any $j,\sigma(j)$,

$$(f_i)_{\phi_i}(f_\phi(f_\phi^{-1}((F_j,A)))) \leq (f_i)_{\phi_i}((F_j,A)),$$

$$\delta_0(f_\phi^{-1}((F_j,A)),f_\phi^{-1}((G_{\sigma(j)},A)) \leq \delta_i((f_i)_{\phi_i}((F_j,A)),(f_i)_{\phi_i}((G_{\sigma(j)},A))).$$

Since $(F,A) \leq \odot_{j=1}^{p}f_\phi^{-1}((F_j,A))$, we have, for all $j,\sigma(j)$ and $i \in \Gamma$,

$$\begin{aligned}\delta_0((F,A),(G,A)) &\leq \bigwedge_{\sigma\in K}\Big\{\oplus_{j=1}^{p}\delta_0(f_\phi^{-1}((F_j,A)),f_\phi^{-1}((G_{\sigma(j)},A)))\Big\} \\ &\quad \text{(by Remark 2.8(2))} \\ &\leq \bigwedge_{\sigma\in K}\Big\{\oplus_{j=1}^{p}\bigwedge_{i\in\Gamma}\delta_i((f_i)_{\phi_i}((F_j,A)),(f_i)_{\phi_i}((G_{\sigma(j)},A)))\Big\}\end{aligned}$$

Hence $\delta_0((F,A),(G,A)) \leq \delta(f_\phi((F,A)),f_\phi((G,A)))$. □

From Remark 2.8(3) and Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $(L,\odot,\leq)$ be an idempotent complete residuated lattice. Let $\{(X_i,A_i,\delta_i) \mid i \in \Gamma\}$ be a family of soft L-fuzzy pre-proximity spaces. Let X be a* *set and, for each $i \in \Gamma$, $f_i : X \to X_i$ a mapping. Define the function $\delta : S(X,A) \times S(X,A) \to L$ on X by*

$$\delta((F,A),(G,A)) = \bigwedge \Big\{ \oplus_{j=1}^{p} \Big(\bigwedge_{i \in \Gamma} \delta_i((f_i)_{\phi_i}((F_j,A)),(f_i)_{\phi_i}((G_k,A))) \Big) \Big\},$$

where the first $\bigwedge$ is taken over all two finite families $\{(F_j,A) \mid (F,A) = \odot_{j=1}^{p}(F_j,A)\}$ and $\{(G_k,A) \mid (G,A) = \oplus_{k=1}^{q}(G_k,A)\}$. Then δ is the coarsest soft L-fuzzy preproximity on X which for each $i \in \Gamma$, $(f_i)_{\phi_i}$ is a fuzzy proximity soft map.

Let **SPROX** be a category with object (X, A, δ_X) where δ_X is a soft L-fuzzy preproximity with a morphism $f_\phi : (X, A, \delta_X) \to (Y, B, \delta_Y)$ is a fuzzy proximity soft map. Let **SET** be a category with object (X, f) where X is a set with a morphism $f : X \to Y$ is a function.

Theorem 3.3. *The forgetful functor U : **SPROX** $\to$ **Set** defined by $U(X,A,\delta) = X$ and $U(f) = f$ is topological.*

Proof. From Theorem 3.1, every U-structured source $(f_i : X \to U(X_i, A_i, \delta_i))_{i\in\Gamma}$ has a unique U-initial lift $(f_i : (X,A,\delta) \to (X_i,\delta_i))_{i\in\Gamma}$ where δ in Theorem 3.1. □

Corollary 3.4. *Let (Y, B, δ_Y) be a soft L-fuzzy pre-proximity space. Let X be a set , $f : X \to Y$ and $\phi : A \to B$ mappings. Define the function $\delta : S(X,A)\times S(X,A) \to L$ on X by*

$$\delta((F,A),(G,A)) = \bigwedge \Big\{ \bigwedge_{\sigma \in K} \Big\{ \oplus_{j=1}^{p} \Big(\delta_Y(f_\phi((F_j,A)), f_\phi((G_{\sigma(j)},A))) \Big) \Big\} \Big\},$$

where the first $\bigwedge$ is taken over all two finite families $\{(F_j,A) \mid (F,A) = \odot_{j=1}^{p}(F_j,A)\}$, $\{(G_{\sigma(j)},A) \mid (G,A) = \oplus_{j=1}^{p}(G_{\sigma(j)},A)\}$ and

$$A = \{\sigma \mid \sigma : \{1,...,p\} \to \{1,...,p\} \quad \textit{is a bijective function}\}.$$

Then δ is the coarsest soft L-fuzzy pre-proximity on X which f_ϕ is a fuzzy proximity soft map such that

$$\delta((F,A),(G,A)) = \delta_Y(f_\phi((F,A)), f_\phi((G,A))).$$

Proof. From Theorem 3.1 and the definition of $\delta((F,A),(G,A))$, we only show:

$$\delta((F,A),(G,A)) \geq \delta_Y(f_\phi((F,A)), f_\phi((G,A))).$$

Suppose $\delta((F,A),(G,A)) \not\geq \delta_Y(f_\phi((F,A)), f_\phi((G,A)))$. Then there exist two finite families $\{(F_j,A) \mid (F,A) = \odot_{j=1}^{p}(F_j,A)\}$, $\{(G_{\sigma(j)},A) \mid (G,A) = \oplus_{j=1}^{p}(G_{\sigma(j)},A)\}$

and $\sigma \in K$ such that

$$\oplus_{j=1}^{p}\Big(\delta_Y(f_\phi((F_j,A)),f_\phi((G_{\sigma(j)},A)))\Big) \not\geq \delta_Y(f_\phi((F,A)),f_\phi((G,A))).$$

On the other hand, since $\odot_{j=1}^{p} f_\phi((F_j,A)) \geq f_\phi(\odot_{j=1}^{p}(F_j,A))$ and $\oplus_{j=1}^{p} f_\phi((G_{\sigma(j)},A)) \geq f_\phi(\oplus_{j=1}^{p}(G_{\sigma(j)},A))$ from Lemma 2.6(9,10), we have

$$\begin{aligned}&\oplus_{j=1}^{p}\Big(\delta_Y(f_\phi((F_j,A)),f_\phi((G_{\sigma(j)},A)))\Big) \geq \delta_Y(\odot_{j=1}^{p} f_\phi((F_j,A)),\oplus_{j=1}^{p} f_\phi((G_{\sigma(j)},A)))\\ &\geq \delta_Y(f_\phi(\odot_{j=1}^{p}(F_j,A)),f_\phi(\oplus_{j=1}^{p}(G_{\sigma(j)},A))) = \delta_Y(f_\phi((F,A)),f_\phi((G,A))).\end{aligned}$$

It is a contradiction. Hence the result holds. □

Definition 3.5. Let (X,A,δ_X) be a soft L-fuzzy pre-proximity space , $Z \subset X$ and $C \subset A$. The pair (Z,C,δ) is said to be a *subspace* of (X,A,δ_X) if it is endowed with the initial soft L-fuzzy pre-proximity with respect to $(Z,i,(X,\delta_X))$ where i is the inclusion function. From Corollary 3.8, we define the function $\delta : L^Z \times L^Z \to L$ on A by

$$\delta((F,A),(G,A)) = \delta_X(i_i((F,A)),i_i((G,A))).$$

Definition 3.6. Let $X = \prod_{i\in\Gamma} X_i$ be the product of the sets from family $\{(X_i,A_i,\delta_i)| i \in \Gamma\}$ of soft L-fuzzy pre-proximity spaces. The initial soft L-fuzzy pre-proximity $\delta = \otimes\delta_i$ on X with respect to the family $\{\pi_i : X \to (X_i,A_i,\delta_i) \mid i \in \Gamma\}$ of all projection maps is called the *product soft L-fuzzy pre-proximity* of $\{\delta_i \mid i \in \Gamma\}$, and $(X,\prod_{i\in\Gamma} A_i,\otimes\delta_i)$ is called the *product soft L-fuzzy pre-proximity space.*

Example 3.7. Let $U = \{h_i \mid i = \{1,...,6\}\}$ with h_i=house and $E = \{e,b,w,c,i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings. Define a binary operation $\odot$ on $[0,1]$ by

$$x \odot y = \max\{0,x+y-1\},\ x \to y = \min\{1-x+y,1\}$$

$$x \oplus y = \min\{1,x+y\},\ x^* = 1-x$$

Then $([0,1],\wedge,\to,0,1)$ is a complete residuated lattice (ref. [5, 6]). Let $A = \{b,c,i\} \subset E$ and $X = \{h^1,h^4,h^5,h^6\}$. Put (H,A) be a fuzzy soft set as follow:

(H,A)	h^1	h^4	h^5	h^6	$(H,A)\odot(H,A)$	h^1	h^4	h^5	h^6
b	0.5	0.6	0.2	0.6	b	0.0	0.2	0.0	0.2
c	0.1	0.5	0.5	0.6	c	0.0	0.0	0.0	0.2
i	0.4	0.6	0.6	0.5	i	0.0	0.2	0.2	0.0

(H^*, A)	h^1	h^4	h^5	h^6
b	0.5	0.4	0.8	0.4
c	0.9	0.5	0.5	0.4
i	0.6	0.4	0.4	0.5

$(H^*, A) \oplus (H^*, A)$	h^1	h^4	h^5	h^6
b	1.0	0.8	1.0	0.8
c	1.0	1.0	1.0	0.8
i	1.0	0.8	0.8	1.0

(K, A)	h^1	h^4	h^5	h^6
b	0.6	0.5	0.4	0.6
c	0.7	0.4	0.6	0.6
i	0.5	0.3	0.3	0.7

$(H, A) \odot (K, A)$	h^1	h^4	h^5	h^6
b	0.1	0.1	0.0	0.2
c	0.0	0.0	0.1	0.2
i	0.0	0.0	0.0	0.2

(1) We define soft L-fuzzy preproximities $\delta_1, \delta_2 : S(X, A) \times S(X, A) \to L$ as

$$\delta_1((F,A),(G,A)) = \begin{cases} 0, & \text{if } (F,A) = (0_X, A) \text{ or } (G,A) = (0_X, A) \\ 0.4, & \text{if } (F,A) \leq (H,A) \leq (G,A)^*, \\ & (F,A) \not\leq (H,A) \odot (H,A) \\ 0.7, & \text{if } \ (0_X, A) \neq (F,A) \leq (H,A) \odot (H,A) \\ & \leq (G,A)^*, (H,A) \not\leq (G,A)^*, \\ 1, & \text{otherwise,} \end{cases}$$

$$\delta_2((F,A),(G,A)) = \begin{cases} 0, & \text{if } (F,A) = (0_X, A) \text{ or } (G,A) = (0_X, A) \\ 0.5, & \text{if } (F,A) \leq (K,A) \leq (G,A)^*, \\ 1, & \text{otherwise,} \end{cases}$$

But δ_i for $i = 1, 2$, is not a soft L-fuzzy quasi-proximity because

$$1 = \bigwedge_{(F,A)\in S(X,A)} (\delta_1((H,A)\odot(H,A),(F,A)) \oplus \delta_1((F^*,A),(H,A)^* \oplus (H,A)^*)) \\ \not\leq \delta_1((H,A)\odot(H,A),(H,A)^* \oplus (H,A)^*) = 0.7.$$

$$1 = \bigwedge_{(F,A)\in S(X,A)} (\delta_1((H,A)\odot(H,A),(F,A)) \oplus \delta_1((F^*,A),(H,A)^* \oplus (H,A)^*)) \\ \not\leq \delta_1((H,A)\odot(H,A),(H,A)^* \oplus (H,A)^*) = 0.7.$$

(2) By Theorem 3.1, let $f_1 = f_2 : X \to S$ and $\phi_1 = \phi_2 : A \to A$ be identity maps. We obtain the coarsest soft L-fuzzy preproximity $\delta : S(X, A) \times S(X, A) \to L$ which is finer than δ_i, $i = 1, 2$, as follows

$$\delta((F,A),(G,A)) = \begin{cases} 0, & \text{if } (F,A) = (0_X, A) \text{ or } (G,A) = (0_X, A) \\ 0.4, & \text{if } (F,A) \leq (H,A) \leq (G,A)^*, \\ & (F,A) \not\leq (H,A) \odot (H,A) \\ 0.5, & \text{if } (F,A) \leq (K,A) \leq (G,A)^*, \\ & (F,A) \not\leq (H,A) \odot (K,A) \\ 0.7, & \text{if } \ (0_X, A) \neq (F,A) \leq (H,A) \odot (H,A) \\ & \leq (G,A)^*, (H,A) \not\leq (G,A)^*, \\ 0.9, & \text{if } \ (0_X, A) \neq (F,A) \leq (H,A) \odot (K,A) \\ & \leq (G,A)^*, (H,A) \not\leq (K,A)^*, \\ 1, & \text{otherwise.} \end{cases}$$

REFERENCES

1. K.V. Babitha & J.J. Sunil: Soft set relations and functions. *Compu. Math. Appl.* **60**(2010), 1840-1849.
2. N. Cağman, S. Karatas & S. Enginoglu: Soft topology. *Comput. Math. Appl.* **62** (2011), no. 1, 351-358.
3. D. Čimoka & A.P. Šostak: L-fuzzy syntopogenous structures, Part I: Fundamentals and application to L-fuzzy topologies, L-fuzzy proximities and L-fuzzy uniformities. *Fuzzy Sets and Systems* **232** (2013), 74-97.
4. F. Feng, X. Liu, V.L. Fotea & Y.B. Jun: Soft sets and soft rough sets. *Information Sciences* **181** (2011), 1125-1137.
5. P. Hájek: *Metamathematices of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht (1998).
6. U. Höhle & S.E. Rodabaugh: *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory.* The Handbooks of Fuzzy Sets Series 3, Kluwer Academic Publishers, Boston, 1999.
7. Y.C. Kim & J.M. Ko: Soft L-topologies and soft L-neighborhood systems. *J. Math. Comput. Sci.* (to appear).
8. ______: Soft L-uniformities and soft L-neighborhood systems. *J. Math. Comput. Sci.* (to appear).
9. ______: Soft L-fuzzy quasi-uniformities and soft L-fuzzy topogenous orders. *Submit to J. Intelligent and Fuzzy Systems.*
10. R. Lowen: Fuzzy uniform spaces. *J. Math. Anal. Appl.* **82** (1981), 370-385.
11. D. Molodtsov: Soft set theory. *Comput. Math. Appl.* **37** (1999), 19-31.
12. Z. Pawlak: Rough sets. *Int. J. Comput. Inf. Sci.* **11** (1982), 341-356.
13. ______: Rough probability. *Bull. Pol. Acad. Sci. Math.* **32** (1984), 607-615.
14. A.A. Ramadan, E.H. Elkordy & Y.C. Kim: Perfect L-fuzzy topogenous space, L-fuzzy quasi-proximities and L-fuzzy quasi-uniform spaces. *J. Intelligent and Fuzzy Systems* **28** (2015), 2591-2604.
15. M. Shabir & M. Naz: On soft topological spaces. *Comput. Math. Appl.* **61** (2011), 1786-1799.
16. B. Tanay & M.B. Kandemir: Topological structure of fuzzy soft sets. *Comput. Math. Appl.* **61** (2011), no. 10, 2952-2957.
17. Hu Zhao & Sheng-Gang Li: L-fuzzifying soft topological spaces and L-fuzzifying soft interior operators. *Ann. Fuzzy Math. Inform.* **5** (2013), no. 3, 493-503.
18. Í. Zorlutuna, M. Akdag, W.K. Min & S. Atmaca: Remarks on soft topological spaces. *Ann. Fuzzy Math. Inform.* **3** (2012), no. 2, 171-185.

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